Adinkras and the Dynamics
of Superspace Prepotentials

C. F. Doran

Department of Mathematics, University of Washington
Seattle, WA 98105, USA
doran@math.washington.edu

M. G. Faux

Department of Physics, State University of New York
Oneonta, NY 13825, USA
fauxmg@oneonta.edu

S. J. Gates, Jr.

Center for String and Particle Theory, Department of Physics
University of Maryland, College Park, MD 20472, USA
gatess@wam.umd.edu

T. Hübsch [1]

Department of Applied Mathematics and Theoretical Physics
Delaware State University, Dover, DE 19901, USA
thubsch@desu.edu

K. M. Iga

Natural Science Division, Pepperdine University, Malibu, CA 90263, USA
Kevin.Iga@pepperdine.edu

G. D. Landweber

Department of Mathematics, Bard College
Annandale-on-Hudson, NY 12504-5000, USA
gregland@bard.edu

---

[1]On leave from Dept. of Physics and Astronomy, Howard University, Washington, DC.

Abstract

We demonstrate a method for describing one-dimensional N-extended supermultiplets and building supersymmetric actions in terms of unconstrained prepotential superfields, explicitly working with the Scalar supermultiplet. The method uses intuitive manipulations of Adinkras and $\mathcal{GR}(\mathrm{d}, N)$ algebras, a variant of Clifford algebras. In the process we clarify the relationship between Adinkras, $\mathcal{GR}(\mathrm{d}, N)$ algebras, and superspace.

Mathematics Subject Classification: Primary: 81Q60; Secondary: 15A66, 70S05

Keywords: off-shell supersymmetry, superfields, superpotentials, Lagrangian dynamics

1   Introduction

During the second half of the twentieth century, quantum field theory in general and Yang-Mills theories in particular provided a fertile and important arena for speculating about fundamental laws of nature. A fundamental aspect of these theories is that the constituent fields comprise representations of various symmetries. In these theories, most notably the standard model of particle physics, which has garnered spectacular experimental verification, the elementary fields describe off-shell representations of internal symmetries, where the important qualifier *off-shell* indicates that the symmetry representation of the fields is independent of their four-momentum configuration. The most important contemporaneous arena for attempting to reconcile particle physics with gravitation is perturbative string theory, its developing non-perturbative generalizations, such as M-theory, and its effective descriptions in terms of supergravity theories, all of which involve supersymmetry in one way or another. However, there remains a noteworthy fundamental structural distinction between string-inspired physics and Yang-Mills theories concerning the way the respective inherent symmetries are represented. Superstring theories, and their effective descriptions in terms of ten- or eleven-dimensional supergravity, are formulated *on-shell*: the supersymmetry is realized only when the basic fields satisfy classical equations of motion. This discrepancy indicates that the current understanding of supersymmetry is as yet incomplete, and it motivates the investigation of how off-shell supersymmetry can be realized generally in quantum field theories.

A traditional approach to classifying irreducible supersymmetry representations relies on the fact that all known off-shell supermultiplets can be formulated using Salam-Strathdee superfields subject to differential constraints and

gauge transformations. Distinctions between supermultiplets can be encoded using different ways to pose such restrictions. This approach has an appealing elegance to it. Unfortunately, for cases with more than a few supersymmetries, the range of possible constraints is large, and no compelling rhyme nor reason has emerged as a means for organizing these. For instance, the importantly influential $N = 4$ Super Yang-Mills theory in four dimensions has never been described off-shell, and if there exists a constrained $N = 4$ superfield description of this supermultiplet, this has not yet been discovered. We believe that in order to resolve this dilemma, we must go beyond ordinary superspace techniques, instead developing new approaches based on emerging facts about the mathematical underpinnings of supersymmetry.

In previous papers [1,2,3] we described a re-conceptualization for organizing the mathematics associated with supersymmetry representation theory which is complementary to, but logically independent of, the popular Salam-Strathdee superspace methods. Our approach provides fresh insight and additional leverage from which to attack the off-shell problem. One of our motivating desires is to determine an off-shell field theory description of four-dimensional $N = 4$ Super Yang-Mills theory and the ten- and eleven-dimensional supergravity theories. Our investigations are predicated on two related themes: The first purports that the mathematical content of supersymmetry in field theories of arbitrary spacetime dimension is fully encoded in the seemingly restricted context of one-dimensional field theories, *i.e.*, within supersymmetric quantum mechanics. The second is the observation that the representations of one-dimensional superalgebras admit a classification in terms of graph theory, using diagrams called "Adinkras" which we have been incrementally developing.

