String Theory on Elliptic Curve Orientifolds and KR-Theory

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Abstract: We analyze the brane content and charges in all of the orientifold string theories on space-times of the form $E \times \mathbb{R}^8$, where E is an elliptic curve with holomorphic or anti-holomorphic involution. Many of these theories involve "twistings" coming from the B-field and/or sign choices on the orientifold planes. A description of these theories from the point of view of algebraic geometry, using the Legendre normal form, naturally divides them into three groupings. The physical theories within each grouping are related to one another via sequences of T-dualities. Our approach agrees with both previous topological calculations of twisted KR-theory and known physics arguments, and explains how the twistings originate from both a mathematical and a physical perspective.

Contents

1. Introduction

The purpose of this paper is to study type IIA and IIB string theories on all possible orientifold backgrounds for which the underlying spacetime manifold X is $\mathbb{T}^2 \times \mathbb{R}^8$. The $\mathbb{T}^2$ factor should be equipped with a complex structure, making it into an elliptic curve (over $\mathbb{C}$), as well as with a holomorphic or anti-holomorphic involution ι, which defines

the orientifold structure.[1] (We extend the involution ι to X by making it trivial on the $\mathbb{R}^8$ factor.) We discover that, and also explain why, the orientifold theories on elliptic curves are naturally divided into three groupings, with the theories in each grouping related to one another by sequences of T-dualities.

This is quite a natural problem for a variety of reasons. Compactifying string theories on elliptic curves is motivated by the fact that they are the simplest compact Calabi–Yau manifolds (complex manifolds with a global non-vanishing holomorphic volume form). Working with orientifolds is natural also—the orientifold construction generalizes the Gliozzi–Scherk–Olive projection and encompasses most of the standard supersymmetric string theories.

1.1. Motivation. The sigma-model of orientifold string theory on a spacetime X with involution ι involves *equivariant* maps $\varphi\colon \Sigma \to X$, so that $\iota \circ \varphi = \varphi \circ \Omega$. Here Σ is an oriented two-manifold, possibly with boundary (the case of open strings), called the *string worldsheet*, and Ω, called the *worldsheet parity operator*, is an orientation-reversing involution on Σ. We require Σ/Ω, though not necessarily Σ itself, to be connected. (Thus an allowable possibility is $\Sigma = \Sigma_0 \sqcup \overline{\Sigma}_0$, where Σ_0 is a connected oriented surface, $\overline{\Sigma}_0$ is the same surface with orientation reversed, and Ω interchanges the two.) See for example [10]; there some extra twisting data, which we are ignoring for the moment, is also taken into account, and the notation is slightly different.

Orientifold string theories include all of the standard theories of types IIA, IIB, and I, as well as a number of variants sometimes denoted IA, $\tilde{\text{I}}$ and $\widetilde{\text{IA}}$. We analyze all possible T-duality relationships between these theories when X is the product of an elliptic curve with flat 8-space. It will be apparent from the results below that all of these theories should be considered together, since they are linked to one another by T-duality.

For orientifold theories, as explained in [15,16] and [32, §5.2], D-brane charges are given by KR-theory in the sense of Atiyah [3]. We compute the relevant KR-groups in all cases, and relate these groups to the actual branes that arise. We also study how the KR-groups and branes are related under T-duality and mirror symmetry. In this context it is useful to quote from [21, §6]: "Since T-duality is related to the Fourier transform, and since the Fourier transform of a real function is not necessarily real, a theory of T-duality in type I string theory necessarily involves KR-theory, or Real K-theory in the sense of Atiyah".

There is already a fair body of literature on orientifold compactifications on S^1, and there is even some literature on T^2 orientifolds (e.g., [13, §7.2]). However, to our knowledge, this is the first attempt at a systematic study of all type II orientifold string theories on $\mathbb{T}^2 \times \mathbb{R}^8$ that includes a calculation of all the KR groups and a study of all possible T-dualities. We also take into account all possibilities for the complex structure, using the classification in [4]. Considering the complex structure is important, since elliptic curves are the simplest case for checking predictions of mirror symmetry. Understanding elliptic curve orientifolds will also be the first step in attacking orientifolds on higher-dimensional Calabi–Yau manifolds, such as abelian varieties, K3 surfaces, and most of all, Calabi–Yau threefolds. For example, a large class of interesting K3 surfaces come with elliptic curve fibrations.

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[1] Note that in some of the literature, the word "orientifold" is used to denote the quotient space X/ι, but it is really essential to keep track of the *pair* (X, ι) and not just the quotient.

1.2. Outline of the paper. This paper begins in Sect. 2 with a review of the classification of holomorphic and anti-holomorphic involutions on elliptic curves, taken from [4]. The classification of anti-holomorphic involutions is equivalent to the classification of elliptic curves defined over $\mathbb{R}$, found in [1]. Next, in Sect. 3 we review the KR-theory of Atiyah and all its twisted versions (including those coming from a sign choice on the components of the fixed set). We then record all the groups that occur for the various possible involutions and twistings. Most of these calculations are taken from [12], but we also relate the results to earlier calculations made in [18,28] and to classifications of twistings of KR by Moutuou [23,24].

The heart of this paper consists of Sects. 4, 5, and 6. We begin by describing the T-dualities that relate the various orientifold string theories on elliptic curves (with a holomorphic or anti-holomorphic involution). Most of these theories only live on a certain portion of the moduli space of elliptic curves with Kähler structure and B-field. This moduli space is described by two parameters τ (describing the complex structure) and ρ (describing the Kähler form and B-field), which are interchanged under T-duality. It turns out that the orientifold theories break into three groupings, and iterated T-dualities relate all of the theories in a single grouping. This fact was known before (e.g., in [13], though some cases go back to [8,16,33]), but our description of what happens to the involutions is more explicit. In Sect. 5, we attack the problem of how to explain the three T-duality groupings in purely geometric terms, without recourse to physical arguments. Here it turns out that algebraic and complex geometry plays a crucial role; the T-duality groupings can be explained perfectly in terms of the Legendre normal forms of real elliptic curves, and the uniformization of these curves in terms of Jacobi elliptic functions. Finally, in Sect. 6, we give a complete description of the D-brane and O-plane charges in the various theories, and explain how these transform under T-duality.

2. The Classification of Holomorphic and Anti-Holomorphic Involutions

A torus $\mathbb{T}^2$ with a complex structure can be identified with $\mathbb{C}/\Lambda$ for some lattice Λ. The holomorphic maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ are given by complex affine maps $z \mapsto \gamma z + \delta$ sending Λ into Λ'. Thus we can rotate and scale so that the lattice Λ is generated by 1 and a complex number τ with Im $\tau > 0$. Note that

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$$

leaves the torus invariant up to holomorphic isomorphism. For applications to string theory we want our torus to be equipped with a Kähler form $J \sim \sqrt{G} dx \wedge dy$ and the NS–NS two-form B-field B, which combine to give an invariant $\rho = \int_{\mathbb{T}^2}(B + iJ)$ in the upper half-plane. T-duality along with the gauge invariance $\rho \mapsto \rho + 1$ implies

$$\rho \mapsto \frac{a\rho + b}{c\rho + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$$

also leaves the torus invariant. Therefore, the quantum moduli space of $\mathbb{T}^2$ (with its geometry as given by ρ) is given by a product of two copies of the quotient of the upper half-plane by $\mathrm{PSL}(2, \mathbb{Z})$. In this context, mirror symmetry [30] corresponds to the interchange $(\tau, \rho) \mapsto (\rho, \tau)$.

In [4], the authors look at all holomorphic and anti-holomorphic involutions of $\mathbb{T}^2$ combined with the worldsheet parity operator, which correspond to the possible orientifold structures for type IIB and type IIA theories, respectively.

Table 1. Table of anti-holomorphic involutions

Case	τ	$j(\tau)$	α	β	s	Fixed pts
(a)	$i\tau_2$ with $\tau_2 > 1$	$j > 1$	1	0	2	$\mathrm{Im}(z) = 0$; $\mathrm{Im}(z) = \tau_2/2$
			-1	0	2	$\mathrm{Re}(z) = 0$; $\mathrm{Re}(z) = 1/2$
			1	$1/2$	0	
			-1	$\tau/2$	0	
(b)	i	1	$1 \sim -1$	0	2	$\mathrm{Im}(z) = 0$; $\mathrm{Im}(z) = 1/2$
			$i \sim -i$	0	1	$z = re^{i\pi/4}, r \in \mathbb{R}$
			$1 \sim -1$	$1/2$	0	
(c)	$e^{i\theta}$ with $\pi/3 < \theta < \pi/2$	$(0,1)$	τ	0	1	$z = re^{i\theta/2}, r \in \mathbb{R}$
			$-\tau$	0	1	$z = ire^{i\theta/2}, r \in \mathbb{R}$
(d)	$e^{i\pi/3}$	0	$1 \sim e^{2i\pi/3} \sim e^{4i\pi/2}$	0	1	$\mathrm{Im}(z) = 0, \sqrt{3}/2$
			$e^{i\pi/3} \sim -1 \sim e^{5i\pi/3}$	0	1	$\mathrm{Re}(z) = 0, 1/2$
(e)	$\frac{1}{2} + i\tau_2$ with $\tau_2 > \frac{1}{2}\sqrt{3}$	$j < 0$	1	0	1	$\mathrm{Im}(z) = 0, \tau_2$
			-1	0	1	$\mathrm{Re}(z) = 0, 1/2$

The fixed set of a holomorphic involution on a complex elliptic curve E is a closed complex submanifold, hence is either empty, a finite non-empty set, or everything. Holomorphic involutions are always of the form $z \mapsto \pm z + \delta$. When we choose the $+$ sign, δ is a 2-torsion point in E, hence is 0 (giving the trivial involution) or an element of E of order precisely 2 (giving a free involution). When we choose the $-$ sign, δ can be any point in E and there are exactly four fixed points (the 2-torsion points in E shifted by $\delta/2$).

An anti-holomorphic involution φ of $\mathbb{T}^2$ must be induced by a self-map $z \mapsto \alpha\bar{z} + \beta$ of $\mathbb{C}$ preserving Λ and of order 2 modulo translation by an element of Λ. All of the anti-holomorphic involutions of $\mathbb{T}^2$ were worked out in [1] and they are given by Table 1. A necessary and sufficient condition for an elliptic curve to admit an anti-holomorphic involution is for its j-invariant to be real, $j(\tau) \in \mathbb{R}$.

Table 1 gives the invariant known as the *species*, s, of each involution. The species gives the number of components of the fixed point locus of the involution. The authors of [4] show that the species also gives the charges of the O-planes present. The classification in Table 1 also has an interpretation in terms of algebraic geometry. Any complex torus of complex dimension 1 is automatically a smooth projective variety and an elliptic curve E defined over $\mathbb{C}$. An anti-holomorphic involution ι makes this into a *real* elliptic curve; i.e., E is defined over $\mathbb{R}$ and ι corresponds to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $E(\mathbb{C})$. The fixed set E^ι is the set of real points $E(\mathbb{R})$; topologically it is just a disjoint union of s circles. The fact that $s \leq 2$ is just a special case (since elliptic curves have genus 1) of Harnack's curve theorem, and the classification by species is familiar from the theory of real elliptic curves [1]. The classification of IIA orientifold theories by species was pointed out by Sagnotti [29].

As we said earlier, when we combine the involutions in Table 1 with the worldsheet parity operator, D-brane and O-plane charges should be classified by KR-theory. In [18, Example A.5] the authors calculate the KR-theory for involutions with non-trivial species, i.e., $s = 1$ or 2. They show

$$KR^0(\mathbb{T}^2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2^{s-1},$$
$$KR^{-1}(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2^{s+1}.$$

Note that when the fixed locus has two components, $KR^{-1}(\mathbb{T}^2 \times \mathbb{R}^8)$ is isomorphic to $KO(\mathbb{T}^2 \times \mathbb{R}^8)$. At the moment, this might appear accidental, but we will see that this can be explained by a chain of T-duality isomorphisms.

3. KR with a Sign Choice and Calculations for Tori

It was proposed in [22,32], and is now generally accepted, that D-brane charges in string theory should be classified by some variant of K-theory. In *orientifold* theories, charges should be classified by some variant of KR-theory, as described by Witten in [32]. However, classical KR-theory can only apply when all O-planes have the same charge. When O-planes with opposite charges are present, the appropriate substitute is KR-*theory with a sign choice*, which we described in the companion paper [12]. In this section, we will briefly review KR-theory and KR-theory with a sign choice, as well as certain twisted variants. All these twistings of KR-theory were discussed and classified by Moutuou [23,24,26], though this may not be readily apparent because of the great generality of Moutouou's framework. (Moutuou deals with $\mathbb{Z}_2$-graded algebras over Real groupoids, but here we only need the case where the grading is trivial and the groupoid reduces to a Real space.)

We will also discuss some of the different notations appearing in the literature and the relations between them, and review and further amplify the calculations from [12] for the case of 2-torus orientifolds. This section is purely topological; we temporarily ignore geometrical structures such as Riemannian metrics, complex structures, and Kähler forms, except insofar as they illuminate the topology.

KR-theory, in the sense of Atiyah [3], is the cohomology theory that classifies stable isomorphism classes of virtual Real vector bundles on a "Real" space (X, ι). A Real space is a locally compact (Hausdorff) space X, together with a self-homeomorphism ι of X of period 2. A Real vector bundle on such a space is a complex vector bundle E, together with a conjugate-linear bundle automorphism of E of period 2, covering ι. If X is compact, $KR(X)$ is the group of formal differences $[E] - [F]$, where E and F are Real vector bundles over X, and we identify $[E] - [F]$ with $[E'] - [F']$ if there is an isomorphism of Real bundles $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ for some Real bundle G over X. When X is only *locally* compact, $KR(X)$ is defined similarly, but with E and F required to be trivialized and isomorphic in a neighborhood of infinity.

For string theory on a smooth manifold X, the charges of D-branes are classified by pairs of vector bundles (E, F), the Chan–Paton bundles on the branes and anti-branes, modulo the equivalence $(E, F) \sim (E \oplus H, F \oplus H)$. D-branes on orientifolds of the form $X/(\iota \cdot \Omega)$, where X is a smooth manifold, ι is an involution on X, and Ω is the world sheet parity operator, are classified by vector bundles on X that are equivariant under the action of $\iota \cdot \Omega$. Ω sends a vector bundle E to its complex conjugate $\bar{E}$. Therefore, a vector bundle E is $\iota \cdot \Omega$-equivariant if there exists an isomorphism, φ, from the pullback $\iota^* E$ to $\bar{E}$ such that $(\varphi \iota^*)^2 = 1$, which is exactly the Reality condition of Atiyah. Thus we naturally arrive at the group $KR(X)$ (the spacetime involution ι being understood).

More generally, D-brane charges are classified by $KR^{-j}(X)$, where the index j depends on the dimension of the brane. To define the higher KR-groups we must first introduce some notation. Let $\mathbb{R}^{p,q} = \mathbb{R}^p + i\mathbb{R}^q$ with the involution ι given by complex conjugation, and let $S^{p,q}$ be the unit sphere (of dimension $p + q - 1$) in $\mathbb{R}^{p,q}$. (In this notation, the roles of p and q are the reverse of those in the notation used by Atiyah in [3], but the same as the notation in [6,20,27].) We define

$$KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q}).$$

This obeys the periodicity condition [3, Theorem 2.3]

$$KR^{p,q}(X) \cong KR^{p+1,q+1}(X),$$

where the isomorphism is given by cup product with the Bott class. Since $KR^{p,q}$ only depends on the difference $p - q$, we can define

$$KR^{q-p}(X) = KR^{p,q}(X).$$

$KR^j(X)$ is periodic with period 8.

When we compactify string theory on on an m-dimensional space M, so that the spacetime manifold is $\mathbb{R}^{10-m,0} \times M$, we are interested in the charges of D-branes in the non-compact dimensions. So we want to consider Dp-branes of codimension $9 - m - p$ in $\mathbb{R}^{9-m,0}$. These can arise from both Dp-branes located at a particular point in M or higher dimensional D-branes that wrap non-trivial cycles in M. Furthermore, we only want to consider systems with finite energy, so we only want to classify systems that are asymptotically equivalent to the vacuum in the transverse space $\mathbb{R}^{9-m-p,0}$. That means that the system must be equivalent to the vacuum on an entire copy of M at infinity. Mathematically this means we want to add a copy of M at infinity (i.e., take the product with M of the one-point compactification of $\mathbb{R}^{9-m-p,0}$) and consider bundles on $S^{10-m-p,0} \times M$ that are trivialized on the copy of M at infinity. Such bundles are classified by $KR^{-i}(S^{10-m-p,0} \times M, M)$. This can be related to the KR-theory of M via the isomorphism

$$KR^{-i}(S^{10-m-p,0} \times M, M) \cong KR^{p+m-9-i}(M). \tag{1}$$

Dp-brane charges are classified by $KR^{p+m-9-i}(M)$ where i will depend on the string theory and M. We are considering the case when M is a Real elliptic curve, so $m = 2$.

This classification of D-brane charges includes the usual classification of type I brane charges by KO-theory and type II brane charges by complex K-theory. The type I theory is obtained by letting $\iota = 1$. This corresponds to the well known fact that the type I theory is the type IIB theory divided out by the action of Ω. In terms of the KR-theory classification, being equivariant means that E is isomorphic (in a way fixing the base X) to $\bar{E}$, or that E is real. The classification of equivariant Real bundles on X is thus the same as that of real bundles on X, giving the well known mathematical result [3]

$$KR(X) \cong KO(X)$$

when ι is trivial. To obtain the usual type II classification of D-branes in a spacetime X, we use the result from [3, Proposition 3.3]

$$KR\left(X \overset{\frown}{\sqcup} X\right) \cong K(X),$$

where the involution exchanges the two copies of X.

Often, when studying the K-theory classification of D-branes for the type II theories on a smooth manifold, the full indexing of $K^{-i}(X)$ is ignored, since it has period 2. While this is often most convenient for the purposes of mathematical calculations, to determine the brane content it is often more useful to use the relative K-theory given by the isomorphism (1). For the trivial case of type IIB D-branes in Minkowski spacetime, the distinction between $K^0(\text{pt})$ to classify $D9$-branes and $K^{-2}(\text{pt})$ is inconsequential. However, for our current purposes, the distinction is very important. So we will want to keep track of the full $\mathbb{Z}/8$-graded group $KR^*(X)$.

KR-theory with a sign choice, introduced in [12], is a variant of KR-theory for a Real space (X, ι) with a choice α of $\pm$ signs, one for each component of the fixed set X^ι. This theory needs to be defined via noncommutative geometry, and we refer the reader to [12] for the precise definition, but it has the property that on a component F of X^ι with positive sign choice, $KR_\alpha^*(F) = KO^*(F)$, the usual K-theory of real vector bundles, whereas on a component F of X^ι with negative sign choice, $KR_\alpha^*(F) = KSp^*(F)$, the K-theory of quaternionic vector bundles. This is precisely what is appropriate if F is an O^+- (resp., O^--) plane. (Note that there is some disagreement in the literature about what should be called an O^+-plane and what should be called an O^--plane, but we are following the convention in [33, §2.3]. As Witten points out, the associated tadpoles have *opposite* sign.)

The basic facts about KR-theory can be found in [3] or in [20, §1.10]—note that these sources use opposite indexing conventions and that we are following Lawson-Michelsohn, not Atiyah, so that $\mathbb{R}^{p,q} = \mathbb{R}^p \oplus i\mathbb{R}^q$ with involution fixing the $\mathbb{R}^p$ summand and multiplying by -1 on the $\mathbb{R}^q$ summand. For locally compact but non-compact Real spaces, we always use KR-theory with compact supports. For any real space (X, ι) (often we will suppress the involution in the notation), $KR^j(\mathbb{R}^{1,0} \times X) \cong KR^{j-1}(X)$ and $KR^j(\mathbb{R}^{0,1} \times X) \cong KR^{j+1}(X)$. If X is compact and has an ι-fixed point x_0, then the inclusion $\{x_0\} \hookrightarrow X$ is equivariant and equivariantly split, so $KR^j(X) \cong \widetilde{KR}^j(X) \oplus KO^j$, where KO^j means $KO^j(\text{pt})$ and $\widetilde{KR}^j(X) = KR^j(X \smallsetminus \{x_0\})$. Thus $KR^j(S^{1,1}) \cong KR^j(\mathbb{R}^{0,1}) \oplus KO^j \cong KO^{j+1} \oplus KO^j$, and $KR^j(S^{2,0}) \cong KR^j(\mathbb{R}^{1,0}) \oplus KO^j \cong KO^{j-1} \oplus KO^j$. We also have $KR^j(S^{0,2} \times X) \cong KSC^j(X)$, the self-conjugate K-theory of Anderson [2] and Green [14], by [3, Proposition 3.5].