Our two most recent previous papers on this subject [1,2] have concentrated on formal mathematical aspects of this approach and were devoted to developing precise terminology, developing mathematical theorems associated with our Adinkra diagrams, and describing part of a supermultiplet classification scheme using the language of Adinkras. In this paper we use these techniques to elucidate instead some of the physics of supersymmetry rather than the mathematics. In particular we address the question of how the the special class of irreducible one-dimensional arbitrary N-extended supermultiplets known as Scalar supermultiplets can be described in terms of unconstrained superfields[2], known as prepotentials, and how these can be used to build supersymmetric action functionals for these supermultiplets. We focus on Scalar supermultiplets because these provide the simplest non-trivial context in which to illustrate our techniques. Similar techniques can be brought to bear on a wide class of interesting supermultiplets; we intend to produce followup papers

---

[2]Theorem 7.6 of Ref. [1] guarantees this, and gives a general algorithm to this effect; part of our present task then is to show the concrete details of this construction.

in the near future addressing some of these questions.

A familiarity with the basic techniques described in [1,3], which in turn are predicated on developments appearing in [5,6,7], is an absolute prerequisite for following our subsequent discussion. Central to these are the relevance of $\mathcal{GR}(\mathrm{d}, N)$ algebras to supersymmetry representations, the meaning and the significance of Adinkra diagrams, and the basic idea concerning how automorphisms on the space of supermultiplets may be coded in terms of raising and lowering operations on Adinkras.

This paper is structured as follows:

In Section 2 we describe a special class of Adinkra diagrams, known as Base Adinkras, which are the graphical counterparts of Clifford algebra superfields. We explain how these diagrams, which are in general reducible, can be used as fundamental tools for constructing irreducible supermultiplets via geometric vertex raising operations. In Section 3 we review another special class of Adinkras, known as Top Adinkras, which are the graphical counterparts of Salam-Strathdee superfields; these provide the connection between our technology and more traditional techniques. We explain how Top Adinkras can be obtained from Base Adinkras via extreme application of vertex raising operations, and we review a relationship between superspace differentiation and vertex raising. We explain how Top Adinkras can be used to organize the construction of superspace operators useful for projecting onto subspaces corresponding to irreducible representations; this method supplies a graphical counterpart to the organization of superspace differential projection operators. In Section 4 we review the concept of garden algebras, and we explain in algebraic terms what is meant by a Clifford algebra superfield. This section describes algebraically many of the diagrammatic facts appearing in Section 2. In Section 5 we review the definition of Scalar supermultiplets and we develop the rudiments of an algorithm for discerning a prepotential description of these, which is implemented in the balance of the paper. In Section 6 we focus on the special case of $N = 2$ Scalar supermultiplets and methodically develop the corresponding prepotential superfields and a manifestly supersymmetric action built as a superspace integral involving these. This allows for a clean exposition regarding superspace gauge structures endemic to similar prepotential descriptions in the context of general N-extended supersymmetry. In Section 7 we describe the main computational result of the paper, by generalizing the $N = 2$ analysis presented in Section 6 to the case of general Scalar superfields for any value of N. An important output of this analysis is that we provide the first descriptions of 1D, N-arbitrary superprojectors which naturally are associated with Scalar supermultiplets. We then briefly summarize our results with concluding remarks.

2 Adinkrammatics

The mathematical data of one-dimensional N-extended supermultiplets can often be conveniently described in terms of bipartite graphs known as Adinkras, introduced in [3]. The vertices of these graphs correspond to the component fields of the supermultiplet, while the edges encode the supersymmetry transformations. In addition, each vertex of an Adinkra comes with an integral height assignment corresponding to twice the engineering dimension [3] of the corresponding component field. A subset of vertices, called sinks, correspond to local maxima; these connect via edges *only* to vertices with lower height.[4] Another subset of vertices, known as sources, correspond to local minima; these connect via edges *only* to vertices with greater height. In [1] we proved the so-called "Hanging Gardens Theorem", which states that an Adinkra is fully determined by specifying the underlying graph together with the set of sinks and the heights of those sinks. An Adinkra can then be envisioned as a latticework or a macramé, hanging from its sinks, and we alternatively refer to the sinks as "hooks".