Note that since our spacetime manifolds will always be of the form $X \times \mathbb{R}^{8,0}$, where X is a two-dimensional Real space, and since KR-theory has Bott periodicity of period 8, there is a natural isomorphism $KR^j(X \times \mathbb{R}^{8,0}) \cong KR^j(X)$, and we can ignore the $\mathbb{R}^8$ factor for purposes of this section. (However, it will be needed in Sect. 4 when we talk about specific branes.)

In [12], we computed the KR with a sign choice for all possible holomorphic or antiholomorphic involutions on complex elliptic curves X. In fact there are not that many different topological possibilities.

3.1. Holomorphic involutions.
If the involution is holomorphic, either it is trivial, or X is homeomorphic to $S^{1,1} \times S^{1,1}$ as a Real space, or the involution is free and X is homeomorphic to $S^{0,2} \times S^{2,0}$.

3.1.1. Trivial involutions.
For spaces with trivial involution, KR-theory reduces to KO-theory. Topologically, a $\mathbb{T}^2$ with trivial involution is just the Real space $S^{2,0} \times S^{2,0}$, and $KR^j(S^{2,0} \times S^{2,0}) \cong KO^j(S^1 \times S^1) \cong KO^j(S^1) \oplus KO^{j-1}(S^1) \cong KO^j \oplus KO^{j-1} \oplus KO^{j-1} \oplus KO^{j-2}$. The associated physical theory is the type I string theory on $\mathbb{T}^2$.

Just for completeness, note that if E is an elliptic curve with trivial involution, we can put a holomorphic involution on $E \amalg E$ that simply interchanges the two factors. This space is $\mathbb{T}^2 \times S^{0,1}$ as a Real space, and $KR^j(\mathbb{T}^2 \times S^{0,1}) \cong K^j(\mathbb{T}^2)$, which is $\cong \mathbb{Z}^2$ in each degree. The associated physical theory is ordinary Type IIB theory on E (with no involution).

Table 2. $KO^j(T^2, \widetilde{w}_2)$, KR^j (species 1), and $KR^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$

$j \bmod 8$	$KO^j(T^2, \widetilde{w}_2)$	KR^j (species 1)	$KR^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$
0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2$	$\mathbb{Z}$
-1	$\mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2$
-2	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$
-3	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$
-4	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
-5	0	$\mathbb{Z}$	$\mathbb{Z}^2$
-6	$\mathbb{Z}$	0	$\mathbb{Z}$
-7	$\mathbb{Z}^2$	$\mathbb{Z}$	0

3.1.2. Four fixed points. A $\mathbb{T}^2$ with a holomorphic involution with four fixed points is topologically just $S^{1,1} \times S^{1,1}$. And we obtain

$$
\begin{aligned}
KR^j(S^{1,1} \times S^{1,1}) &\cong KR^j(S^{1,1}) \oplus KR^j(\mathbb{R}^{0,1} \times S^{1,1}) \\
&\cong KR^j(S^{1,1}) \oplus KR^{j+1}(S^{1,1}) \\
&\cong KO^j \oplus KO^{j+1} \oplus KO^{j+1} \oplus KO^{j+2}.
\end{aligned}
$$

When there are four fixed points, there are two other interesting possible assignments of signs. When the sign choice is $(+, +, -, -)$, we can identify X with

$$
S^{1,1}_{(+,-)} \times S^{1,1} \cong (S^{1,1}_{(+,-)} \times \{\text{pt}\}) \amalg (S^{1,1}_{(+,-)} \times \mathbb{R}^{0,1})
$$

and we obtain

$$
KR^j_{(+,+,-,-)}(S^{1,1} \times S^{1,1}) \cong KSC^{j+2} \oplus KSC^{j+1},
$$

as was shown in [12].

The sign choice $(+, +, +, -)$ requires a more complicated calculation which was done in [12]; the result appears in Table 2.

Note that mathematically we could also consider the sign choice $(-, -, -, +)$. This however does not make physical sense. If the net O-plane charge is negative, then tadpole cancellation would require adding anti-branes, which would violate supersymmetry. For mathematical completeness, we note that the relevant KR-groups can be obtained from $KR^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$ by shifting the index by 4.

3.1.3. Free involutions. A $\mathbb{T}^2$ with a holomorphic involution with no fixed points is topologically just $S^{0,2} \times S^{2,0}$ (which is equivalent to $S^{0,2} \times S^{0,2}$ as will be discussed later). And we obtain

$$
\begin{aligned}
KR^j(S^{0,2} \times S^{2,0}) &\cong KSC^j(S^1) \\
&\cong KSC^j \oplus KSC^{j-1}.
\end{aligned}
$$

Note that in this case the groups are periodic with period 4, which is in accordance with [18, Proposition 1.8], though in general that statement is false ($S^{0,4}$ provides a counterexample, as one can see from [3]).

3.2. Antiholomorphic involutions. The study of KR-theory for antiholomorphic involutions is a special case of the study of KR-theory for real algebraic curves. This has been studied extensively in [18, 28], which provide methods of calculation, though we will need to correct two misprints in those papers. We can also take the antiholomorphic involution on $E \sqcup \bar{E}$ that interchanges the two factors, and we again get complex K-theory $K^j(\mathbb{T}^2)$, but this time with a focus on odd-dimensional D-branes. The associated physical theory is ordinary Type IIA theory on E (with no involution).

3.2.1. Species 2. A $\mathbb{T}^2$ with an antiholomorphic involution of species 2 is topologically just $S^{1,1} \times S^{2,0}$. And we obtain

$$\begin{aligned} KR^j(S^{1,1} \times S^{2,0}) &\cong KO^j(S^1) \oplus KO^{j+1}(S^1) \\ &\cong KO^j \oplus KO^{j-1} \oplus KO^{j+1} \oplus KO^j. \end{aligned}$$

In case of species 2, there is also the sign choice $(+, -)$, in which case we obtain

$$\begin{aligned} KR^j_{(+,-)}(S^{1,1} \times S^{2,0}) &\cong KR^j_{(+,-)}(S^{1,1}) \oplus KR^{j-1}_{(+,-)}(S^{1,1}) \\ &\cong KSC^{j+1} \oplus KSC^j. \end{aligned}$$

3.2.2. Species 0. A $\mathbb{T}^2$ with an antiholomorphic involution of species 0 is topologically just $S^{0,2} \times S^{1,1}$. And we obtain

$$\begin{aligned} KR^j(S^{0,2} \times S^{1,1}) &\cong KSC^j(S^{1,1}) \\ &\cong KSC^j \oplus KSC^j(\mathbb{R}^{0,1}) \cong KSC^j \oplus KSC^{j+1}. \end{aligned}$$

Note that in this case the groups are periodic with period 4. Furthermore, the final result is in accordance with [18, Example A.3] with genus $g = 1$. [There is a small misprint in [18]; the calculation of $KR^{-*}(X)$ is correct and does follow from collapse of the spectral sequence $H^p_G(X; KR^q) \Rightarrow KR^{p+q}(X)$, but $E_2^{2,-2} = H^2(X/G; \mathbb{Z}(-1)) \cong \mathbb{Z}$, not 0. For purposes of our present application, $G = \mathbb{Z}_2$ and $X = \mathbb{T}^2$, X/G is a Klein bottle, and $H^2(X/G; \mathbb{Z}(-1)) \cong H_0(X/G; \mathbb{Z}) \cong \mathbb{Z}$ by (twisted) Poincaré duality.]

3.2.3. Species 1. The calculation of $KR^j(X)$ when X is a real elliptic curve of species 1 is a bit tricky and was done in [12, Theorem 4]. The result is that

$$KR^j(X) \cong \left(KO^j\right)^2 \oplus K^{j-1},$$

and also appears in Table 2.

It is interesting to compare this calculation with [18, Corollary 4.2], that says that the natural map $K_j(X; \mathbb{Z}_2) \to KR^{-j}(X; \mathbb{Z}_2)$ sending algebraic to topological K-theory is an isomorphism for j sufficiently large ($j \geq 1$ in fact will do). Here K-theory or KR-theory with $\mathbb{Z}_2$ coefficients is related to the integral theory by a universal coefficient or Bockstein exact sequence

$$\begin{aligned} 0 &\to KR^{-j}(X)/2 \to KR^{-j}(X; \mathbb{Z}_2) \to {}_2KR^{-j+1}(X) \to 0, \\ 0 &\to K_j(X)/2 \to KR_j(X; \mathbb{Z}_2) \to {}_2K_{j-1}(X) \to 0, \end{aligned} \tag{2}$$

where ${}_2KR^{-j+1}(X)$ denotes the 2-torsion in $KR^{-j+1}(X)$, and similarly for K_j. The torsion subgroup of $K_j(X)$ was computed in [28], but there is a small typo in the

Table 3. Algebraic K-theory mod 2 for a real elliptic curve of species 1

$j \bmod 8$	Order of $K_j(X; \mathbb{Z}_2) \cong KR^{-j}(X; \mathbb{Z}_2)$
0	2^2
1	2^3
2	2^4
3	2^3
4	2^2
5	2
6	0
7	2

statement of [28, Main Theorem 0.1]. $K_2(X)_{\text{tors}}$ should contain $\nu + 1$ copies of $\mathbb{Z}_2$ (here ν is the species), not ν copies as written. (This result was miscopied from [28, Theorem 4.6].) The K-theory with $\mathbb{Z}_2$ coefficients, or the KR theory with $\mathbb{Z}_2$ coefficients, is then as given in Table 3.

3.3. Twisted groups. Finally, in the case of the trivial involution, we also have twisted groups with a non-zero twist $\widetilde{w}_2 \in H^2(\mathbb{T}^2, \mathbb{Z}_2)$. Such twisted KO-theory was introduced in [11], and can be identified with the topological K-theory of a noncommutative algebra that is locally, but not globally, isomorphic to continuous functions on $\mathbb{T}^2$ with values in a matrix algebra over $\mathbb{R}$, since the automorphism group of $M_n(\mathbb{R})$ has the homotopy type of $PO(n)$ and $BPSO(2n)$ approximates $K(\mathbb{Z}_2, 2)$ in low dimensions. The twisted KO-groups also appear in Table 2 and in Witten's "theory with no vector structure" [33].

Twistings and sign choices in KR-theory have been unified in work of Moutuou [23, 24]. He constructs and computes a *graded Brauer group* [26] of graded real continuous-trace algebras over a Real space (X, ι). The equivalence relation is Morita equivalence over X and the group operation is graded tensor product (over X). For our purposes we don't need the grading, so we get a *Brauer group* of (ungraded) real continuous-trace algebras, which turns out to be

$$\text{BrR}(X, \iota) \cong H^0(X^\iota, \mathbb{Z}_2) \oplus H_\iota^2(X, \mathcal{S}), \tag{3}$$

where the first summand is the group of *sign choices* and the second group is *equivariant* sheaf cohomology (this is discussed in greater detail in [25]) for the Real sheaf $\mathcal{S}$ of germs of S^1-valued continuous functions and we use the complex conjugation involution on S^1. The second summand encodes the (*Real*) *Dixmier-Douady class*; in the notation of [24, Proposition 4.4.9], this is the ungraded analogue $\text{BrR}_0(X)$ of $\widehat{\text{BrR}_0}(X)$. In the same notation, the first summand is $H^0(X, \text{Inv } \mathcal{K})$, where $\text{Inv } \mathcal{K}$ is the ungraded analogue of the sheaf $\text{Inv } \widehat{\mathcal{K}}$. But it is easy to see that this sheaf is supported on the fixed set, where it has stalks ± 1 corresponding to the two possible local sign choices (orthogonal type or symplectic type), thus giving (3).

4. T-Duality

In this section we will discuss how the various orientifolds classified in Sect. 2 are related via T-duality. These relationships have already been discussed in [13]. In [12]

we showed that you need to include a sign choice and that twisting KR-theory by this sign choice correctly classifies charges in T-dual theories. While the need for using KR-theory and the geometric meaning of the twisting caused by the B-field are well understood, there is no purely geometric explanation of why T-duality requires a sign choice. In this section we will simply review the various T-duality relationships. We will give a geometric description for all of the possible T-dualities between elliptic curve orientifolds, including an explanation for all sources of twisting (both sign choice and B-field) in the following section. In Sect. 6 we will describe the brane content in the different theories using the K-theoretic analysis of [12] together with this geometric description.

Since the right- and left-movers have the same chirality in the type IIB theory, only holomorphic involutions are compatible with the type IIB theory. Similarly the type IIA theory is only compatible with antiholomorphic involutions since the left- and right-movers have opposite chirality. Since T-duality (on a single circle factor) interchanges the type IIA string theory with the type IIB theory, it also exchanges holomorphic and antiholomorphic involutions. The various theories can be broken into 3 groups, with the theories in a single group related via T-duality. Note that real elliptic curves (the spacetimes for type IIA orientifold theories) are generally grouped by their species. However, as we saw in [12], the type IA and $\widetilde{IA}$ theories are both defined on species 2 real elliptic curves but cannot be related by a T-duality. Our geometric description will show that the type $\widetilde{IA}$ theory should be grouped with the species 0 real elliptic curves even though it is species 2. Since we have not yet defined a mathematical way to define the different T-duality groupings, we will classify them in terms of their physical theories for now.

The first group contains the type I theory as well as as the type IIA theory on an annulus (known as the type I$'$ or IA theory) and the type IIB theory on $\mathbb{T}^2/\mathbb{Z}_2$ with four fixed points (and four $O7^+$-planes). The second group contains the type $\tilde{I}$ and type $\widetilde{IA}$ theories as well as the type IIA theory on a Klein bottle and the type IIB theory with four fixed points corresponding to two $O7^-$-planes and two $O7^+$-planes. The third group contains the type I theory without vector structure described in [33] (the type I theory with non-trivial B-field), the type IIA theory on a Möbius strip, and the type IIB theory with one $O7^-$-plane and three $O7^+$-planes. The fact that the last two of these theories belong in the same T-duality grouping was already pointed out in [8]. Note that each of the three groups contains one type IIB theory with four fixed points—such theories are classified the net O-plane charge—and also contains one type IIA theory on a quotient of the torus by an orientation-reversing involution. This would provide two natural ways to classify the groups, but instead we choose to refer to them as the type I, type Ĩ, and "type I without vector structure" groups.

The physical moduli space for string theory on a real elliptic curve is determined by the complex structure, τ, and the Kähler modulus, ρ. The moduli space for the type IIA theories (real elliptic curves with an antiholomorphic involution) is shown in Fig. 1. This picture appeared already in [7, Figs. 2 and 3]. As can be seen from the figure, the complex structure is constrained, while the Kähler modulus is free. After a T-duality transformation, we obtain the type IIB theory with the roles of τ and ρ reversed. Therefore, the complex structure is free and the Kähler modulus is constrained for holomorphic involutions. The constraints on ρ in the type IIB theory confirm the known result that there are two possible values of the B-field in the type I theory, $B = 0, \frac{1}{2}$, corresponding to the two vertical legs in the first factor of Fig. 1.
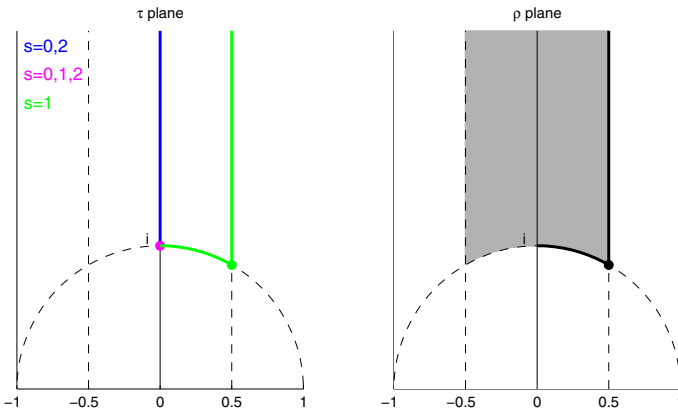
Fig. 1. Physical moduli space of string theory on a real elliptic curve with antiholomorphic involution corresponding to the type IIA theory

At first glance, the arc $\tau = e^{i\theta}$, with $\frac{\pi}{3} < \theta < \frac{\pi}{2}$ would seem to imply the T-dual IIB theory would have an unallowable value of B, since $0 < \operatorname{Re} \rho < \frac{1}{2}$. If we let $u = \sin\theta$ then the arc is described by $\tau = \sqrt{1 - u^2} + iu$, with $\frac{\sqrt{3}}{2} < u < 1$. Performing the SL $(2, \mathbb{Z})$ transformations $\tau \mapsto \tau - 1$ and then $\tau \mapsto -\frac{1}{\tau}$ sends τ to

$$
\begin{aligned}
\widetilde{\tau} &= \frac{1}{1 - \tau} \\
&= \frac{1}{1 - \sqrt{1 - u^2} - iu} \\
&= \frac{1}{2} + i\frac{u}{2(1 - \sqrt{1 - u^2})}.
\end{aligned}
$$

Since τ and $\widetilde{\tau}$ are related by an SL $(2, \mathbb{Z})$ transformation, they describe equivalent elliptic curves. This shows us that for any real elliptic curve (elliptic curve with antiholomorphic involution) there is a representative with $\operatorname{Re} \tau = 0$ or $\frac{1}{2}$. This matches with the fact that that the only possible values of the B-field ($\operatorname{Re} \rho$) for type IIB theories on elliptic curve orientifolds are 0 and $\frac{1}{2}$.

Proposition 1. *The stable D-brane charges in type IIA orientifold theories on elliptic curves do not not depend on the B-field.*

Proof. This follows immediately the observation in [4] that a non-trivial B-field only affects O-planes that wrap the entire elliptic curve. All of the type IIA theories contain either no O-planes or O-planes that wrap a 1-cycle in the elliptic curve. Alternatively, to put this in purely mathematical terms, twisting of KR-theory by the B-field amounts to a $\tilde{w}_2$ twist in H^2 (with $\mathbb{Z}_2$ coefficients) of a component of the fixed set of ι, and for this to be non-zero requires $\iota \equiv 1$.

This can also be seen in the physical moduli space for type IIB $\mathbb{T}^2$ orientifolds versus the one for type IIA $\mathbb{T}^2$ orientifolds (Fig. 1). For type IIB theories, the Kähler modulus, ρ, is constrained while the complex modulus, τ, is unconstrained. The opposite is true for the type IIA theories. Changing the B-field for a type IIA elliptic curve orientifold theory changes the complex structure of the T-dual IIB elliptic curve orientifold theory.

As can be seen in Fig. 1 and the list of holomorphic involutions given at the beginning of Sect. 2, all of the type IIB elliptic curve orientifold theories are well defined for all complex structures. Therefore the D-brane charges in the T-dual IIB theory are independent of the choice of complex structure, and stable D-brane charges in the original IIA theory must not depend on the B-field.

Proposition 1 can be viewed as saying that the stable D-brane charges in the type IIB elliptic curve orientifold theories do not depend on whether or not you compactify on a rectangular or a non-rectangular torus. It is important to note, however, that while the Dp-brane *charges* will remain unchanged, the actual *sources* could be affected by a non-trivial B-field. This is because a non-trivial B-field can affect D-branes that wrap both compact directions [4]. Another way to see this is that T-dualizing in a direction that is not tangential or normal to the direction wrapped by a brane will affect the resulting brane. We will find in Sect. 6 that the direction of T-duality is usually constrained in the IIB theories to well defined directions relative to the branes, making the brane content clear in a manner that is independent of the choice of complex structure (or B-field for the IIA theories).

Using Proposition 1 and the following discussion, we will generally assume that all type IIA theories have trivial B-field, or equivalently that all IIB theories are compactified on rectangular tori. This assumption does not affect any of our results. We will first consider the two groupings containing only type IIB theories with trivial B-fields together, and then consider the inclusion of non-trivial B-fields. We do this to clarify the difference between twistings by the B-field and twistings by the sign choice.

4.1. T-duality for elliptic curve orientifolds with trivial B-field.

Two of the three T-duality groups only contain type IIB theories with trivial B-field. They are the group containing the type I theory and the group containing the type $\tilde{\text{I}}$ theory. These are the two groups whose IIA theories only exist on rectangular tori. Let us first consider the group containing the type I theory with trivial B-field.

4.1.1. The type I theory.

The type I theory compactified on $\mathbb{T}^2$ corresponds to the type IIB orientifold theory compactified on $S^{2,0} \times S^{2,0}$. In [12] we described how the chain of T-dualities starting from this theory can be obtained by compactifying the type IIB theory on $S^{2,0}$ [6,27] on an additional circle.

Beginning with the type IIB theory compactified on $S^{2,0} \times S^{2,0}$, which is just the type I theory compactified on $\mathbb{T}^2 \cong S^1 \times S^1$, T-dualizing a single copy of $S^{2,0}$ will transform it to $S^{1,1}$. Therefore, T-dualizing one circle of the type IIB theory on $S^{2,0} \times S^{2,0}$ (corresponding to the involution $z \mapsto z$) will give the type IIA theory on either $S^{2,0} \times S^{1,1}$ (corresponding to the involution $z \mapsto \bar{z}$), or $S^{1,1} \times S^{2,0}$ (corresponding to the involution $z \mapsto -\bar{z}$). This accounts for all of the species 2 antiholomorphic involutions (see Table 1).

As can be seen from Table 1, the involutions $z \mapsto \pm\bar{z}$ only correspond to $S^{1,1} \times S^{2,0}$ if the complex modulus is $\tau = i\tau_2$ with $\tau_2 \geq 1$. This tells us that we must have a rectangular torus. Our 2-torus is also equipped with a Kähler form $J \equiv \sqrt{G}dx \wedge dy$ and the NS-NS 2-form B-field B, which combine to give the Kähler modulus $\rho = \int_{\mathbb{T}^2}(B + iJ)$. T-duality exchanges τ and ρ. Since τ is purely imaginary in the type IIA theory, ρ must be purely imaginary in the T-dual theory. Therefore, the type I theory compactified on a 2-torus cannot have any B-field (the only non-zero possibility for a B-field is $B = \frac{1}{2}$, which gives the type I theory without vector structure as described in [33], and will be
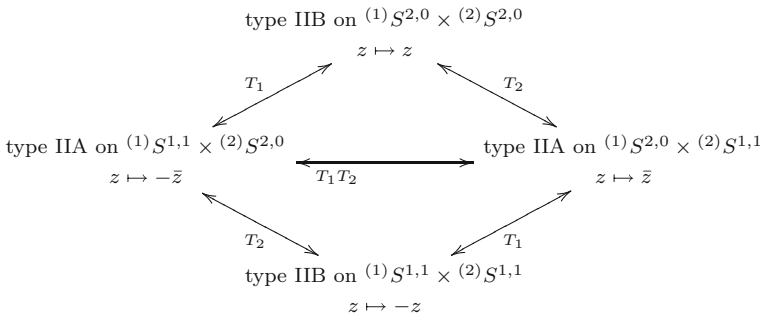
type IIB on $^{(1)}S^{2,0} \times {}^{(2)}S^{2,0}$

$$z \mapsto z$$

T_1 T_2

type IIA on $^{(1)}S^{1,1} \times {}^{(2)}S^{2,0}$ type IIA on $^{(1)}S^{2,0} \times {}^{(2)}S^{1,1}$

$$z \mapsto -\bar{z} \qquad\qquad T_1 T_2 \qquad\qquad z \mapsto \bar{z}$$

T_2 T_1

type IIB on $^{(1)}S^{1,1} \times {}^{(2)}S^{1,1}$

$$z \mapsto -z$$

Fig. 2. Chain of T-dualities connecting the various theories related to the type I theory with trivial B-field. T_i represents T-duality on the indicated *circle*

discussed later). As always, we only consider the case where the type IIA theory has zero B-field so that the T-dual IIB theory is on a rectangular torus as well.[2]

After T-dualizing one of the two circles in the type I theory we can T-dualize the other circle. This corresponds to T-dualizing the copy of $S^{2,0}$ in the type IIA theory on $S^{1,1} \times S^{2,0}$ or equivalently, simultaneously T-dualizing both circles in the original type I theory. This gives the type IIB theory on $S^{1,1} \times S^{1,1}$ which corresponds to the spacetime involution $z \mapsto -z$. This can be easily seen by composing the involutions that describe the 2 individual T-dualities, $z \mapsto \bar{z}$ and $z \mapsto -\bar{z}$. The type IIB theory on $S^{1,1} \times S^{1,1}$ has 4 $O7^+$-planes located at the four fixed points of $z \mapsto -z$ which correspond to the 2-torsion points of the elliptic curve: 0, $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1}{2} + \frac{\tau}{2}$. This chain of dualities can be neatly displayed as in Fig. 2. At the Gepner point corresponding to $\tau = i$ in the type IIA theory, there is a rotational symmetry under multiplication by i, so the involutions $z \mapsto \pm\bar{z}$ are equivalent. This collapses the horizontal line in Fig. 2, corresponding to the fact that the torus is square and there is no difference between the two circles.

4.1.2. The type $\tilde{I}$ theory. There are a couple of ways we can compactify the type $\tilde{I}$ theory on an elliptic curve. The first way is to compactify the type $\tilde{I}$ theory on a single circle, and then on another circle with trivial involution. For a single compact dimension, the type $\tilde{I}$ theory is the type IIB orientifold $(\mathbb{R}^9 \times S^1)/(\iota \cdot \Omega)$ where ι is the spacetime involution that rotates the compact direction π radians. In our notation, this is the type IIB theory on $\mathbb{R}^{9,0} \times S^{0,2}$. The T-dual of the type $\tilde{I}$ theory is the type $\widetilde{IA}$ theory [13, §7.2]. The type IA theory contains two $O8^+$-planes. The type $\widetilde{IA}$ theory is obtained from the type IA theory by replacing one of the $O8^+$-planes with an $O8^-$-plane. Using the notation $S_\alpha^{p,q}$ of [12], where α is the sign choice on the components of the fixed set, the compactification manifold for the type $\tilde{I}$ theory is $S_{(+,-)}^{1,1}$.

Let x be the coordinate of the compact direction in the type $\tilde{I}$ theory. Considering the circle as $\mathbb{R}/\mathbb{Z}$, we see that $S^{0,2}$ is the circle mod the involution

$$x \mapsto x + \frac{1}{2}.$$

Under T-duality this becomes the dual circle mod the involution

$$\tilde{x} \mapsto -\tilde{x} + \frac{1}{2}.$$

---

[2] Assuming the B-field is trivial in the type IIA theories does not affect our end results by Proposition 1.
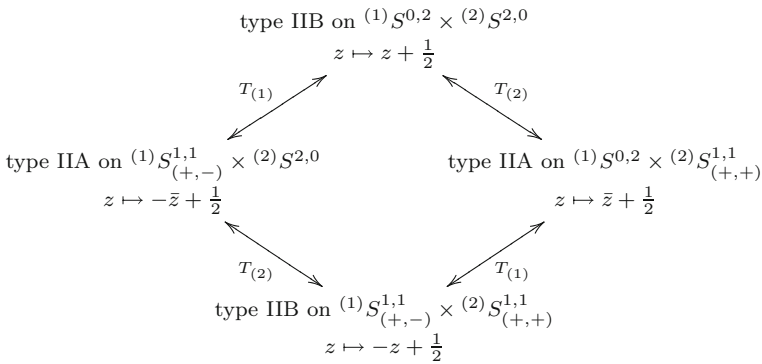
type IIB on $^{(1)}S^{0,2} \times {}^{(2)}S^{2,0}$

$z \mapsto z + \frac{1}{2}$

$T_{(1)}$ $T_{(2)}$

type IIA on $^{(1)}S^{1,1}_{(+,-)} \times {}^{(2)}S^{2,0}$ type IIA on $^{(1)}S^{0,2} \times {}^{(2)}S^{1,1}_{(+,+)}$

$z \mapsto -\bar{z} + \frac{1}{2}$ $z \mapsto \bar{z} + \frac{1}{2}$

$T_{(2)}$ $T_{(1)}$

type IIB on $^{(1)}S^{1,1}_{(+,-)} \times {}^{(2)}S^{1,1}_{(+,+)}$

$z \mapsto -z + \frac{1}{2}$

Fig. 3. Chain of T-dualities connecting the various theories related to the species 0 antiholomorphic involution $z \mapsto \bar{z} + \frac{1}{2}$. T_i represents T-duality on the indicated *circle*

The two fixed points of this involution are located at $x = \frac{1}{4}, \frac{3}{4}$. We see that the O-planes are no longer located at the 2-torsion points $x = 0$ and $x = \frac{1}{2}$, as they are with the involution $x \mapsto -x$, but have been shifted. Every involution $x \mapsto -x + \delta$, $\delta \in \mathbb{R}$, gives $S^{1,1}$ with two $O8^+$-planes except the case $\delta = \pm\frac{1}{2}$. What makes $\delta = \frac{1}{2}$ unique is that the involution exchanges the 2-torsion points. For all other values of δ, the two 2-torsion points are mapped to distinct points. The fact that the 2-torsion points are exchanged for $\delta = \frac{1}{2}$ corresponds physically to the fact that the O-plane charges corresponding to the 2-torsion points can annihilate. However, the locations of the O-planes are shifted, so we end up with an $O^+ - O^-$-plane pair. This provides a heuristic way of viewing the need for a twisting corresponding to a sign choice, as will be discussed further in the following section. In the language of [19], we should consider $S^{0,2}$ as a circle with a crosscap attached and then we see that the T-dual of a crosscap is an $O^+ - O^-$ plane pair.

Compactifying the type $\tilde{I}$ theory on another circle with trivial involution is the type IIB theory on $\mathbb{R}^{8,0} \times S^{0,2} \times S^{2,0}$. This corresponds to the involution $z \mapsto z + \frac{1}{2}$ on $\mathbb{C}/\Lambda$, with $S^{0,2}$ being the circle that is the image of $[0, 1]$ and $S^{2,0}$ being the circle that is the image of $\tau \cdot [0, 1]$.

Now if we T-dualize the circle that is the image of $[0, 1]$, $S^{0,2}$, we get the type $\widetilde{IA}$ theory compactified on another circle with trivial involution. In our notation this is the type IIA theory on $\mathbb{R}^{8,0} \times S^{1,1}_{(+,-)} \times S^{2,0}$ and corresponds to the involution $z \mapsto -\bar{z} + \frac{1}{2}$. If we now T-dualize the copy of $S^{2,0}$, we get the type IIB theory on $S^{1,1}_{(+,-)} \times S^{1,1}_{(+,+)} = (S^{1,1} \times S^{1,1})_{(+,+,-,-)}$. $(S^{1,1} \times S^{1,1})_{(+,+,-,-)}$ is $\mathbb{C}/\Lambda$ with the involution $z \mapsto -z + \frac{1}{2}$. It has four fixed points corresponding to 2 $O7^-$-planes and 2 $O7^+$-planes. Let us now consider what happens if we perform the T-dualities in the opposite order.

If we first T-dualize the copy of $S^{2,0}$ in the type IIB theory on $S^{0,2} \times S^{2,0}$, we get the type IIA on $S^{0,2} \times S^{1,1}_{(+,+)}$ corresponding to the species 0 antiholomorphic involution $z \mapsto \bar{z} + \frac{1}{2}$. If we now T-dualize the copy of $S^{0,2}$ we will get the type IIB theory on $S^{1,1}_{(+,-)} \times S^{1,1}_{(+,+)}$. This chain of dualities is shown in Fig. 3.

The group of theories related by T-duality pictured in Fig. 3 doesn't contain all of the species 0 antiholomorphic involutions and therefore does not have the symmetry we saw with the group containing the type I theory with trivial B-field (Fig. 2). We can
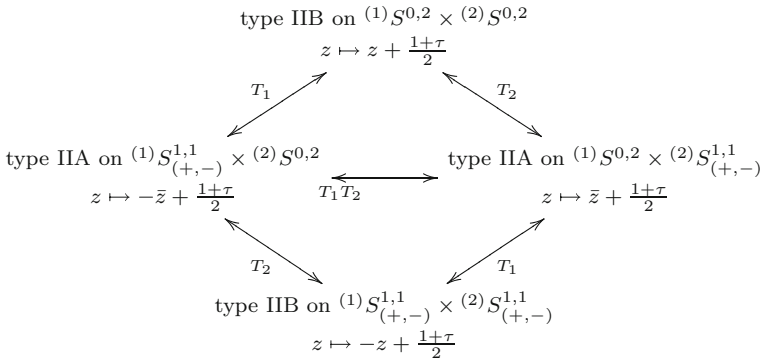
type IIB on $^{(1)}S^{0,2} \times {}^{(2)}S^{0,2}$

$$z \mapsto z + \frac{1+\tau}{2}$$

T_1 T_2

type IIA on $^{(1)}S^{1,1}_{(+,-)} \times {}^{(2)}S^{0,2}$ type IIA on $^{(1)}S^{0,2} \times {}^{(2)}S^{1,1}_{(+,-)}$

$$z \mapsto -\bar{z} + \frac{1+\tau}{2} \qquad\qquad T_1 T_2 \qquad\qquad z \mapsto \bar{z} + \frac{1+\tau}{2}$$

T_2 T_1

type IIB on $^{(1)}S^{1,1}_{(+,-)} \times {}^{(2)}S^{1,1}_{(+,-)}$

$$z \mapsto -z + \frac{1+\tau}{2}$$

Fig. 4. Chain of T-dualities connecting the various theories related to the species 0 antiholomorphic involution $z \mapsto \bar{z} + \frac{1+\tau}{2}$

easily obtain a picture containing the species 0 antiholomorphic map $z \mapsto -\bar{z} + \frac{\tau}{2}$ by just reversing the roles of τ and 1 in Fig. 3 by starting with the type IIB theory with involution $z \mapsto z + \frac{\tau}{2}$. This, however, requires multiple groupings and is not as satisfying a picture.

This can be resolved by taking half shifts in both the real and imaginary directions simultaneously. By this we mean starting with type IIB theory with involution $z \mapsto z + \frac{1+\tau}{2}$. This is the type IIB theory compactified on $S^{0,2} \times S^{0,2}$. Performing a single T-duality in different directions will give the type IIA theory on $z \mapsto \pm\bar{z} + \frac{1+\tau}{2}$. We are being purposefully vague about the direction of T-duality as it is not as simple as in the previous cases and we will discuss it further shortly. This corresponds to the type IIA theory on $S^{0,2} \times S^{1,1}_{(+,-)}$ or $S^{1,1}_{(+,-)} \times S^{0,2}$.

The involution $z \mapsto \bar{z} + \frac{1+\tau}{2}$ is equivalent to $z \mapsto \bar{z} + \frac{1}{2}$, while the involution $z \mapsto -\bar{z} + \frac{1+\tau}{2}$ is equivalent to $z \mapsto -\bar{z} + \frac{\tau}{2}$. Here, two involutions, ι_1 and ι_2, are equivalent if $(\mathbb{C}/\Lambda)/\iota_1$ and $(\mathbb{C}/\Lambda)/\iota_2$ are dianalytically equivalent as explained in [1]. This means ι_1 and ι_2 are equivalent if there exists an analytic automorphism, δ, of $\mathbb{C}/\Lambda$ such that

$$\iota_1 = \delta\iota_2\delta^{-1}. \tag{4}$$

Finally, T-dualizing the other direction gives us the type IIB theory with involution $z \mapsto -z + \frac{1+\tau}{2}$ corresponding to $S^{1,1}_{(+,-)} \times S^{1,1}_{(+,-)} = (S^{1,1} \times S^{1,1})_{(+,+,-,-)}$. This is again the type IIB theory with four fixed points corresponding to two O^--planes and two O^+-planes. This chain of dualities is pictured in Fig. 4.

Note how, as opposed to Fig. 3, Fig. 4 is symmetric and both possible species 0 antiholomorphic involutions occur in a single diagram. Also as in the T-duality group pictured in Fig. 2, the horizontal line in Fig. 4 collapses at the Gepner point $\tau = i$, signifying that the torus is square. Since the species 0 antiholomorphic involutions only exist for τ purely imaginary, we see that the T-dual type IIB theories must have trivial B-fields. As always, we are assuming that the type IIA theories have trivial B-fields, so that the type IIB tori are also rectangular. At first glance it might seem that the two type IIB theories appearing at the top of Figs. 3 and 4 are related by a change of complex structure, so that the type IIA theories appearing in Fig. 3 differ by a choice of B-field from those appearing in Fig. 4. However, by Proposition 1 we know that the B-field does not affect the stable D-brane charge in the type IIA theories, so the B-field could not

differentiate the IIA theories appearing in Figs. 3 and 4. We will see in Sect. 5.1.1 that the theories in Figs. 3 and 4 are related by the direction of T-duality. This could be viewed as a change of complex structure, but as we will show the two complex structures are related by an $SL(2, \mathbb{Z})$ transformation. Therefore they are defined on equivalent elliptic curves and their T-dual theories will have the same B-fields. Changing the direction of T-duality will, however, alter the sources for the stable brane charges. This effect was crucial in our understanding the sign choice as a source of twisting.

As an example of Proposition 1 and to better explain our assumption that all type IIA theories involved have trivial B-fields, let us look at the T-duality group containing the type I theory a little more closely. Proposition 1 says that the assumption that the type IIA theories have trivial B-field does not affect our results about O-plane and D-brane charges and their relationship under T-duality. If we included a non-trivial B-field in the type IIA theory on $S^{1,1} \times S^{2,0}$ for example, then it will still be T-dual to the type I theory with trivial B-field compactified on a 2-torus. The only difference is that the torus will no longer be rectangular, but this does not affect the stable brane charges, nor the O-plane content (which determines the overall theory). While the brane content in the type I theory will be the same as for the case when the torus is rectangular, since the direction of T-duality is no longer orthogonal to (or in the same direction as) the direction the branes wrap, the sources in the T-dual theory could be affected. A brane that wraps both compact directions is usually obtained from a brane that wraps a single compact direction via T-duality by T-dualizing the direction orthogonal to the direction wrapped by the brane. For a non-rectangular torus, T-dualizing one leg can send a brane that wraps a single cycle to a brane that wraps a different cycle. While the source might change, the important feature is that the stable charge remains the same. This will become clearer in Sect. 6 when we discuss the brane content in the various theories.

4.2. *T-duality for elliptic curve orientifolds with non-trivial B-field.*

The only possible nonzero value for the B-field is $B = \frac{1}{2}$. Furthermore, this is only a possibility for the type I theory. The type Ĩ theory cannot have a non-trivial B-field. If it did, its T-dual would be a non-rectangular torus, and as we just saw, the type Ĩ theory is T-dual to the species 0 real elliptic curve, which only exists for rectangular tori. So we are left only to consider the case of the type I theory with non-trivial B-field, or as it is more commonly referred to, the type I theory without vector structure [33].

Consider the type I theory without vector structure on a rectangular torus. Its T-dual theory will be a type IIA theory on an elliptic curve with Re $\tau = \frac{1}{2}$ and an antiholomorphic involution, and trivial B-field.[3] The involution on the T-dual type IIA theory must be a species 1 anti-holomorphic involution because those are the only possible antiholomorphic involutions when Re $\tau = \frac{1}{2}$.

The torus with the species 1 involution is the only torus orientifold that cannot be split into the product of two invariant circles and is the only truly new case we get from considering compactifications on two circles versus one. As can be seen from Table 1, the species 1 antiholomorphic involutions only exist for $\tau = e^{i\theta}$, $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$, with involution $z \mapsto \pm\tau\bar{z}$, or $\tau = \frac{1}{2} + i\tau_2$ with $\tau_2 \ge \frac{\sqrt{3}}{2}$ and involution $z \mapsto \pm\bar{z}$. As noted previously, every real elliptic curve with $\tau = e^{i\theta}$ is equivalent to a real elliptic curve with $\tau = \frac{1}{2} + i\tau_2$, and since this is the case we get from the obvious way of T-dualizing

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[3] As usual the assumption that the type IIA theories have trivial B-field does not affect our final result by Proposition 1.
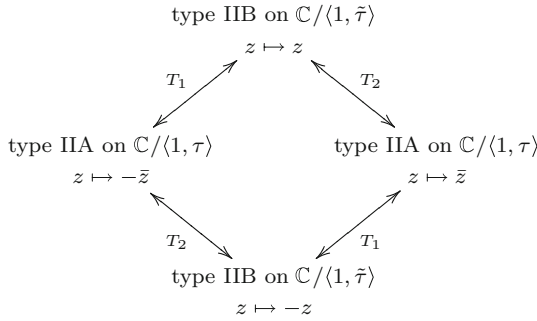
$$\text{type IIB on } \mathbb{C}/\langle 1, \tilde{\tau} \rangle$$
$$z \mapsto z$$

T_1 T_2

$$\text{type IIA on } \mathbb{C}/\langle 1, \tau \rangle \qquad\qquad \text{type IIA on } \mathbb{C}/\langle 1, \tau \rangle$$
$$z \mapsto -\bar{z} \qquad\qquad\qquad\qquad z \mapsto \bar{z}$$

T_2 T_1

$$\text{type IIB on } \mathbb{C}/\langle 1, \tilde{\tau} \rangle$$
$$z \mapsto -z$$

Fig. 5. Chain of T-dualities connecting the various theories related to the type I theory without vector structure. $\tau = \frac{1}{2} + i\tau_2$ and $\tilde{\tau}$ is purely imaginary

the type I theory without vector structure, it is the case we will consider. We will briefly consider the case of $\tau = e^{i\theta}$ in the following section to motivate our understanding of the sign choice as a twisting.

So far we have seen that the type I theory without vector structure is T-dual to the type IIA theory with $\tau = \frac{1}{2} + i\tau_2$, $\rho = iV$ and a species 1 antiholomorphic involution. Performing another T-duality in the other compact direction gives the type IIB theory with four fixed points, but now with three O^+-planes and one O^--plane.

Without having our geometric description of the sign choice, it is easiest to see this by noting that the species 1 antiholomorphic involutions give a Möbius strip which can viewed as a cylinder with a cross-cap. T-duality transforms the cross-cap into an O^+-, O^--plane pair, while the boundary is transformed to two O^+-planes [19]. We can put this chain of dualities into a diagram similar to the one for the species 0 and 2 groups, and the result is given in Fig. 5.

Note that the involutions appearing in Fig. 5 are the same as those appearing in the chain of dualities connecting the group containing the type I theory (Fig. 2). This shows that the two vertical legs in the left-hand side of Fig. 1 both describe the T-dualities of the type I theory, with the Re $\tau = 0$ leg corresponding to trivial B-field, and the Re $\tau = \frac{1}{2}$ branch corresponding to $B = \frac{1}{2}$. In [4] (and actually already in [7]) the authors note that the species obstructs continuous deformation from the large limit branch $\tau = i\infty$ to the large limit branch $\tau = \frac{1}{2} + i\infty$, leading to two disjoint large volume type IIB torus orientifolds. We see that these two large volume type IIB theories correspond to the type I theory with the two possible values for the B-field. While this shows a physical relationship between the species 2 and 1 groups, the mathematical relationship is described by Alling [1]. This chain of dualities is considered in [19].

We still have not considered the arc associated to species 1 appearing in Fig. 1. This corresponds to the type IIA theory with $\tau = e^{i\theta}$, $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, and involution $z \mapsto \pm\tau\bar{z}$. We have noted that these elliptic curves are equivalent to ones with $\tau = \frac{1}{2} + i\tau_2$. Moreover, under continuous transformations from the species 1 antiholomorphic involutions with $\tau = e^{i\theta}$ to the ones with $\tau = \frac{1}{2} + i\tau_2$ through the Gepner point, the j-invariant goes from positive to negative through 0 at the Gepner point $\tau = e^{\frac{i\pi}{3}}$. This shows us that the brane content and dual theories stay the same except for values of the parameters. While considering the type IIA theories with $\tau = \frac{1}{2} + i\tau_2$ is often more convenient, the case with $\tau = e^{i\theta}$ will be useful later so we show the chain of T-dualities related to this case in Fig. 6.
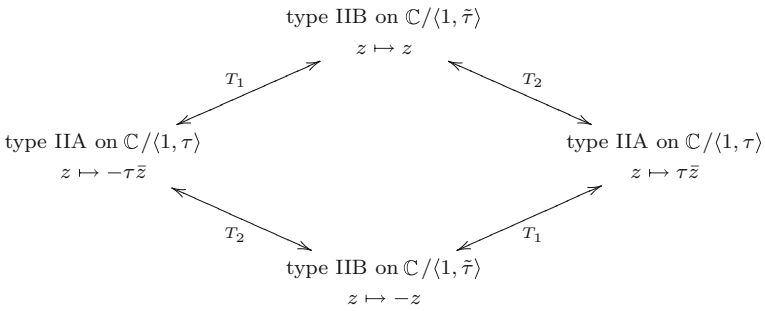
$$\text{type IIB on } \mathbb{C}/\langle 1, \tilde{\tau}\rangle$$
$$z \mapsto z$$

T_1 T_2

type IIA on $\mathbb{C}/\langle 1, \tau\rangle$ type IIA on $\mathbb{C}/\langle 1, \tau\rangle$
$z \mapsto -\tau\bar{z}$ $z \mapsto \tau\bar{z}$

T_2 T_1

$$\text{type IIB on } \mathbb{C}/\langle 1, \tilde{\tau}\rangle$$
$$z \mapsto -z$$

Fig. 6. Chain of T-dualities connecting the various theories related to the species 1 antiholomorphic involutions with $\tau = e^{i\theta}$. $\tilde{\tau}$ is purely imaginary

5. Jacobi Functions, Real Elliptic Curves, and T-Duality

The different T-duality groupings described in the previous section all contain spaces that are topologically equivalent, but have distinct brane content. This is what motivated us to define KR-theory with a sign choice in [12]. While there are physical explanations for the sign choices of the O-planes, so far we have seen no mathematical explanation for why T-duality forces this type of twisting on KR-theory.

The standard twisting of K-theory by the B-field is usually interpreted geometrically in terms of the angle the direction being T-dualized makes with the cycles wrapped by branes. In this section we will briefly mention how the distinction between the chains of dualities pictured in Figs. 3 and 4, as well as the two branches of species 1 antiholomorphic involutions, hints at such an interpretation. However, we will not delve too far into this interpretation and instead give a more general description of sign choices by describing canonical normal forms for the elliptic curves appearing in the three different T-duality groups encoding this information.

5.1. A heuristic description of a sign choice as a twisting.

5.1.1. Different ways of T-dualizing the type $\tilde{I}$ theory. Let us first consider the distinction between the two different T-duality groups containing the type $\tilde{I}$ theory we discussed in Sect. 4 (Figs. 3, 4). When we discussed the chain of T-dualities pictured in Figs. 3 and 4 we were purposefully vague about the directions of T-duality. For the type IIB theory with involution $z \mapsto z + \frac{1}{2}$ (type IIB on $S^{0,2} \times S^{2,0}$) the circle given by the projection of $[0, 1]$ is equivariant and corresponds to the factor of $S^{0,2}$; the circle corresponding to the projection of $\tau \cdot [0, 1]$, however, is not equivariant. The circle $\tau \cdot [0, 1]$ is sent by the involution to the parallel circle $\tau \cdot [0, 1] + \frac{1}{2}$. By T-dualizing $S^{0,2}$ we get the type IIA theory on $S^{1,1}_{(+,-)} \times S^{2,0}$. Now, the circle $[0, 1]$ corresponds to $S^{1,1}_{(+,-)}$ and is still equivariant. Also, the perpendicular circles, $\tau \cdot [0, 1] + \frac{1}{4}$ and $\tau \cdot [0, 1] + \frac{3}{4}$ are also equivariant, making it clear what it means to T-dualize in the τ direction.

For the type IIB theory on $S^{0,2} \times S^{0,2}$ the equivariant circles are the diagonal and anti-diagonal. In the T-dual theory $S^{0,2} \times S^{1,1}_{(+,-)}$, the involution $z \mapsto \bar{z} + \frac{1+\tau}{2}$ exchanges the diagonal and anti-diagonal. This shows that branes that wrap the diagonal and anti-diagonal are not independent, and more importantly, branes wrapping the real and imaginary axis are not independent. It is still well defined to T-dualize in the real and imaginary
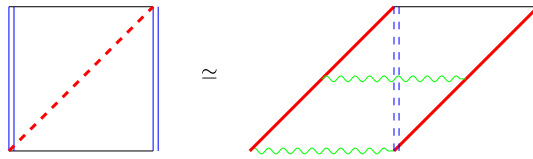
Fig. 7. $S^{0,2} \times S^{0,2}$, corresponding to the involution $z \mapsto z + \frac{1+\tau}{2}$, on the *left* can be viewed as $S^{2,0} \times S^{0,2}$ on the *right* where the 2 *green squiggly lines* are copies of $S^{2,0}$ that get exchanged under $z \mapsto z + \frac{1+\tau}{2}$ (color figure online)

directions, since the circles that go through the fixed points in the type IIB theory with involution $z \mapsto -z + \frac{1+\tau}{2}$ are all equivariant. This, combined with the dependence between wrappings of the real and imaginary directions, shows that branes that wrap $S^{0,2}$ in the non-symmetric case should now wrap the diagonal (which is still a copy of $S^{0,2}$) and continue to T-dualize in the real and imaginary directions.

From the above discussion we are motivated to look at the equivalent elliptic curve with $\tilde{\tau} = 1 + \tau$. $S^{0,2} \times S^{0,2}$ corresponds to a torus with legs 1 and it, and involution $z \mapsto z + \frac{1+it}{2}$. We could instead consider the equivalent real elliptic curve given by a torus with legs 1 and $1 + it$, and the same involution $z \mapsto z + \frac{1+it}{2}$. As pictured in Fig. 7 this decomposes $S^{0,2} \times S^{0,2}$ into $S^{2,0} \times S^{0,2}$, where the copy of $S^{2,0}$ is generated by 1 and the copy of $S^{0,2}$ is generated by $1 + it$. It might at first seem unsatisfactory that the copy of $S^{2,0}$ is not equivariant, but recall that for the involution $z \mapsto z + \frac{1}{2}$ the generator of $S^{2,0}$, $\tau \cdot [0, 1]$, is also not equivariant but gets equated with the circle $\tau \cdot [0, 1] + \frac{1}{2}$. In the current situation, as pictured on the right-hand side of Fig. 7, the circle $[0, 1]$ is equated with the circle $[0, 1] + \frac{\tilde{\tau}}{2} = [0, 1] + \frac{1+it}{2}$.

The non-symmetric species $\tilde{0}$ case can be obtained from the symmetric case by T-dualizing in a direction that isn't normal to the cycle the branes wrap. We see that the additional source of twisting beyond the traditional B-field, the sign choice, is also related to the direction of T-duality. The B-field is related to the angle between legs in the T-dual theory. A trivial B-field corresponds to a rectangular torus in the T-dual theory. In our present case all of the tori involved are rectangular (up to equivalence) and thus all of the B-fields are trivial. From Fig. 7 it might appear as though the IIA theories appearing in Fig. 4 will have non-trivial B-field. However, the two elliptic curves in Fig. 7 are equivalent. In fact they are defined by the same lattice, with the only difference being the choice of generators for the lattice. While the distinction between the chain of dualities in Figs. 3 and 4 can't be explained by the B-field, we do expect a B-field-like twisting when we T-dualize a theory containing branes that wrap a cycle that makes a non-right angle with the direction of T-duality. This twisting is clearly accounted for by the sign choice.

Furthermore, note that there are three different T-duality groupings involving the type $\tilde{I}$ theory on a rectangular torus. They correspond to the two asymmetric cases $z \mapsto z + \frac{1}{2}$ and $z \mapsto z + \frac{\tau}{2}$, and the symmetric case $z \mapsto z + \frac{1+\tau}{2}$. Both asymmetric groupings contain one species 0 type IIA theory and one species 2 type IIA theory twisted by a non-trivial sign choice. The species 0 type IIA theory is equivalent to one of the two species 0 type IIA theories appearing in the symmetric grouping. This shows us that the difference between the chain of dualities pictured in Fig. 3 and the one pictured in Fig. 4 is in the direction we T-dualize. This choice of direction decides if we need to include a twisting in one of the type IIA theories or if the twisting only appears in the type IIB theory with four fixed points.

Perhaps more telling is the fact that, in all possible compactifications of the type $\tilde{I}$ theory on a 2-torus, there was only a single equivariant circle. After T-dualizing in the direction of the equivariant circle, a new direction becomes equivariant, making it well-defined to T-dualize in a second direction. But what if we want to perform the T-dualities in the opposite order? Sticking with the case pictured in Fig. 3 for concreteness, if we want to T-dualize the τ direction first, then we must consider pairs of branes that wrap the two cycles $\tau \cdot [0, 1]$ and $\tau \cdot [0, 1] + \frac{1}{2}$ which will correspond to a single brane that wraps one of the new equivariant cycles in $S^{1,1}_{(+,-)} \times S^{2,0}$ twice.

$S^{1,1}_{(+,-)} \times S^{2,0}$ and $S^{1,1}_{(+,-)} \times S^{0,2}$ are related via the annihilation of O^+- and O^--planes. Twisting by the sign choice is related to a topological obstruction to the annihilation of the O^+- and O^--planes appearing in the type IIA theory on $S^{1,1}_{(+,-)} \times S^{2,0}$. As with the case of the type $\tilde{I}$ theory compactified on a circle, we can determine the O-plane content by looking at the 2-torsion points.

Under the involution $z \mapsto z + \frac{1}{2}$ the 2-torsion points transform as

$$0 \longleftrightarrow \tfrac{1}{2} \tag{5}$$

$$\tfrac{\tau}{2} \longleftrightarrow \tfrac{1}{2} + \tfrac{\tau}{2}.$$

As was the case with single compact direction, the exchange of 2-torsion points corresponds to the fact that the O-planes associated with them can annihilate. However, the locations of the O-planes are shifted. The T-dual theory is the type IIA theory with involution $z \mapsto -\bar{z} + \frac{1}{2}$. If the involution were $z \mapsto -\bar{z}$ (the T-dual of $z \mapsto z$), then there would be two $O8^+$-planes wrapping the cycles τ (going through the 2-torsion points 0 and $\frac{\tau}{2}$) and $\tau + \frac{1}{2}$ (going through the 2-torsion points $\frac{1}{2}$ and $\frac{\tau+1}{2}$). But for $z \mapsto z + \frac{1}{2}$, the O-planes in the T-dual theory must have opposite charge since the 2-torsion points transform as in Eq. (5). They do not annihilate each other since they are shifted from the 2-torsion points and wrap the fixed circles $\tau \cdot [0, 1] + \frac{1}{4}$ and $\tau \cdot [0, 1] + \frac{3}{4}$.

For the type $\tilde{I}$ theory, the action on the 2-torsion points is completely determined by the involution, which Atiyah's KR-theory is also sensitive to. T-duality exchanges the complex and Kähler moduli. Therefore the action of the original involution on the 2-torsion is no longer contained in the information of the new T-dual involution, and KR-theory cannot pick this up. This is why we need to add it in as an additional datum in the form of a twisting. Before giving a more general description, let us do a similar analysis for the different branches of the species 1 antiholomorphic involutions.

5.1.2. Different ways of T-dualizing the type I theory without vector structure.

In Sect. 4 we saw that there were two branches to the species 1 antiholomorphic involutions: one with $\tau = e^{i\theta}$, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, and the other with $\tau = \frac{1}{2} + i\tau_2$. We saw that every elliptic curve with $\tau = e^{i\theta}$, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, was equivalent to an elliptic curve with $\tau = \frac{1}{2} + i\tau_2$. Furthermore, we saw that after choosing this representative we could differentiate between the two branches by the sign of the j-invariant. While the representative with $\tau = \frac{1}{2} + i\tau_2$ for all species 1 real elliptic curves was useful for understanding the B-field in T-dual theories, considering the differences in Figs. 5 and 6 is useful for understanding the sign choice.

As before we could determine the signs of the O-planes in the T-dual theories by looking at the action of the antiholomorphic involutions on the 2-torsion points. Let us

first consider the 2-torsion points in the type IIA theory with $\tau = e^{i\theta}$ with involution $z \mapsto \tau\bar{z}$. Under this involution, the 2-torsion points 0 and $\frac{1}{2} + \frac{1}{2}e^{i\theta}$ are fixed while $\frac{1}{2}$ and $\frac{1}{2}e^{i\theta}$ are exchanged. This shows us that in the T-dual type IIB theory with four fixed points there is an O^+- O^--plane pair corresponding to the 2-torsion points that were exchanged and there are two more O^+-planes corresponding to the two fixed 2-torsion points. There are two possible independent T-dualities we could have performed starting with the type IIA theory with $\tau = e^{i\theta}$. The other T-duality would have taken us to a space where the fixed set has a single component. Here the only supersymmetric (physically significant) possibility is giving the component the positive sign choice, corresponding to an orthogonal structure on the Chan–Paton bundle.

Let us now consider the 2-torsion points under the action of the species 1 antiholomorphic involutions for $\tau = \frac{1}{2} + i\tau_2$. Under the involutions $z \mapsto \pm\bar{z}$ the 2-torsion points 0 and $\frac{1}{2}$ are fixed, while $\frac{\tau}{2}$ and $\frac{1}{2} + \frac{\tau}{2}$ are exchanged, giving the same O-plane charge content as for the case of $\tau = e^{i\theta}$.

Here we can again view the need to include the extra twisting of a sign choice in terms of the angle between the direction of T-duality and the equivariant circles. The only equivariant circles for the species 1 antiholomorphic involutions, $z \mapsto \pm\tau\bar{z}$ are the diagonal, S_D, and anti-diagonal, S_A. S_D is the fixed circle for the involution $z \mapsto \tau\bar{z}$ and S_A is the fixed circle for the involution $z \mapsto -\tau\bar{z}$. Therefore, the O-plane wraps either S_D or S_A making an angle $\frac{\pi}{6} < \frac{\theta}{2} < \frac{\pi}{4}$ with the directions we are T-dualizing in, 1 and τ. For the type IIA theory with $\tau = \frac{1}{2} + i\tau_2$ and involution $z \mapsto \pm\bar{z}$ the equivariant circles are parallel to the real and imaginary axes, making non-orthogonal angles with τ.

Note here that the fact that τ is not normal to the real axis is the source of the B-field, but the angle between the equivariant circle parallel to the imaginary axis and τ is the source for this additional twisting of the sign choice. Now let us consider a more general description that does not require analyzing each case separately.

5.2. Normal forms for real elliptic curves and a geometric description of the sign choice. As we just saw, we were able to determine the O-plane charges in a couple of specific examples by looking at the action of the antiholomorphic involutions on the 2-torsion points of the elliptic curve. We could follow a similar argument for each possible case. However, this is not a good way to describe the sign choice. An immediate disadvantage is that we would need to repeat the analysis numerous times just to cover all of the cases in the T-duality grouping containing the type $\tilde{\text{I}}$ theory. Moreover, the description depends on a choice of zero, which is unsatisfactory.[4] We can use the observation about the 2-torsion points, however, to help us determine a more general description.

We will begin with the type IIA theories since these are defined on real elliptic curves[5] which have a canonical description in terms of a defining equation for the field of meromorphic functions on the elliptic curve, $E = \mathbb{R}(x, y)$. The defining equations for elliptic curves are usually given in terms of Weierstrass' elliptic function, $\wp$. This is not the best way to describe real elliptic curves. While $E = \mathbb{R}(\wp, \wp')$ for the species 2 and 1 real elliptic curves, this is not true for the species 0 real elliptic curves since $\wp$ does not have period $\frac{1}{2}$. Therefore $\wp \notin E$ [1].

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[4] From the point of view of the physics, the group structure on the elliptic curve is not natural, since it requires fixing a distinguished basepoint in spacetime, violating Mach's principle, and we should therefore just keep the structure of a principal homogeneous space over the Jacobian.

[5] Here we take a more general definition of real elliptic curve, including both classical and non-classical elliptic curves in the sense of [1, §0.10].

As noted by Whittaker and Watson [31, Ch. XX], it is easiest to use elliptic functions of order 2 when proving general theorems about elliptic functions, due to the behavior of their singularities. There are two classes of order 2 elliptic functions. The first class contains order 2 elliptic functions with a single double pole in each fundamental domain. $\wp$ is in this class, and the fact that it has a single pole is what prevents it from working in the species 0 case. The second class contains functions with two simple poles whose residues sum to zero in each cell. Clearly we would like to give the defining equation for the species 0 real elliptic curves in terms of elliptic functions in class 2, so that the shift of $\frac{1}{2}$ can exchange the two poles (accounting for the exchange of the 2-torsion points).

As explained in [1], the defining equation of the species 0 real elliptic curves can be written in terms of the standard Jacobian elliptic function *sinus amplitudinis* (*sinam*), denoted sn. Note that if we let the quarter period of sn be $K = \frac{1}{4}$ and the half-period of sn be $K' = \frac{ti}{2}$ so that $\tau' = \frac{K'}{K} = 2ti$, then we can write sn in terms of theta functions as

$$\text{sn}\,(u) = \frac{\theta_0(0)\theta_1(2u)}{2\theta_1'(0)\theta_0(2u)}.$$

This makes sense since then sn has zeros at the points of Λ and $\Lambda + \frac{1}{2}$, and poles at the points of $\Lambda + \frac{ti}{2}$ and $\Lambda + \frac{1+it}{2}$. Note that sn has the same periods, $(1, ti)$, as the elliptic curve with $\tau = ti$ that the species 0 involution is defined on.

An immediate benefit of using the Jacobian elliptic functions instead of the theta functions is that the theta functions are only periodic, while the Jacobian elliptic functions are doubly periodic. Now following [1], we describe the defining equations for the species 0 real elliptic curves.

Consider the real degree 4 polynomial

$$L_{u,v,w}(x, k) \equiv (-1)^u(1 - (-1)^v x^2)(1 - (-1)^w k^2 x^2), \tag{6}$$

with $u, v, w \in \{0, 1\}$, $0 < k \leq 1$, and if $v = w$ then $k < 1$. Equations of the form

$$y^2 = L_{u,v,w}(x, k)$$

are said to be in generalized Legendre form.

The defining equation for the species 0 real elliptic curves can be put generalized Legendre form as

$$(i\,\text{sn}'(z))^2 = L_{1,1,1}(i\,\text{sn}(z), k), \tag{7}$$

where as usual, k is the Legendre modulus

$$k = \frac{\theta_2^2}{\theta_3^2},$$

where $\theta_i = \theta_i(0)$. Letting $x = i\,\text{sn}(z)$ and $y = i\,\text{sn}'(z)$, (7) is

$$y^2 = -(1 + x^2)(1 + k^2 x^2). \tag{8}$$

Before discussing how this relates to T-duality and sign choices, we note that the species 2 and 1 defining equations can also be put in generalized Legendre form. Even though we can give a cubic defining equation in terms of $\wp$ for the species 2 and 1 real elliptic curves, it will be more useful to use the quartic Legendre form both for comparison to the species 0 case and for general T-duality analysis as well.

The defining equation for the species 2 antiholomorphic involutions can be put in the generalized Legendre form

$$(\mathrm{sn}'(z))^2 = L_{0,0,0}(\mathrm{sn}(z), k). \tag{9}$$

Letting $x = \mathrm{sn}(z)$ and $y = \mathrm{sn}'(z)$ this is

$$y^2 = (1 - x^2)(1 - k^2 x^2). \tag{10}$$

The species 1 real elliptic curves do not have purely imaginary τ, so we need to define sn in terms of different quarter- and half-periods. For species 1, let the quarter-period of sn in one direction be $\frac{1}{4}$ while the half-period in the other direction is $K' = \frac{\tau}{2} = \frac{1}{4} + \frac{i\tau_2}{2}$, so $\tau' = \frac{K'}{K} = 1 + 2i\tau_2$. This way sn has the same periods as the elliptic curve the species 1 involution is defined on, $(1, \frac{1}{2} + i\tau_2)$. In this case the Legendre modulus, k, is purely imaginary as described in [1]. We can put the defining equation for the species 1 real elliptic curves in the generalized Legendre form as

$$(\mathrm{sn}'(z))^2 = L_{0,0,1}(\mathrm{sn}(z), -ik). \tag{11}$$

Letting $x = \mathrm{sn}$ and $y = \mathrm{sn}'$ this is

$$y^2 = (1 - x^2)(1 - k^2 x^2). \tag{12}$$

It might at first appear that Eqs. (10) and (12) are the same, but for Eq. (10), $k^2 > 0$, while for Eq. (12), $k^2 < 0$.

Both the species 0 and 2 antiholomorphic involutions only exist on elliptic curves with purely imaginary complex structure. Therefore, Eqs. (8) and (10) describe rectangular tori (as is clear by the choice of the period) and it should be possible to perform both species 0 and 2 involutions on either one. This naturally leads to the question of why Eq. (8) is associated with species 0 while Eq. (10) is associated with species 2. The answer (which turns out to be a crucial ingredient in understanding T-duality and sign choice) is determined by the effect of the different involutions on x and y.

As we will see, Eq. (10) is associated with species 2 because it is the elliptic curve where the meromorphic functions x and y on E (the generators of the function field, which we will simply call the meromorphic coordinates) transform in the standard way $(x, y) \mapsto (\bar{x}, \bar{y})$ under the standard species 2 involution $z \mapsto \bar{z}$, and not for any other involutions, while Eq. (8) is associated with species 0 since it is the elliptic curve for which the meromorphic coordinates of E transform in the standard way $(x, y) \mapsto (\bar{x}, \bar{y})$ for the species 0 involution $z \mapsto \bar{z} + \frac{1}{2}$, and not for any other involutions. In this way, Eqs. (7) and (9) can be thought of as the canonical normal forms for the species 0 and 2 real elliptic curves, respectively. While the normal form for the species 1 involutions is distinguished by the complex structure (seen here by the quarter periods), we could also view it as the canonical transformation for the species 1 involutions since only under the standard involution $z \mapsto \bar{z}$ do the meromorphic coordinates of E transform in the standard way $(x, y) \mapsto (\bar{x}, \bar{y})$.

Now that we have a canonical choice of a normal form for each species of anti-holomorphic involution, we can give a general geometric description of the T-duality relationships we saw in Sect. 4. We will immediately see each T-duality grouping should be classified by the generalized Legendre form of the elliptic curve the involutions are defined on.
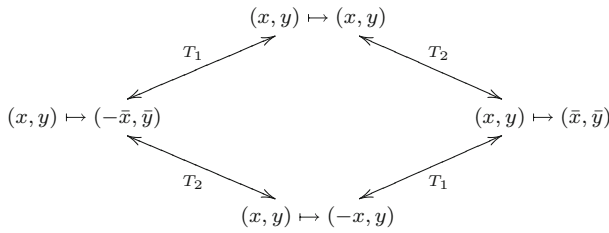
Fig. 8. Chain of T-dualities connecting the various theories related to the type I theory with trivial B-field (*top of the diagram*). All theories are defined on the elliptic curve $y^2 = (1 - x^2)(1 - k^2 x^2)$. Holomorphic involutions correspond to type IIB theories and antiholomorphic involutions correspond to type IIA theories

At the Gepner point $\tau = i$ there exist antiholomorphic involutions for all three species. This fact allows one to find some non-trivial identities for the elliptic functions by performing involutions of a certain species on the elliptic curve not canonically associated with that species.

5.2.1. The T-duality group defined by $y^2 = (1 - x^2)(1 - k^2 x^2)$. As we saw above, the elliptic curve defined by

$$y^2 = L_{0,0,0}(x, k),$$

with $x = \mathrm{sn}(z)$ and $y = \mathrm{sn}'(z)$ has purely imaginary complex structure parameter τ. Therefore Eq. (9) has to be associated to either the T-duality group containing the type I theory with trivial B-field or the one containing the type $\tilde{\mathrm{I}}$ theory. To determine which, we need to find the involution that sends $(\mathrm{sn}(z), \mathrm{sn}'(z))$ to $(\overline{\mathrm{sn}(z)}, \overline{\mathrm{sn}'(z)})$.

Using the addition formulas for $\mathrm{sn}(z)$ and $\mathrm{sn}'(z) = \mathrm{cn}(z)\,\mathrm{dn}(z)$, as well as the fact that sn is odd while cn and dn are even, it can be easily shown that under complex conjugation

$$\mathrm{sn}(\pm \bar{z}) = \pm \overline{\mathrm{sn}(z)} \quad \text{and} \quad \mathrm{sn}'(\pm \bar{z}) = \overline{\mathrm{sn}'(z)}. \tag{13}$$

Having shown that $z \mapsto \bar{z}$ sends (x, y) to $(\bar{x}, \bar{y})$ we see that the elliptic curve defined by Eq. (8) should be associated to the T-duality grouping that contains $z \mapsto \bar{z}$ on a rectangular torus, i.e., to the grouping containing the type I theory (Fig. 2).

We should view all of the theories appearing in Fig. 2 as being defined on the elliptic curve

$$y^2 = (1 - x^2)(1 - k^2 x^2),$$

differing only in the involution defining their orientifold (real) structure and the Legendre modulus, which depends on τ. With this viewpoint we can describe the chain of dualities pictured in Fig. 2 in terms of the induced action on x and y alone as pictured in Fig. 8.

The only information we haven't yet specified is the sign choice. We saw in Sect. 5.1 that this information was obtained from the action on the 2-torsion points. This was, however, an unsatisfactory description, since as is clear from Fig. 7 the description depended on distinguishing between equivalent elliptic curves. This same information is contained in a *canonical way* in the zeros of the Jacobi Legendre normal form.

The zeros of Eq. (10) are $x = \mathrm{sn}(z) = \pm 1$ and $\pm \frac{1}{k}$. The fact that all of the zeros are real tells us that all of the components of the fixed sets of any of the involutions must have an orthogonal structure, or a positive sign choice. The fact that the zeros are real

means that they are fixed under conjugation, corresponding to the fact that all of the 2-torsion points are fixed in this T-duality grouping, making the link between the action on the 2-torsion points and the type of zeros of y explicit.

Note that the zeros of sn$'(z)$ occur at $z = \frac{1}{4}, \frac{3}{4}, \frac{1}{4} + \frac{ti}{2}$ and $\frac{3}{4} + \frac{ti}{2}$. Furthermore, sn is real and distinct at all four of those values. In particular,

$$\text{sn}\left(\frac{1}{4}\right) = 1$$

$$\text{sn}\left(\frac{3}{4}\right) = -1$$

$$\text{sn}\left(\frac{1}{4} + \frac{ti}{2}\right) = \frac{1}{k}$$

$$\text{sn}\left(\frac{1}{4} + \frac{ti}{2}\right) = -\frac{1}{k}. \tag{14}$$

Therefore we see that the T-duality grouping containing the type I theory is associated to the normal form $(\text{sn}')^2 = L_{0,0,0}(\text{sn}, k)$ and the involutions pictured in Fig. 8. We should note that all the theories are defined by the same normal form. T-duality changes the value of τ, and hence the Legendre modulus. T-duality leaves the canonical normal form invariant, but changes the Legendre modulus and defining periods. Now let us turn our attention to the group containing the type Ĩ theory.

5.2.2. The T-duality group defined by $y^2 = -(1+x^2)(1+k^2x^2)$. We have already seen that the elliptic curve defined by Eq. (7) should be associated with a species 0 involution, and so must be associated with one of three possible versions of the T-duality grouping that contains the type Ĩ theory. Based on the choice of x and y, it should be associated with the type Ĩ theory defined by a half-period shift in the real direction. Given our definition of sn ($K = \frac{1}{4}$, $K' = \frac{it}{2}$),

$$\text{sn}(z + \frac{1}{2}) = \text{sn}(z + 2K) = -\text{sn}(z),$$

$$\text{sn}'(z + 2K) = -\text{sn}'(z). \tag{15}$$

Therefore under $z \mapsto \bar{z} + \frac{1}{2}$

$$(x, y) = (i \, \text{sn}(z), i \, \text{sn}'(z)) \mapsto \left(i \, \text{sn}\left(\bar{z} + \frac{1}{2}\right), i \, \text{sn}'\left(\bar{z} + \frac{1}{2}\right)\right) \tag{16}$$

$$= (-i \, \text{sn}(\bar{z}), -i \, \text{sn}'(\bar{z}))$$

$$= (-i \, \overline{\text{sn}(z)}, -i \, \overline{\text{sn}'(z)})$$

$$= (\bar{x}, \bar{y}), \tag{17}$$

showing that the elliptic curve defined by Eq. (7) should be associated with the T-duality grouping pictured in Fig. 3. Again, we can view this T-duality group in terms of the induced action on x and y alone, as pictured in Fig. 9.

As before, we can determine the sign choice from the zeros of Eq. (7). The zeros are $x = \pm i$ and $\pm \frac{i}{k}$. Since $x = i \, \text{sn}(z)$ we see that the zeros occur at the same places as for Eq. (9): $z = \frac{1}{4}, \frac{3}{4}, \frac{1}{4} + \frac{ti}{2}$ and $\frac{3}{4} + \frac{ti}{2}$. However, now instead of the all of the zeros being real,
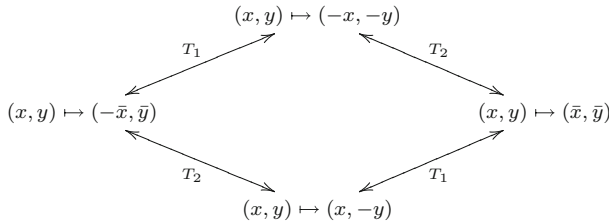
Fig. 9. Chain of T-dualities connecting the various theories related to the type $\tilde{I}$ theory on $S^{0,2} \times S^{2,0}$ (*top of the diagram*). All theories are defined on the elliptic curve $y^2 = -(1+x^2)(1+k^2x^2)$. Holomorphic involutions correspond to type IIB theories and antiholomorphic involutions correspond to type IIA theories

they are imaginary and come in complex conjugate pairs. This tells us that the charges of the O-planes associated to these points must have opposite charge, corresponding in our previous language to the fact that the corresponding 2-torsion points are exchanged. Let us first consider the theory at the bottom of Fig. 9, the type IIB orientifold with with four fixed points. The fixed points are at the zeros of $i\,\mathrm{sn}'(z)$. The fact that $i\,\mathrm{sn}(\frac{1}{4})$ and $i\,\mathrm{sn}(\frac{3}{4})$ are complex conjugates means that the O-planes located there are opposite in sign. We can define our signs so that the sign of the O-plane at each fixed point is the same as the sign of sn evaluated at that point.

Now let us consider the antiholomorphic involutions, or type IIA theories. The theory corresponding to $(x, y) \mapsto (\bar{x}, \bar{y})$ is fixed point free since $(i\,\mathrm{sn}(z), i\,\mathrm{sn}'(z)) = (-i\overline{\mathrm{sn}(z)}, -i\overline{\mathrm{sn}'(z)})$ has no solutions. The type IIA theory with involution $(x, y) \mapsto (-\bar{x}, \bar{y})$ in Fig. 9 has two O-planes. One wraps $\tau + \frac{1}{4}$ and the other wraps $\tau + \frac{3}{4}$. Parametrize the cycle through $\tau + \frac{1}{4}$ by $s \in [0, 1)$ as $z = \frac{1}{4} + its$. Then it is easy to see sn is real on $\tau + \frac{1}{4}$ and $\tau + \frac{3}{4}$ and has opposite sign on the two cycles. Therefore $x = i\,\mathrm{sn}$ evaluated on the two cycles are complex conjugates of each other, showing that the two O-planes should have opposite sign. This can be seen immediately by noting one of the cycles goes through the two zeros of y with positive sign, while the other goes through the two zeros with a negative sign.

Before considering the symmetric variant of the T-duality group containing the type $\tilde{I}$ theory (Fig. 4), let's consider a half-shift in the imaginary direction alone. That is, consider the involution $z \mapsto z + \frac{\tau}{2}$ where $\tau = it$. Given our definition of sn,

$$\mathrm{sn}\left(z + \frac{it}{2}\right) = \mathrm{sn}(z + K') = \frac{1}{k\,\mathrm{sn}(z)}, \tag{18}$$

$$\mathrm{sn}'(z + K') = \frac{-\mathrm{sn}'(z)}{k(\mathrm{sn}(z))^2}. \tag{19}$$

Therefore, under $z \mapsto z + \frac{\tau}{2}$, $(x, y) \mapsto (-\frac{1}{kx}, \frac{y}{kx^2})$. Note that this in fact an automorphism of the elliptic curve $y^2 = -(1+x^2)(1+k^2x^2)$. In many ways it is more interesting than the other automorphisms we have encountered so far (conjugation, and multiplication by -1). However, for our current purposes, it is unsatisfying that a shift in the imaginary direction is distinguished. This is because we made made a choice of preferred direction by choosing to use sn to define the elliptic curve.

As noted, sn has poles at $\Lambda + \frac{ti}{2}$ and $\Lambda + \frac{1+it}{2}$. We chose sn because its two poles where exchanged by a shift by $\frac{1}{2}$. If we wanted to perform a similar analysis for $z \mapsto z + \frac{\tau}{2}$ we
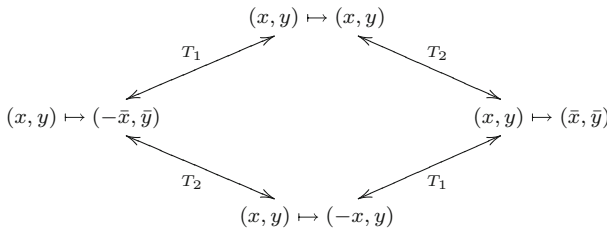
Fig. 10. Chain of T-dualities connecting the various theories related to the type I theory with non-trivial *B*-field (*top of the diagram*). All theories are defined on the elliptic curve $y^2 = (1 - x^2)(1 - k^2 x^2)$, $k^2 < 0$. Holomorphic involutions correspond to type IIB theories and antiholomorphic involutions correspond to type IIA theories

would need to use a Jacobi function with poles that are exchanged by a shift by $\frac{\tau}{2}$. This leads us immediately to sc.

This is equivalent to exchanging the roles of the real and imaginary axes, since

$$i \, \text{sc}(z, k') = \text{sn}(iz, k), \tag{20}$$

where $(k')^2 + k^2 = 1$. Letting $K' = \frac{\tau}{4} = \frac{it}{4}$ and $K = \frac{1}{2}$ we see that the real elliptic curve with involution $z \mapsto -\bar{z} + \frac{\tau}{2}$ is described by

$$(\text{sc}'(z, k))^2 = L_{1,1,1}(\text{sc}(z, k), k'). \tag{21}$$

Letting $y = \text{sc}'(z)$ and $x = \text{sc}(z)$ this becomes

$$y^2 = (1 + x^2)(1 + k'^2 x^2).$$

With this definition, we see that the T-duality group containing the type $\tilde{\text{I}}$ theory defined by a half-period shift in the imaginary direction takes the same form as the T-duality group depicted in Fig. 9. Again the zeros take the same form but with the roles of the real and imaginary axes reversed. That is the zeros occur at $\frac{it}{4}$, $\frac{1}{2} + \frac{it}{4}$, $\frac{3it}{4}$, and $\frac{1}{2} + \frac{3it}{4}$, and they come in two pairs of complex conjugates. Note that we could also view the canonical normal form for the type I T-duality group in terms of sc as

$$(i \, \text{sc}')^2 = L_{0,0,0}(i \, \text{sc}, k).$$

In this way we can view the difference between the species 0 and 2 antiholomorphic involutions as exchanging the roles of the imaginary and real axes (Fig. 10).

Now returning to the symmetric case, we are tempted to use sd with $K' = \frac{\tau}{4}$ and $K = \frac{1}{4}$, so that the poles are exchanged by a shift along the diagonal. However this does not quite work out, as will become clearer when we discuss the T-duality group containing the type I theory without vector structure. The problem is that sd satisfies the differential equation

$$(\text{sd}')^2 = (1 - k'^2 \, \text{sd}^2)(1 + k^2 \, \text{sd}^2),$$

making it clear that there are two real zeros and two imaginary zeros. This is not the correct form we expect for determining the sign choice. This is rectified by instead using $K' = \frac{1}{4} + \frac{it}{4}$ and $K = \frac{1}{4}$, so that $\tau' = \frac{1}{2} + \frac{it}{2}$. Then following the same argument for the species 1 antiholomorphic involution from [1], k is purely imaginary, while

$k' = \sqrt{1-k^2}$ is real. Therefore the zeros have the desired form if we let $x = i\,\mathrm{sd}(z)$ and $y = i\,\mathrm{sd}'(z)$. While this can be done, we see that the asymmetric T-dual group containing the type $\tilde{\mathrm{I}}$ theory is more natural. This is because we have to make a choice of direction no matter which case we consider.

5.2.3. The T-duality group defined by $y^2 = (1 - x^2)(1 - k^2 x^2)$, $k^2 < 0$. As described above the elliptic curve $y^2 = (1 - x^2)(1 - k^2 x^2)$, with $k^2 < 0$, $y = \mathrm{sn}'$ and $x = \mathrm{sn}$ is associated to the T-duality group containing the type I theory without vector structure. As usual we can describe the entire T-duality group (Fig. 5) in terms of the action on x and y. Since x and y have the same definitions as for the T-duality group containing the type I theory, x and y will transform in the same way. However, the normal form now has two real zeros and two complex zeros which are complex conjugates of each other, which determines the sign choice of $(+, +, +, -)$ for the only non-trivial case appearing in this grouping.

The distinction between the T-duality groups containing the type I theory and type I theory without vector structure can be distinguished by whether k is real or imaginary. With this in mind and using the identity

$$\mathrm{sn}(z, ik) = k_1'\, \mathrm{sd}(z/k_1', k_1),$$

where $k_1 = \frac{k}{\sqrt{1+k^2}}$ and $k_1 k_1' = \frac{k}{1+k^2}$, we see that the elliptic curve can be written as

$$y^2 = (1 - k_1'^2 x^2)(1 + k_1^2 x^2),$$

making the form of the zeros clearer.

Now that we have given a geometric description for all of the possible T-duality groups on an elliptic curve and the relevant sign choices, we can describe the brane content in all of the various theories.

6. D-Brane Content in the Various Orientifold Theories

Let's begin by reviewing all the twisted KR groups for the elliptic curve orientifolds. The results are given in Table 4, with the T-duality groupings color-coded. We will want to analyze this table to determine the D-brane content in each theory, and how the D-branes transform under T-duality. Note that this table only includes the *topological* type of each involution, and doesn't include information on the complex structure. When the fixed set of the involution is disconnected, the charges of the various O-planes are indicated.

The first thing one notes about the table is that the torsion-free part of the KR-groups is the same in all cases, except for a degree shift which is accounted for by T-duality. This is perhaps clearer in Table 5, obtained from the KR calculations via Eq. (1). This table will be explained more fully below when we discuss the specific brane content. For example, the torsion-free part of KO^* is $\mathbb{Z}$ in degrees 0 mod 4, so the torsion-free part of $KO^*(T^2)$, which classifies the D-brane charges in the type I theory with trivial B-field, is $\mathbb{Z}$ in degrees 0 and 2 mod 4, and $\mathbb{Z}^2$ in degrees 1 mod 4. This is the same as the torsion-free part of the first column in Table 2, which classifies the D-brane charges in the type I theory with no vector structure, and also the same as the torsion-free part of $KSC^* \oplus KSC^{*-1}$, which classifies D-brane charges in the species 0 case.

The explanation for this is that the *torsion-free part* of the twisted KR-groups classifies the BPS D-branes. These are insensitive to the O-plane charges, since the BPS

Table 4. Summary of the twisted KR groups for all the elliptic curve orientifolds, with the T-duality groupings color-coded

Type	Fixed Set and Twisting	Real Space	KR Groups
IIB (I)	T^2	$S^{2,0} \times S^{2,0}$	$KO^*(T^2)$
IIB (I no vec.)	T^2 with w_2	$S^{2,0} \times S^{2,0}$	$KO^{*-1} \oplus KO^{*-1} \oplus K^*$
IIB (Ĩ)	$\emptyset$	$S^{2,0} \times S^{0,2}$	$KSC^* \oplus KSC^{*-1}$
IIB	$\{++++\}$	$S^{1,1} \times S^{1,1}$	$KO^{*+2}(T^2)$
IIB	$\{+++-\}$	$S^{1,1} \times S^{1,1}$	$KO^{*+1} \oplus KO^{*+1} \oplus K^*$
IIB	$\{++--\}$	$S^{1,1} \times S^{1,1}$	$KSC^{*+2} \oplus KSC^{*+1}$
IIA (species 2)	$S^1 \amalg S^1$	$S^{1,1} \times S^{2,0}$	$KO^{*+1}(T^2)$
IIA	$S^1_+ \amalg S^1_-$	$S^{1,1} \times S^{2,0}$	$KSC^{*+1} \oplus KSC^*$
IIA (species 0)	$\emptyset$	$S^{1,1} \times S^{0,2}$	$KSC^{*+1} \oplus KSC^*$
IIA (species 1)	S^1	not a product	$KO^* \oplus KO^* \oplus K^{*-1}$

Table 5. D-brane charges in all three groups

T-duality group	Type I	Type I no vector	Type Ĩ
Real zeros in normal form	4	2	0
Dp-brane		Charge	
$D7$	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$D6$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
$D5$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
$D4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$D3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$D2$	0	0	$\mathbb{Z}_2$
$D1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$D0$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$D(-1)$	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$

planes are stable near both O^+-planes and O^--planes. Since each T-duality grouping contains a IIB theory with four O-planes, which differ from each other only in the O-plane charges, the BPS spectrum must be the same in all cases.

Alternatively, one can argue that the BPS spectrum, being a torsion-free phenomenon, does not depend on any twistings, either in $H^2(T^2, \mathbb{Z}_2)$ (this type of twisting distinguishes the type I theory without vector structure from the usual type I theory) or depending on a sign choice (since KO and KSp agree up to torsion). We now look at the sources of the different brane charges in all three T-duality groups.

6.1. The T-duality group defined on $y^2 = (1 - x^2)(1 - k^2x^2)$. As described in the previous section, the T-duality group defined on the elliptic curve $y^2 = (1 - x^2)(1 - k^2x^2)$ contains the type I theory compactified on $\mathbb{T}^2$ with trivial B-field. Before looking at this case, it is useful to review the case of compactifying on a single circle. There are many good sources for the KR-theory of single circle and its relation to string theory [6,27].

The type I theory compactified on a circle corresponds to the type IIB theory compactified on $S^{2,0}$ and modded out by the action of Ω. As usual we will not explicitly state that we are modding out by Ω each time, since we will always be modding out by the action of Ω.

Table 6. D-brane charges in the type I theory compactified on a circle and the type IA theory

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	$D(-1)$	Type I on S^1	Type IIA on $S^{1,1}$
KO^{p-8}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$(p+1)$-brane wrapping $S^{2,0}$	Unwrapped p-brane
KO^{p-9}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	Unwrapped p-brane	$(p+1)$-brane wrapping $S^{1,1}$

Dp-brane charges in the type I theory compactified on a circle are classified by

$$KR(S^{9-p,0} \times S^{2,0}, S^{2,0}) \cong KO^{p-8} \oplus KO^{p-9}, \qquad (22)$$

where $KO^{-j} = KO^{-j}(\text{pt})$. The second factor on the right-hand side corresponds to Dp-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from wrapped branes. The complete brane content is given in Table 6.

Since the type IA theory is obtained from the type I theory compactified on a circle by a T-duality, the relevant KR-theory is shifted in index by 1. Therefore, Dp-brane charges in the type IA theory are classified by

$$KR^{-1}(S^{9-p,0} \times S^{1,1}, S^{1,1}) \cong KO^{p-9} \oplus KO^{p-8}, \qquad (23)$$

where the second factor on the right-hand side corresponds to Dp-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from wrapped branes. The complete brane content is given in Table 6. The fact that T-duality exchanges wrapped and unwrapped branes is described by the exchanged roles for KO^{p-8} and KO^{p-9} in the two theories.

$D0$-brane charge in the type I theory receives an integral contribution from a wrapped BPS $D1$-brane and a $\mathbb{Z}_2$ contribution from an unwrapped non-BPS $D0$-brane. Under T-duality the wrapped $D1$-brane gets mapped to an unwrapped $D0$-brane in the type IA theory and the $D0$-brane gets mapped to a wrapped non-BPS $D1$-brane. However, the non-BPS branes are not stable at all points of the moduli space, so we cannot extend this argument to the entire moduli space.

To see how this isomorphism is explained physically when the non-BPS branes are unstable, let us look at the unwrapped $D0$-brane in the type I theory, following [6]. The spectrum of open strings beginning and ending on the $D0$-brane is tachyon free in ten dimensions. However, when we compactify on a circle, the ground state with winding number 1 will have a classical mass squared given by

$$m^2 = -\frac{1}{2} + R^2,$$

in units with $\alpha' = 1$. It will therefore be tachyonic if the radius of the compactification circle is $R < \frac{1}{\sqrt{2}}$. In this situation, the $D0$-brane will decay into a $D1$–$\overline{D1}$ pair that wrap the S^1. The tachyon must have anti-periodic boundary conditions so that above the critical radius it will condense into a stable kink (the $D0$-brane). This requires turning on a $\mathbb{Z}_2$ Wilson line on either the $D1$-brane or $\overline{D1}$-brane. The $\mathbb{Z}_2$ charge of the unwrapped $D0$-brane corresponds to a $\mathbb{Z}_2$ valued Wilson line in its decayed configuration. The same argument shows that the $\mathbb{Z}_2$ charge of the unwrapped $D7$- and $D8$-branes correspond to $\mathbb{Z}_2$ valued Wilson lines on their decay configurations, wrapped $D8$–$\overline{D8}$ and $D9$–$\overline{D9}$

pairs respectively. For the $D(-1)$- and wrapped $D0$-brane you have to compare the instanton action since they are instantonic.

Under T-duality the unwrapped non-BPS Dp-branes with $p = 0, 7, 8$ get mapped to wrapped $D(p+1)$-branes. Since T-duality inverts the radius, these develop a tachyon and become unstable when the T-dual radius $\tilde{R} > \sqrt{2}$. For such radii the wrapped $D(p+1)$-branes decay into unwrapped Dp–$\overline{Dp}$ systems constrained to the $O8^+$-planes. The non-trivial $\mathbb{Z}_2$ Wilson line in the type I theory corresponds to the brane and anti-brane being on different O-planes in the type IA theory. When a $\mathbb{Z}_2$ charged wrapped $D(p+1)$-brane decays in the type IA theory, its charge then corresponds to the $\mathbb{Z}_2$ choice of which O-plane the Dp-brane is located on.

In the region of stability of the type IA theory ($\tilde{R} < \sqrt{2}$) there would, at first glance, seem to be more $\mathbb{Z}_2$ charges than predicted by K-theory. Given the above discussion we would expect the $D0$-brane in the type IA theory to get a $\mathbb{Z}_2$ charge contribution from the choice of which O-plane to locate an unwrapped $D0$-brane and another $\mathbb{Z}_2$ charge contribution coming from a wrapped $D1$-brane (since we are in the region of stability). However, K-theory predicts that there should be only one source of $\mathbb{Z}_2$ $D0$-brane charge. To understand this, consider a stuck $D0$-brane (half of a $D0$-brane) at one O-plane and a wrapped $D1$-brane. This has the same conserved charges as a stuck $D0$-brane at the other O-plane and will decay into the latter configuration. In general a stuck Dp-brane at one O-plane will be transferred to a stuck Dp-brane at the other O-plane by a wrapped non-BPS $D(p+1)$-brane. This is described in [17] as a non-BPS brane stretched between two O-planes switching the type of O-plane between an Op- and $\widetilde{Op}$-plane (an $\widetilde{Op}$-plane can be interpreted as an Op-plane with a stuck Dp-brane). The above brane transfer operation shows that the $\mathbb{Z}_2$ $D0$-brane charge coming from the wrapped $D1$-brane and the contribution from the choice of which O-plane the unwrapped $D0$-brane is located at are not distinct sources of charge and that the K-theory prediction that there is only one distinct source of $\mathbb{Z}_2$ $D0$-brane charge is correct. We have seen that the charge spectrum remains unchanged in and out of the region of stability for the non-BPS branes and that K-theory accurately classifies the charges for the entire moduli space.

Now let us return to the situation of interest, where we compactify two dimensions. The Dp-brane charges in the type IIB theory on $S^{2,0} \times S^{2,0}$ are classified by

$$KR(S^{8-p,0} \times S^{2,0} \times S^{2,0}, S^{2,0} \times S^{2,0}) \cong KO^{p-7} \oplus 2KO^{p-8} \oplus KO^{p-9}. \quad (24)$$

The KO^{p-7} term corresponds to Dp-brane charge coming from $D(p+2)$-branes wrapping the entire $\mathbb{T}^2$. The two copies of KO^{p-8} correspond to $D(p+1)$-branes wrapping the two different circles of $\mathbb{T}^2$ and the KO^{p-9} term corresponds to unwrapped Dp-branes. The complete brane content is given in Table 7.

The Dp-brane charges in the type IIA theory on $S^{2,0} \times S^{1,1}$ are classified by

$$KR^{-1}(S^{8-p,0} \times S^{2,0} \times S^{1,1}, S^{2,0} \times S^{1,1}) \cong KO^{p-7} \oplus 2KO^{p-8} \oplus KO^{p-9}. \quad (25)$$

Now, after performing a T-duality from the previous theory, the KO^{p-7} term corresponds to Dp-brane charge coming from $D(p + 1)$-branes wrapping $S^{2,0}$. One of the copies of KO^{p-8} now corresponds to a $D(p+2)$-branes wrapping $S^{2,0} \times S^{1,1}$, while the other corresponds to unwrapped Dp-branes. The KO^{p-9} term corresponds to $D(p+1)$-branes wrapping $S^{1,1}$.

The Dp-brane charges in the type IIB theory on $S^{1,1} \times S^{1,1}$ are classified by

$$KR^{-2}(S^{8-p,0} \times S^{1,1} \times S^{1,1}, S^{1,1} \times S^{1,1}) \cong KO^{p-7} \oplus 2KO^{p-8} \oplus KO^{p-9}. \quad (26)$$

Table 7. D-brane charges for all of the theories related to the type I theory with trivial B-field on $\mathbb{R}^8 \times \mathbb{T}^2$ by T-duality

Dp-brane	$D7$	$D6$	$D5$	$D4$	$D3$	$D2$	$D1$	$D0$	$D(-1)$	Type I on $S^{2,0}_{(1)} \times S^{2,0}_{(2)}$	Type IIA on $S^{2,0}_{(1)} \times S^{1,1}_{(2)}$	Type IIB on $S^{1,1}_{(1)} \times S^{1,1}_{(2)}$
$KO^{p-7}(\text{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$(p+2)$-brane wrapping $\mathbb{T}^2$	$(p+1)$-brane wrapping $S^{2,0}_{(1)}$	Unwrapped p-brane
$KO^{p-8}(\text{pt})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$(p+1)$-brane wrapping $S^{2,0}_{(1)}$	$(p+2)$-brane wrapping $S^{2,0}_{(1)} \times S^{1,1}_{(2)}$	$(p+1)$-brane wrapping $S^{1,1}_{(2)}$
$KO^{p-8}(\text{pt})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$(p+1)$-brane wrapping $S^{2,0}_{(2)}$	Unwrapped p-brane	$(p+1)$-brane wrapping $S^{1,1}_{(1)}$
$KO^{p-9}(\text{pt})$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	Unwrapped p-brane	$(p+1)$-brane wrapping $S^{1,1}_{(2)}$	$(p+2)$-brane wrapping $S^{1,1}_{(1)} \times S^{1,1}_{(2)}$

Note the shift in index by 2 from the relevant KR-theory for type I on $\mathbb{T}^2$. Performing 2 T-dualities shifts the index by 2. This fact is often overlooked when describing ordinary type IIA/type IIB T-dualities on smooth manifolds. If we perform 2 T-duality transformations on the type IIB theory on $S^1 \times S^1$ we get back the type IIB theory on $S^1 \times S^1$. D-brane charges in the original theory are classified by $K(\mathbb{T}^2)$; in the dual theory they are classified $K^{-2}(\mathbb{T}^2)$. $K^{-2}(\mathbb{T}^2) \cong K(\mathbb{T}^2)$ by Bott periodicity, but it is important to keep track of the index shift for determining the dimensions of the branes contributing the various charges. In our present discussion it is even more important because the relevant KR-theory has period 8 and not 2. For the type IIB theory on $S^{1,1} \times S^{1,1}$, the KO^{p-7} term corresponds to Dp-brane charge coming from unwrapped Dp-branes. The two copies of KO^{p-8} correspond to $D(p+1)$-branes wrapping the different copies of $S^{1,1}$ and the KO^{p-9} term corresponds to $D(p+2)$-branes wrapping $S^{1,1} \times S^{1,1}$.

It is pointed out in [9] that performing 2 T-dualities from the type I theory does not just lead to reflection of both compact directions, but should also be combined with the action of $(-1)^{F_L}$, where F_L is the left-moving spacetime fermion number. As described in [32], D-branes in orientifolds of the type $X/(\iota \cdot \Omega \cdot (-1)^{F_L})$ are classified by $KR_{\pm}$. Using the definition for $KR_{\pm}$ given in [5], $KR_{\pm}(X) \cong KR(X \times \mathbb{R}^{2,0})$, we see that

$$KR_{\pm}(S^{1,1} \times S^{1,1}) \cong KR^{-2}(S^{1,1} \times S^{1,1}). \tag{27}$$

In this way, we can write our T-duality relationship between the type IIB theory on $S^{2,0} \times S^{2,0}$ and on $S^{1,1} \times S^{1,1}$ entirely in terms of $KR_{\pm}$.

$$KR_{\pm}(S^{1,1} \times S^{1,1}) \cong KR_{\pm}^{2}(S^{2,0} \times S^{2,0})$$
$$\cong KH_{\pm}^{-2}(S^{2,0} \times S^{2,0}), \tag{28}$$

where the last line was included for the sake of completeness and to point out that depending on the variant of KR-theory we choose we can make the degree change go in either direction, but it will always be a change of 2. It is also interesting to note that in this example we were able to avoid concerning ourselves with $KR_{\pm}$ and the presence of the $(-1)^{F_L}$ action by taking the appropriate degree shift for KR. $KR_{\pm}$ added no new information beyond using the correct degree for KR-theory.

Again, this description for wrapped and unwrapped branes is not valid in the entire moduli space. Non-BPS D-branes are not stable for all possible radii of the compact dimensions. For determining the non-BPS brane relations under 2 T-dualities between the 2 type IIB theories, it is possible to follow a similar argument as for compactification on a single circle, but higher order brane transfer operations need to be accounted for. Note that even in this case, the regions of stability will not be clear. When performing a single T-duality to the type IIA theory, the situation becomes more ambiguous.

It is unclear if the stability conditions on a single circle can be taken individually for the two circles we now have, or if there is some mixing. Additionally, the effect on the branes will depend on the circle we T-dualize. As an example, consider the non-BPS $D0$-brane in the type I theory compactified on $\mathbb{T}^2$. For compactification on a single circle, we saw that the $D0$-brane will decay into a $D1$–$\overline{D1}$ pair wrapping the circle when the radius becomes too small. An immediate question that arises when going to two compact dimensions is: if the radius of one circle gets too small will the $D0$-brane decay or if the other circle has a large enough radius will it be stable? Another possibility is that when the volume gets too small, the $D0$-brane will decay into a $D1$–$\overline{D1}$ pair wrapping the diagonal. Once we decide which circle the decay $D1$–$\overline{D1}$ pair wraps, we have to consider which circle we're T-dualizing. If we T-dualize the circle the $D1$–$\overline{D1}$ pair wrap, then

Table 8. Unwrapped D-brane charges near the $O8^+$- and $O8^-$-planes in a type IIA orientifold

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	$D(-1)$
KO^{p-8}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
KSp^{p-8}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0

they will map to an unwrapped $D0$–$\overline{D0}$ pair located at the different O-planes. This corresponds to a situation where the non-BPS $D1$-brane in the type IIA theory decays into $D0$–$\overline{D0}$ pair. If, however, we T-dualize the circle orthogonal to the $D1$–$\overline{D1}$ pair instead, they will map to a wrapped $D2$–$\overline{D2}$ pair in the type IIA theory.[6] In this case the non-BPS $D1$-brane in the type IIA theory decays into a wrapped $D2$–$\overline{D2}$ pair.

Most likely all three of these objects: the non-BPS $D1$-brane, the $D0$–$\overline{D0}$ pair, and the $D2$–$\overline{D2}$ pair, are all stable in different regions of the moduli space, while in regions where more than one is stable, brane transfer operations likely show that they are not distinct sources of $D0$-brane charge. This does, however, illustrate an important limitation of the K-theoretic description of brane charges. The K-theory can only tell us there is a stable source of non-BPS $D0$-brane charge. It can not determine what that source is, let alone its regions of stability. To determine this information, one would have to do a full boundary state analysis. Our gained knowledge from the K-theoretic analysis does greatly constrain what boundary states must be looked at, showing the benefit of performing the K-theoretic analysis first.

6.2. The T-duality group defined on $y^2 = -(1+x^2)(1+k^2x^2)$. The series of T-dualities involving the elliptic curve $y^2 = -(1 + x^2)(1 + k^2x^2)$ follows a pattern very similar to the previous case, but involves the type $\tilde{I}$ and $\widetilde{IA}$ theories. Therefore, we will review these two theories and their relation to one another first. The full brane content is given in Table 9.

Dp-brane charges in the type $\tilde{I}$ theory are classified by

$$KR(S^{9-p,0} \times S^{0,2}, S^{0,2}) \cong KSC^{p-8}. \tag{29}$$

KSC doesn't split into pieces from wrapped and unwrapped branes as in the species 2 case. The authors of [6] were still able to determine which charges come from wrapped and unwrapped branes using what we know about T-duality, the type IA theory, and $O8^\pm$-planes. We will follow their argument here.

As described in [12], Dp-brane charges in the type $\widetilde{IA}$ theory are classified by

$$KR_{(+,-)}^{-1}(\mathbb{R}^{9-p,0} \times S^{1,1}, S^{1,1}) = KSC^{p-8}.$$

(This will become clearer as we explore the stability of D-branes near the $O8^+$ and $O8^-$-planes.) We saw in the previous section that unwrapped Dp-branes near an $O8^+$-plane are classified by KO^{p-8} (see Table 6).

Conversely, $O8^-$-planes are quantized with symplectic gauge bundles, classified by $KSp(X) = KO^{-4}(X)$. Therefore, unwrapped Dp-brane charges near the $O8^-$-plane are classified by KSp^{p-8}. See Table 8 for a list of unwrapped Dp-brane charges near $O8^\pm$-planes in a type IIA orientifold.

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[6] For the sake of completeness, we note that if the $D1$–$\overline{D1}$ pair wraps the diagonal, T-dualizing either leg will lead to a wrapped $D2$–$\overline{D2}$ pair with B-field.

Table 9. D-brane charges in the type $\tilde{I}$ and type $\widetilde{IA}$ theories

Dp-brane	KSC^{p-8}	Region of stability	Type $\tilde{I}$	Type $\widetilde{IA}$
$D8$	$\mathbb{Z}$	Stable for all radii	Wrapped $D9$-brane	Unwrapped $D8$-brane
$D7$	$\mathbb{Z}_2$	$R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} > \sqrt{2}$	Wrapped $D8$-brane	Unwrapped $D7$-brane
		$R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} < \sqrt{2}$	Unwrapped $D7$-brane	Wrapped $D8$-brane
$D6$	0			
$D5$	$\mathbb{Z}$	Stable for all radii	Unwrapped $D5$-brane	Doubly wrapped $D6$-brane
$D4$	$\mathbb{Z}$	Stable for all radii	Wrapped $D5$-brane	Unwrapped $D4$-brane
$D3$	$\mathbb{Z}_2$	$R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} > \sqrt{2}$	Wrapped $D4$-brane	Unwrapped $D3$-brane
		$R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} < \sqrt{2}$	Unwrapped $D3$-brane	Wrapped $D4$-brane
$D2$	0			
$D1$	$\mathbb{Z}$	Stable for all radii	Unwrapped $D1$-brane	Doubly wrapped $D2$-brane
$D0$	$\mathbb{Z}$	Stable for all radii	Wrapped $D1$-brane	Unwrapped $D0$-brane
$D(-1)$	$\mathbb{Z}_2$	$R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} > \sqrt{2}$	Wrapped $D0$-brane	Unwrapped $D(-1)$-brane
		$R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\widetilde{IA}} < \sqrt{2}$	Unwrapped $D(-1)$-brane	Wrapped $D0$-brane

Let us first consider BPS branes. Table 9 shows BPS $D8$-branes, but tadpole cancellation in the type $\widetilde{IA}$ theory will require the net $D8$-brane charge to be zero. Table 9 shows that there are unwrapped BPS Dp-branes for $p = 0, 4$. The $\mathbb{Z}$ contribution to these charges coming from both KO and KSp (see Table 8) are equated. This is because two half $D0$-branes on the $O8^+$-plane form a $D0$-brane in the bulk which can then be interpreted as a $D0$-brane on the $O8^-$-plane. Similarly a $D4$-brane on the $O8^+$-plane can be considered as two half $D4$-branes on the $O8^-$-plane. Now half Dp-branes can only live on one of the $O8$-planes and it no longer makes sense to have a brane transfer operation. Therefore there is no longer the $\mathbb{Z}_2$ charge contribution coming from a choice of $O8$-plane that we saw in the type IIA theory on $S^{1,1}$. It is important to note that the BPS branes are stable near both O^--planes and O^+-planes as is apparent from the fact that there are integral contributions coming from both the KO and KSp terms.

We will now consider the unwrapped non-BPS Dp-branes. Unlike the BPS case, we cannot just look locally at stable branes near the different $O8$-planes, but need to take into account global aspects. Table 8 correctly predicts the $\mathbb{Z}_2$ charge contribution coming from unwrapped non-BPS Dp-branes for $p = -1, 3, 7$, but it also seems to predict $\mathbb{Z}_2$ charge contributions from unwrapped non-BPS Dp-branes for $p = 2, 6$ that don't appear in Table 9. This is because the $D6$-brane ($D2$-brane) is stable at the $O8^+$-plane ($O8^-$-plane), but unstable near the $O8^-$-plane ($O8^+$-plane), so not globally stable. Table 9 lists only those charges that are globally stable. Let us look a little closer at why the $D6$-brane is not globally stable. The $D6$-brane can be viewed as $D6$-brane together with its mirror $\overline{D6}$-brane. Near the $O8^+$-plane the orientifold action projects out the tachyon in the system. Near the $O8^-$-plane the projection is different and the tachyon is not removed.

We will now look at the wrapped brane charges following [6], which determines the wrapped brane charges in the type $\widetilde{IA}$ by considering unwrapped brane charges in the type $\tilde{I}$ theory. After going through their description, we will go back and see how we can follow an argument similar to the one we used for the unwrapped branes, by using the appropriately shifted KO and KSp groups. Wrapped $D(p+1)$-branes in type $\widetilde{IA}$ theory correspond to unwrapped Dp-branes in the type $\tilde{I}$ theory, so we will consider unwrapped branes in the type $\tilde{I}$ theory. Since D-branes in the type $\tilde{I}$ theory must obey the

Table 10. Wrapped D-brane charges near the $O8^+$- and $O8^-$-planes in type IIA orientifolds

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	$D(-1)$
KO^{p-9}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
KSp^{p-9}	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0

symmetry we modded the type IIB theory on $\mathbb{R}^9 \times S^1$ out by (which includes a rotation of S^1 by π radians), we must equate an unwrapped Dp-brane with another Dp-brane at the opposite point on the circle for $p = 1, 5$ and a $\overline{Dp}$-brane for $p = -1, 3, 7$. The $D1$- and $D5$-brane configurations are stable and contribute the BPS $D1$- and $D5$-brane charges appearing in Table 9. They correspond to $D2$- and $D6$-branes in the type $\widetilde{IA}$ theory that wrap the compact dimension twice respectively. For $p = -1, 3, 7$ the Dp–$\overline{Dp}$ systems give stable non-BPS states. To see that these states carry $\mathbb{Z}_2$ charge, consider a system consisting of two such states. While each individual Dp–$\overline{Dp}$ pair at opposite points of the circle is stable, the Dp-brane from one state can annihilate with the $\overline{Dp}$-brane from the other state and *vice versa*. This would seem to imply two sources of $\mathbb{Z}_2$ charge in the type $\tilde{I}$ theory; one from the unwrapped Dp-branes with $p = -1, 3, 7$ just described and the other from wrapped $D(p+1)$-branes $p = -1, 3, 7$ corresponding to the unwrapped Dp-branes with $p = -1, 3, 7$ in the type $\widetilde{IA}$ theory via T-duality. The K-theory, however, predicts that there should only be one source of $\mathbb{Z}_2$ Dp-brane charge for $p = -1, 3, 7$. This is because the two different types of states (wrapped and unwrapped branes) are stable in different regions of the moduli space. In the type $\tilde{I}$ theory the unwrapped Dp–$\overline{Dp}$ pair are stable for large R, while the wrapped $D(p+1)$-brane is stable for small R.

We could also have determined the wrapped branes in the type $\widetilde{IA}$ theory by looking at the appropriate K-theory in the vicinity of the O-planes. We saw in the previous subsection that wrapped Dp-brane charge in the type IA theory is classified by KO^{p-9}, so this will classify wrapped Dp-brane charges near the $O8^+$-plane. Near the $O8^-$-plane the orthogonal bundle is replaced with a symplectic bundle, so the wrapped Dp-brane charge will similarly be classified by KSp^{p-9}.

As can be seen from Table 10, this correctly accounts for the BPS $D5$- and $D1$-brane charge coming from wrapped $D6$- and $D2$-branes respectively. It also correctly predicts the non-BPS Dp-brane charge contribution from wrapped $D(p+1)$-branes for $p = -1, 3, 7$. It would also seem to imply the existence of non-BPS Dp-brane charge for $p = 0, 4, 8$ coming from wrapped $D(p+1)$-branes. Just as in the unwrapped case, the non-BPS wrapped $D5$-brane, for example, will be stable near the $O8^+$-plane, but not globally stable. This example shows how KR-theory picks up all global aspects of stable D-brane charges on the orientifold, though the information about wrapped and unwrapped branes is sometimes obscured. We were able to gain that information by looking at the appropriate K-theory that classifies charges locally near each O-plane and then comparing it to the KR-theory to see which locally stable states are globally stable. In fact, with the hindsight of knowing that non-BPS Dp-branes come in pairs (i.e., an unwrapped non-BPS Dp brane will decay into a wrapped non-BPS $D(p+1)$-brane for certain radii), we can determine the Dp-brane charges by comparing the wrapped and unwrapped spectrum.

We see by comparing the first and second lines of Table 8 with the first and second lines of Table 10 that the stable non-BPS D-branes are those that have stable charges in the first line of Table 8 and the first line of Table 10, or in the second line of Table 8

and the second line of Table 10. This corresponds to the fact that both the wrapped and unwrapped brane must be stable since they contribute to the charge in different regions of the moduli space.

Once all of the relevant K-theories are known, much of the D-brane content can be determined from the long exact sequence

$$\cdots \to KSC^{-n-1}(X) \to K^{-n}(X) \to KO^{-n}(X) \oplus KSp^{-n}(X) \to KSC^{-n}(X) \to \cdots,$$
(30)

as suggested in [27].

As an example, consider the segment starting with $K^{-1}(\mathrm{pt})$

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & KO^{-1} \oplus KSp^{-1} & \longrightarrow & KSC^{-1} & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & KO \oplus KSp & \longrightarrow & KSC & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{Z}_2 \oplus 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}
$$
(31)

We see immediately that the 2-torsion in KSC^{-1} comes from KO^{-1}. $KO^{-1}(\mathrm{pt})$ corresponds to the $\mathbb{Z}_2$ charge coming from an unwrapped $D7$-brane or an unwrapped $D(-1)$-brane in the type IIA theory. We can also see immediately that $\alpha = 0$ in Eq. (31). This shows that $KSC(\mathrm{pt})$ is $KO(\mathrm{pt}) \oplus KSp(\mathrm{pt})$ modulo the relation equating the generators of KO and KSp. This corresponds to the BPS D-brane charge coming from an unwrapped $D8$-brane, unwrapped $D4$-brane, and unwrapped $D0$-brane.

Again, this method cannot tell us anything about regions of stability, or really anything about the sources. We were able to determine the sources in this situation because of previous knowledge about the relationship between the physical sources near $O8^{\pm}$-planes and KO and KSp.

Now that we've reviewed the type $\tilde{\mathrm{I}}$ and $\widetilde{IA}$ theories we can easily obtain the species 0 cases we are interested in by compactifying on another circle.

It is easiest to first discuss the brane content for a half shift in only one direction, as is pictured in Fig. 3. This is because Fig. 3 corresponds to compactifying the type $\tilde{\mathrm{I}}$ and type $\widetilde{IA}$ theories on another circle. Dp-brane charges in the type IIB on $S^{0,2} \times S^{2,0}$ are classified by

$$KR(S^{8-p} \times S^{0,2} \times S^{2,0}, S^{0,2} \times S^{2,0}) \cong KSC^{p-7} \oplus KSC^{p-8}.$$
(32)

We can determine the D-brane content by compactifying the type $\tilde{\mathrm{I}}$ theory on a copy of $S^{2,0}$. Now Dp-branes in the type $\tilde{\mathrm{I}}$ theory can wrap $S^{2,0}$ and we see that KSC^{p-8} classifies branes in the type $\tilde{\mathrm{I}}$ theory that do not wrap $S^{2,0}$, while KSC^{p-7} classifies branes from the type $\tilde{\mathrm{I}}$ theory that now wrap $S^{2,0}$. For example, $D7$-brane charge is classified by $KSC \oplus KSC^{-1}$. The $\mathbb{Z}$ charge coming from KSC corresponds to the integral $D8$-brane charge from the type $\tilde{\mathrm{I}}$ theory now wrapping $S^{2,0}$. Since the BPS $D8$-brane charge in the type $\tilde{\mathrm{I}}$ theory came from a $D9$-brane wrapping $S^{0,2}$, wrapping it additionally on $S^{2,0}$ shows that the BPS $D7$-brane charge comes from a $D9$-brane wrapping the entire compact space. The $\mathbb{Z}_2$ charge coming from KSC^{-1} corresponds to the $D7$-brane charge in the type $\tilde{\mathrm{I}}$ theory that does not wrap $S^{2,0}$. For these branes there are stability conditions (not present with BPS branes) that cannot be determined by the K-theory analysis.

We saw that for the type $\tilde{\mathrm{I}}$ theory, the non-BPS $D7$ brane charge corresponds to an unwrapped $D7$-brane for large radius and a wrapped $D8$-brane for small radius (see Table 9). For one compactification circle, the stability of the $D7$-brane required a large

radius because in the covering circle the $D7$-brane is a $D7$–$\overline{D7}$ pair located at antipodal points of the circle. This argument continues to make sense when we compactify on an additional circle; however, it is unclear how the stability of the unwrapped $D7$-brane or wrapped $D8$-brane will depend on the radius of $S^{2,0}$. Determining the non-BPS brane stability only in terms of the size of the underlying type $\tilde{I}$ theory,[7] the full brane content is given in Table 11. Determining the full stability conditions for the non-BPS branes would again require doing a full boundary state analysis. The brane content for the other theories involved can be determined via T-duality and is also shown in Table 11.

The Dp-brane charges in the type IIA theory compactified on $S^{0,2} \times S^{1,1}$ are classified by

$$KR^{-1}(S^{8-p} \times S^{0,2} \times S^{1,1}, S^{0,2} \times S^{1,1}) \cong KSC^{p-7} \oplus KSC^{p-8}. \qquad (33)$$

Here, KSC^{p-7} classifies brane that don't wrap $S^{1,1}$ and KSC^{p-8} classifies branes that do wrap $S^{1,1}$, since this theory is obtained from the IIB theory on $S^{0,2} \times S^{2,0}$ by T-dualizing $S^{2,0}$. The complete brane content is listed in Table 11.

For the type IIA theory on $S^{1,1}_{(+,-)} \times S^{2,0}$ there had been no description of the brane content in terms of the KR-theory of the topological compactification space, $S^{1,1} \times S^{2,0}$. This led us to define KR-theory with a sign choice in [12]. Dp-brane charges are classified by

$$KR^{-1}_{(+,-)}(S^{8-p} \times S^{2,0} \times S^{1,1}, S^{2,0} \times S^{1,1}) \cong KSC^{p-7} \oplus KSC^{p-8}. \qquad (34)$$

Here KSC^{p-7} classifies branes that wrap $S^{2,0}$, KSC^{p-8} classifies branes that don't wrap $S^{2,0}$, and the branes that wrap $S^{1,1}_{(+,-)}$ are the same as those that wrap $S^{1,1}$ in the type IIA theory on $S^{0,2} \times S^{1,1}$ by 2 T-dualities.

Finally, Dp-branes in the type IIB theory on $S^{1,1}_{(+,-)} \times S^{1,1}_{(+,+)}$ are classified by

$$KR^{-2}_{(+,+,-,-)}(S^{8-p} \times S^{1,1} \times S^{1,1}, S^{1,1} \times S^{1,1}) \cong KSC^{p-7} \oplus KSC^{p-8}, \qquad (35)$$

with KSC^{p-7} corresponding to branes that don't wrap $S^{1,1}$ and KSC^{p-8} corresponding to branes that do.

Now let's turn our attention to the case where we shift both the real and imaginary directions by a half (Fig. 4). The 2 type IIA theories occurring in Fig. 4 are $S^{1,1}_{(+,-)} \times S^{0,2}$ and $S^{0,2} \times S^{1,1}_{(+,-)}$, which are dianalytically equivalent to $S^{1,1} \times S^{0,2}$ and $S^{0,2} \times S^{1,1}$, respectively. Therefore the Dp-brane charges are classified by $KR^{-1}(S^{8-p} \times S^{0,2} \times S^{1,1}, S^{0,2} \times S^{1,1})$. The type IIB theory with four fixed points is the same as before, but we have introduced a new ambiguity for the type IIB theory with no fixed points.

As an example, consider the non-BPS $D7$ charge in the type IIB theory with no fixed points. We saw that when we shifted in one direction, the source for this charge was a $D8$-brane wrapping $S^{0,2}$ (at least in some region of the moduli space). $S^{0,2} \times S^{0,2}$ is topologically equivalent to $S^{0,2} \times S^{2,0}$, so we would expect the D-brane content to be the same. There are now two copies of $S^{0,2}$, however, so it is no longer immediately clear which one the $D8$-brane should wrap. This is related to determining what direction we should T-dualize in as was discussed earlier.

As noted, Dp-brane charges in the type IIA theory on $S^{1,1}_{(+,-)} \times S^{0,2}$ are classified by

$$KR^{-1}(S^{8-p} \times S^{0,2} \times S^{1,1}, S^{0,2} \times S^{1,1}),$$

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[7] This corresponds to assuming the radius of $S^{2,0}$ is large.

Table 11. D-brane charges in the type $\widetilde{I}$ and type $\widetilde{I}A$ theories assuming R_2 large

Dp-brane	$\dfrac{KSC^{p-7}(\mathrm{pt})}{KSC^{p-8}(\mathrm{pt})}$	Region of stability	Type IIB on $S^{0,2}_{R_1}\times S^{2,0}_{R_2}$	Type IIA on $S^{1,1}_{(+,-)}\times S^{2,0}$	Type IIA on $S^{0,2}\times S^{1,1}$	Type IIB on $S^{1,1}_{(+,-)}\times S^{1,1}$
D7	$\dfrac{\mathbb{Z}}{\mathbb{Z}_2}$	Stable for all radii	Wrapped $D9$-brane	$D8$-brane wrapping $S^{2,0}$	$D8$-brane wrapping $S^{0,2}$	Unwrapped $D7$-brane
		$R_1 < \frac{1}{\sqrt{2}}$	$D8$-brane wrapping $S^{0,2}$	Unwrapped $D7$-brane	Wrapped $D9$-brane	$D8$-brane wrapping $S^{1,1}$
		$R_1 > \frac{1}{\sqrt{2}}$	Unwrapped $D7$-brane	$D8$-brane wrapping $S^{1,1}_{(+,-)}$	$D8$-brane wrapping $S^{1,1}$	Wrapped $D9$-brane
D6	$\mathbb{Z}_2$	$R_1 < \frac{1}{\sqrt{2}}$	Wrapped $D8$-brane	$D7$-brane wrapping $S^{2,0}$	$D7$-brane wrapping $S^{0,2}$	Unwrapped $D6$-brane
		$R_1 > \frac{1}{\sqrt{2}}$	$D7$-brane wrapping $S^{2,0}$	Wrapped $D8$-brane	Unwrapped $D6$-brane	$D7$-brane wrapping $S^{1,1}_{(+,-)}$
D5	$\dfrac{0}{0}$	Stable for all radii	Unwrapped $D5$-brane	$D6$-brane wrapping $S^{1,1}_{(+,-)}$ twice; Doubly wrapped $D6$-brane	$D6$-brane wrapping $S^{1,1}$	Wrapped $D7$-brane
D4	$\dfrac{\mathbb{Z}}{\mathbb{Z}_2}$	Stable for all radii	$D5$-brane wrapping $S^{2,0}$	Unwrapped $D4$-brane	Unwrapped $D4$-brane	$D5$-brane wrapping $S^{1,1}_{(+,-)}$
		Stable for all radii	$D5$-brane wrapping $S^{0,2}$	$D4$-brane wrapping $S^{2,0}$	Wrapped $D6$-brane	$D5$-brane wrapping $S^{1,1}$
D3	$\dfrac{\mathbb{Z}}{\mathbb{Z}_2}$	Stable for all radii	Wrapped $D5$-brane	Unwrapped $D3$-brane	$D4$-brane wrapping $S^{0,2}$	Unwrapped $D3$-brane
		Stable for all radii	$D4$-brane wrapping $S^{0,2}$	$D4$-brane wrapping $S^{1,1}_{(+,-)}$	Wrapped $D5$-brane	$D4$-brane wrapping $S^{1,1}$
		$R_1 < \frac{1}{\sqrt{2}}$	Unwrapped $D3$-brane	$D3$-brane wrapping $S^{2,0}$	$D4$-brane wrapping $S^{1,1}$	Wrapped $D5$-brane
		$R_1 > \frac{1}{\sqrt{2}}$	Wrapped $D4$-brane	Wrapped $D4$-brane	$D3$-brane wrapping $S^{0,2}$	Unwrapped $D2$-brane
D2	$\mathbb{Z}_2$	$R_1 < \frac{1}{\sqrt{2}}$	$D3$-brane wrapping $S^{2,0}$	$D2$-brane wrapping $S^{1,1}_{(+,-)}$	Unwrapped $D2$-brane	$D3$-brane wrapping $S^{1,1}_{(+,-)}$
		$R_1 > \frac{1}{\sqrt{2}}$	Unwrapped $D1$-brane	Wrapped $D2$-brane	$D2$-brane wrapping $S^{1,1}$	Wrapped $D3$-brane
D1	$\dfrac{0}{0}$	Stable for all radii	$D1$-brane wrapping $S^{2,0}$	Unwrapped $D0$-brane	Unwrapped $D0$-brane	$D1$-brane wrapping $S^{1,1}_{(+,-)}$
		Stable for all radii	$D1$-brane wrapping $S^{0,2}$	$D0$-brane wrapping $S^{2,0}$	Wrapped $D2$-brane	$D1$-brane wrapping $S^{1,1}$
D0	$\dfrac{\mathbb{Z}}{\mathbb{Z}_2}$	Stable for all radii	Wrapped $D1$-brane	Unwrapped $D(-1)$-brane	$D0$-brane wrapping $S^{0,2}$	Unwrapped $D(-1)$-brane
		Stable for all radii	$D0$-brane wrapping $S^{0,2}$	$D0$-brane wrapping $S^{0,2}$	Wrapped $D1$-bane	$D0$-brane wrapping $S^{1,1}$
D(−1)	$\dfrac{\mathbb{Z}}{\mathbb{Z}_2}$	$R_1 < \frac{1}{\sqrt{2}}$; $R_1 > \frac{1}{\sqrt{2}}$	Unwrapped $D(-1)$-brane	$D0$-brane wrapping $S^{0,2}$	$D0$-brane wrapping $S^{1,1}$	Wrapped $D1$-brane

which is the same as for the non-symmetric case. For the non-symmetric case, the double T-duality between the 2 type IIA theories related two different theories. For the symmetric case, it relates the same theory.

If R_1 and R_2 are both large (or both small large) then the 2 IIA theories in the symmetric case will be in the same regions of stability for the non-BPS branes. For concreteness, consider the case where R_1 and R_2 are both large. The brane content under this assumption for the non-symmetric case is given in Table 11. In this region the non-BPS brane charge comes from an unwrapped $D7$-brane in the type IIB theory on $S^{0,2} \times S^{0,2}$. In the 2 T-dual IIA theories this comes from a $D8$-brane wrapping the copy of $S^{1,1}_{(+,-)}$ in $S^{1,1}_{(+,-)} \times S^{0,2}$ or $S^{0,2} \times S^{1,1}_{(+,-)}$. In both cases $S^{1,1}_{(+,-)}$ has a small radius and $S^{0,2}$ has a large radius, so the IIA theories are truly symmetric. If R_1 is small and R_2 large (or vice versa), then the two IIA theories are in different regions of the moduli space. So if we start with the type IIA theory on $S^{1,1}_{(+,-)} \times S^{0,2}$ where the torus has small volume, the double T-dual will give the type IIA theory on $S^{0,2} \times S^{1,1}_{(+,-)}$ where the torus has large volume. According to Table 11 we would expect the non-BPS $D7$-charge to be given by a wrapped $D9$-brane in the type IIA theory on $S^{1,1}_{(+,-)} \times S^{0,2}$ when both compact directions have small radii. Under a double T-duality we would expect the non-BPS $D7$-brane charge to come from an unwrapped $D7$-brane in the type IIA theory on $S^{0,2} \times S^{1,1}_{(+,-)}$ with large volume. It is reasonable to expect the unwrapped $D7$-brane to be stable for large volume based on what we know about type IIA circle orientifolds, but without performing a full boundary state analysis we cannot be sure how the stability conditions for non-BPS branes on $S^{0,2}$ and $S^{1,1}_{(+,-)}$ combine in our current case.

We cannot extend the results in Table 11 to the symmetric case following the prescription described above for $D6$- and $D2$-brane charge. Let us consider the case of $D6$-brane charge. For the type IIA theory on $S^{0,2} \times S^{1,1}_{(+,-)}$, we would expect the non-BPS $D6$-brane charge to come from an unwrapped $D6$-brane when $S^{0,2}$ has a large radius and $S^{1,1}_{(+,-)}$ has a small radius, by comparison to the non-symmetric case. This, however, does not make sense. Under 2 T-dualities this would map to a $D8$-brane wrapping $S^{1,1}_{(+,-)} \times S^{0,2}$, where again $S^{0,2}$ has a large radius and $S^{1,1}_{(+,-)}$ has a small radius. This would imply that both unwrapped $D6$-branes and wrapped $D8$-branes are stable (and dependent) sources of non-BPS $D6$-brane charge in this region of the moduli space of $S^{0,2} \times S^{1,1}_{(+,-)}$. We could then expect the wrapped $D8$-brane to be stable in $S^{0,2} \times S^{1,1}$. By T-duality this would imply a stable unwrapped $D6$-brane in the type IIA theory on $S^{1,1}_{(+,-)} \times S^{2,0}$. We know this cannot be possible since $D6$-branes are unstable near O^--planes in type IIA theories. The problem is seen more easily by noting that if an unwrapped $D6$-brane was stable in the type IIA theory on $S^{0,2} \times S^{1,1}_{(+,-)}$, then under T-duality there would be a stable $D7$-brane wrapping $S^{0,2}$ in the type IIB theory on $S^{0,2} \times S^{0,2}$, which is not possible. The problem for both the $D2$ and $D6$ charges is that in the non-symmetric case there is a region where the charge comes from a $D(p+1)$-brane, $p = 2, 6$, wrapping $S^{2,0}$ which is not stable wrapping $S^{0,2}$ (see Table 9).

One possible solution to this is simply to say that the only source for non-BPS $D6$-brane charge in the type IIB theory of $S^{0,2} \times S^{0,2}$ is a wrapped $D8$-brane, but there are several unsatisfactory consequences of this. This would preclude the possibility of a stable $D7$-brane wrapping $S^{1,1}_{(+,-)}$ in the type IIB theory with four fixed points and assume the unwrapped $D6$-brane is stable everywhere in the moduli space. We would expect
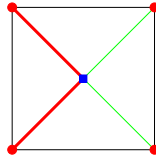
Fig. 11. The *red thick* and *green thinner lines* show equivariant copies of $S^{0,2}$ in $S^{1,1}_{(+,-)} \times S^{0,2}$, which intersect at the *blue square* and red circles (note that all *red circles* are equated) (color figure online)
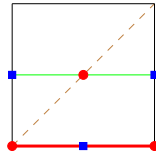

Fig. 12. The *red* and *green lines* show a copy $S^{2,0}$ in $S^{0,2} \times S^{0,2}$, relative to $\tilde{\tau} = \tau + 1$, shown by the *brown dashed line* (color figure online)

the unwrapped $D6$-brane in the type IIB theory on $S^{1,1}_{(+,-)} \times S^{1,1}_{(+,-)}$ to be unstable for small volume and a $D7$-brane wrapping $S^{1,1}_{(+,-)}$ to be stable there (more on this below). Furthermore, we know there is a copy of $S^{2,0}$ in $S^{0,2} \times S^{0,2}$ from Fig. 7.

As another possible resolution to the sources of $D6$-brane charge, consider the theory with involution $z \mapsto -\bar{z} + \frac{1+\tau}{2}$ with $\tau = it$, the type IIA theory on $S^{1,1}_{(+,-)} \times S^{0,2}$. We mentioned earlier that in the T-dual theory $S^{0,2} \times S^{0,2}$, branes that wrap $S^{0,2}$ should wrap the diagonal since it is equivariant. The diagonal is no longer equivariant in $S^{1,1}_{(+,-)} \times S^{0,2}$—it is exchanged with the antidiagonal. Instead we should consider pairs of branes that wrap the equivariant copies of $S^{0,2}$ pictured in Fig. 11. The reason we need to consider pairs should become apparent momentarily. Note that the red circle and green circle each wrap the imaginary direction corresponding to $S^{0,2}$, but do not wrap the real direction corresponding to $S^{1,1}_{(+,-)}$. It is hard to see what happens to this pair of branes under T-duality, but notice that we can decompose them as the diagonal and antidiagonal.

If we then T-dualize in the imaginary direction to get $S^{0,2} \times S^{0,2}$, branes that wrap the diagonal and anti-diagonal will map to branes that wrap the real direction; see Fig. 12.

As described in the discussion of Fig. 7, the pair of red and green lines in Fig. 12 (T-dual to the pair of red and green lines in Fig. 11) together define a copy of $S^{2,0}$ in $S^{0,2} \times S^{0,2}$ relative to the equivalent complex modulus $\tilde{\tau} = \tau + 1$. To describe the unwrapped $D6$- and wrapped $D8$-branes that appear in the non-symmetric case, we rely on the previous observation that the non-symmetric case can be obtained from the symmetric case by instead T-dualizing in the $\tilde{\tau}$ direction. Note that when T-dualizing along the diagonal, a brane that wraps to the diagonal will map to an unwrapped brane, while a brane that wraps the antidiagonal will map to a wrapped brane. While we cannot give the sources for all of the non-BPS charges, the K-theory analysis greatly constrains what boundary states need to be considered.

Finally, we make one last note about the D-branes in the type IIB theory with involution $z \mapsto -z + \frac{1+\tau}{2}$. This is the same as the type IIB theory with involution $z \mapsto -z + \frac{1}{2}$ (which we considered previously), with the only difference being which 2-torsion points are exchanged. The exchange of 2-torsion points corresponds to an O^+–O^--plane pair,

so the only difference between the two theories is the relative location of the O^+- and O^--planes. Therefore we can easily convert our previous discussion of D-brane content. For example, in a certain region of the moduli space we found there was a stable $D8$-brane wrapping $S^{1,1}$. In general this corresponds to a $D8$-brane stretched between the two O^+-planes.

6.3. The T-duality group defined on $y^2 = (1 - x^2)(1 - k^2 x^2)$, $k^2 < 0$. Letting M be any of the species 1 real elliptic curves, Dp-brane charges in the type IIA theory with species 1 are classified by

$$KR^{-1}(S^{8-p,0} \times M, M) \cong KR^{p-8}(M). \tag{36}$$

The calculation of these KR-groups is given in Sect. 3 and results in terms of D-brane charges are given in Table 5.

The KR-groups do not split into wrapped and unwrapped terms as in the previous two cases. Before discussing what we can determine about the sources, let us briefly discuss the charge classifications in the type IIB theories. Dp-brane charges in the type I theory without vector structure live in $KO^{p-7}(T^2, \widetilde{w}_2)$, where $\widetilde{w}_2 \in H^2(T^2, \mathbb{Z}_2)$ is non-zero (see the first column in Table 2). The Dp-brane charges in the type IIB theory with three O^+-planes and one O^--plane are classified by $KR_{(+,+,+,-)}^{-2}(S^{8-p,0} \times S^{1,1} \times S^{1,1}, S^{1,1} \times S^{1,1})$. (See the third column in Table 2. [8])

In our calculation of $KR^{-j}(M)$ in [12] we used the exact sequence:

$$\cdots \to KO^j \xrightarrow{\rho} KO^{j-1} \oplus KO^j \to \widetilde{KR}^j(M) \to KO^{j+1} \xrightarrow{\rho} KO^j \oplus KO^{j+1} \to \cdots. \tag{37}$$

The connecting maps ρ are given by cup product with a class in $KO^{-1} \cong \mathbb{Z}_2$ (into the first summand), which turned out to be non-zero (see Sect. 3.2.3), and a class in $KO^0 \cong \mathbb{Z}$ (into the second summand), which turned out to be zero. Note that if the connecting map were trivial then we would obtain the short exact sequence

$$0 \to KO^{j-1} \oplus KO^j \to \widetilde{KR}^j(M) \to KO^{j+1} \to 0. \tag{38}$$

This would give

$$\widetilde{KR}^j(M) \cong KO^{j-1} \oplus KO^j \oplus KO^{j+1}, \tag{39}$$

or

$$KR^j(M) \cong KO^{j-1} \oplus 2KO^j \oplus KO^{j+1}, \tag{40}$$

since $KR^j(M) \cong \widetilde{KR}^j(M) \oplus KO^j(M)$. This is just the KR-theory for the type I theory with trivial B-field. So mathematically, we see that the difference in the brane classification for the type I theory with non-trivial B-field from that with trivial B-field comes from the non-triviality of the connecting maps ρ in Eq. (37), and thus must be related to the twisting (which is 2-torsion).

Now let us return to the D-brane sources. A lot of information can be gained by looking at the brane charges for the three groups side by side; see Table 5.

As noted previously, the BPS spectrum is the same for all three groups. As an example, consider the BPS $D7$-brane charge. As with all the other cases, in the type I theory

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[8] One might expect the need to add an additional twisting due to the B-field, but as already noted, non-trivial B-fields do not affect O-planes that do not wrap the compact directions. The affect of the non-trivial B-field is already encoded in the sign choice.

with $B = \frac{1}{2}$ this corresponds to a wrapped $D9$-brane. In the type IIA theories it corresponds to a $D8$-brane wrapping the fixed circle, and in the type IIB theory with four fixed points it corresponds to an unwrapped $D7$-brane. As before, two half $D7$-branes located at the $O7^-$-plane can form a $D7$-brane in the bulk, which can be explained as a $D7$-brane at one of the O^+-planes, showing why the BPS spectrum is unchanged. Note that the only cases where there could be any possible confusion are the values of p for which there are two sources of BPS charge. This happens for $D4$- and $D0$-branes. In both cases there are $D(p+1)$-branes wrapping 1-cycles in the type IIB theories, where there are two distinct 1-cycles to wrap. In the type IIA theories, where there is only one 1-cycle that can be wrapped by a BPS brane, we have a wrapped $D(p+2)$-brane and an unwrapped Dp-brane.

Determining the non-BPS sources is more complicated, but we can draw some conclusions by comparing the three groups that still need to be verified by a boundary state analysis. There are only three values of p for which the Dp-brane charge contains torsion; they are $p = 7, 6$, and -1.

The $p = 7$ and $p = -1$ cases are related by Bott periodicity, so we will only describe the situation for the non-BPS $D7$-brane charge. Then the $D(-1)$-brane charge source can be obtained by shifting the degree by 8. We will also only describe the situation for the type IIA theories, since the IIB theory can be obtained following the T-dualities described. There are three sources for non-BPS $D7$-brane charges in the species 2 type IIA theories: an unwrapped $D7$-brane, a wrapped $D9$-brane, and a $D8$-brane wrapping $S^{1,1}$. For species 0 there is one source of non-BPS $D7$-brane charge. This can correspond to a $D9$-, $D8$-, or $D7$-brane depending on where in the moduli space we are. The important feature here is that the unwrapped $D7$-brane is related to the wrapped $D8$-brane based on the radius of $S^{1,1}_{(+,-)}$. As noted in the calculation of the KR-theory for the species 1 case in Sect. 3, the KR-theory for M with the fixed circle removed gives KSC, showing that away from the fixed circle the species 1 case should contain the species 0 charges. Let's first consider the $D8$-brane wrapping $S^{1,1}$, which appears in both the species 2 and 0 groups. It seems safe to assume that this is a source for non-BPS $D7$-brane charge for the species 1 group when one compact direction is small and the other is large, for the same reason that it contributed non-BPS charge in the other cases. The copy of $S^{1,1}$ it wraps in the species 1 case is the circle perpendicular to the fixed circle (for $\tau = e^{i\theta}$ this is the diagonal, S_D, or antidiagonal, S_A). T-dualizing both directions will exchange S_D and S_A, sending the $D8$-brane wrapping the copy of $S^{1,1}$ in one IIA theory to a $D8$-brane wrapping the copy of $S^{1,1}$ in the T-dual IIA theory, which also has one large compact direction and one small one. This shows that if the $D8$-brane wrapping $S^{1,1}$ is stable for a species 1 type IIA theory, it must also be stable for the doubly T-dual IIA theory. Now in the species 0 case the $D8$-brane wrapping $S^{1,1}$ and wrapped $D9$-brane are stable in different regions of the moduli space, so it would not make sense to include a wrapped $D9$-brane and $D8$-brane wrapping $S^{1,1}$ in the same region of stability. However, if we include an unwrapped $D7$-brane in the other T-dual type IIA theory, we will have a wrapped $D9$-brane. Therefore all that is left that the second source can be is a $D8$-brane wrapping $S^{1,1}_{(+,-)}$. For $\tau = e^{i\theta}$ the copy of $S^{1,1}_{(+,-)}$ that is wrapped is parallel to the fixed circle, but shifted by a half. It is easy to show, following similar arguments, that it is not possible to construct a consistent situation where the $D8$-brane wrapping $S^{1,1}$ is not stable, since unwrapped $D7$-branes and wrapped $D9$-branes are stable in different regions. Therefore we see that in both the large and small volume limit the non-BPS $D7$-brane sources are a $D8$-brane wrapping $S^{1,1}$ and a $D8$-brane wrapping $S^{1,1}_{(+,-)}$. The important

feature that leads to this conclusion is that from the previous two cases we saw that a $D8$-brane is stable whether or not it wraps $S^{1,1}$ or $S^{1,1}_{(+,-)}$, unlike the other branes involved.

For the non-BPS $D6$-brane charge there is also a unique possibility for consistent T-dual sources. As noted, away from the fixed circle we would expect the species 1 real elliptic curve to contain the sources of non-BPS brane charges from the species 0 real elliptic curves. If this source was a $D7$-brane wrapping $S^{0,2}$, after performing a double T-duality we would get back a wrapped $D7$-brane. This would not leave any room for the second source of non-BPS charge since the wrapped $D8$-brane and unwrapped $D6$-branes are stable in different regions. This implies that the two sources of non-BPS $D6$-brane charge should be an unwrapped $D6$-brane and a wrapped $D8$-brane.

As one last interesting note on the non-BPS brane charges, consider the source for non-BPS $D3$- and $D2$-brane charge that appears in the species 0 group, but not in the species 1 or 2 groups. The source in the species 0 IIA theory with an $O8^+$- and an $O8^-$-plane is an unwrapped $D3$-brane located at the $O8^-$-plane. This corresponds to a $D4$-brane stretched between the two $O7^-$-planes in the T-dual type IIB theory with four fixed points. Since the type IIB theories with four fixed points for the species 1 and 2 groups do not have two $O7^-$-planes, this charge cannot exist in these theories and does not appear in their K-theory spectra.

7. Conclusion

Let us summarize what we have accomplished in this paper. We have studied *all* orientifold string theories on space-times of the form $E \times \mathbb{R}^{8,0}$, where E is an elliptic curve with holomorphic or anti-holomorphic involution. These are quite natural spacetimes to consider since elliptic curves are the only compact Calabi–Yau manifolds of complex dimension 1, and only holomorphic or anti-holomorphic involutions will preserve supersymmetry. These theories divide into three groups, and all the theories within each group are related to one another by sequences of T-dualities. For each theory, there is a corresponding twist (given by the sign choice on the O-planes and/or the B-field), and the twisted KR-theory classifies the D-brane charges. We determine not only the charge groups but also the precise brane content for each theory. To the best of our knowledge, the brane content of the type I theory without vector structure was not previously known.

It is worth pointing out a few key points:

1. The torsion-free part of the KR-groups classifies the BPS spectrum and does not depend on the twists. Twisting only affects the 2-torsion in the KR-groups, not the torsion-free part of the groups.
2. Each T-duality grouping includes precisely one IIB theory with four O-planes. The signs of these O-planes can be read off from the Legendre normal form of the corresponding real elliptic curve with involution, and are reflected in the uniformization of the elliptic curve via Jacobi functions.
3. Each T-duality grouping also includes a unique variant of type I string theory, or in other words, a IIB theory where the holomorphic involution on E is either trivial or free. Possibilities for this theory are the conventional type I theory, the type I theory without vector structure, and the type $\tilde{\text{I}}$ theory.
4. A full stability analysis of the various classes of branes still remains to be done in some cases, but what we have done here is a first approximation based on understanding of theories compactified on a circle. For the "type I theory without vector structure", our understanding is already complete.

5. The T-duality groupings can be understood from either purely mathematical or purely physical points of view. It is quite dramatic that the calculations of the twisted KR-groups (which is pure algebraic topology) and the classifications via Legendre normal forms (which is pure algebraic/analytic geometry) both confirm what had been conjectured by physicists many years ago.

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