By performing various geometric operations on their Adinkras, we can transform one supermultiplet into another. In earlier papers we have described some of these operations as part of an ongoing endeavor to describe a mathematically rigorous supersymmetry representation theory. In a "vertex raising" operation[5], we take a vertex which is a local minimum or source and increase its height by two, physically lifting it two levels on the page. In terms of the supermultiplet, such an operation replaces the corresponding component field with a new component field of engineering dimension one greater. In [1] we explained how such vertex raising operations, when applied to superfields, can be implemented via superspace derivatives.

A central construct in this system is a particular supermultiplet known as the "Clifford algebra superfield", whose component fields correspond to a basis of the Clifford algebra $Cl(N)$. This supermultiplet, which in general is reducible, has 2^{N-1} boson fields sharing a common engineering dimension and 2^{N-1} fermion fields of common engineering dimension one-half unit greater than the bosons. All supermultiplets can be obtained from Clifford algebra superfields via a combination of operations analogous to vertex raising and

---

[3]A field with engineering dimension δ has units of $(\text{ mass })^\delta$ in a system where $\hbar = c = 1$.

[4]In [3], the edges were directed with arrows indicating the placement of time derivatives in the supersymmetry transformations. Here, we assume that the fields have well-defined engineering dimensions, and that the Adinkra vertices have a height assignment. In this case, all arrows point towards the vertex of greater height. With this convention, our use of sources and sinks agrees with the standard usage in graph theory.

[5]This operation is related to the "automorphic duality" transformation of [3,7], which we also referred to as a "vertex raise" in [1]. The corresponding transformation of supermultiplets is called "dressing" in [9].

quotients or projections derived from the symmetries of the corresponding Adinkra.

In the balance of this section we provide a graphical description of the Clifford algebra superfield and some related and relevant diagrammatic operations. It should be understood that there exists an algebraic context for these methods, and that a full appreciation requires a synergistic understanding of the diagrams and their underlying algebraic structure. The algebraic context for the diagrams are reviewed below in Section 4. (Subsection 4.1 in particular provides an especially useful context for understanding the ways in which vertices may be coalesced.)

2.1   The Base Adinkra

The Adinkra corresponding to the Clifford algebra superfield is called the "Base Adinkra". For example, the $N = 4$ Clifford algebra superfield includes eight bosons sharing a common engineering dimension and eight fermions with engineering dimension one-half unit greater than the bosons. The $N = 4$ Base Adinkra can be drawn as follows:
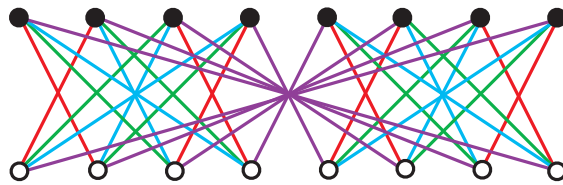

$$(2.1)$$

where each of the four supersymmetries corresponds to a unique edge color, each bosonic vertex corresponds to a boson field while each fermionic vertex corresponds to a fermion field. We use throughout this paper the convention promulgated in [1] whereby the vertical placement of Adinkra vertices correlates faithfully with the height assignment. Thus, higher components, having larger engineering dimension, appear closer to the top of the diagram. The existence of the height assignment to each vertex implies that Adinkras also provide a natural realization of an abelian symmetry, whose generator (denoted by **d**) is realized on each vertex by multiplication of the vertex by one-half times the height assignment of the vertex. This is the basis of the filtration discussed in [2].

In (2.1), a left-right symmetry is apparent: the Adinkra remains unchanged if it is reflected about a vertical axis passing through its center. This symmetry of the Adinkra gives rise to a projection of the $N = 4$ Clifford algebra superfield onto an irreducible submultiplet corresponding to the $N = 4$ Scalar Adinkra. This projection is described in detail in Subsection 2.4 below.

Another way to draw the $N = 4$ Base Adinkra is as follows:

(2.2)
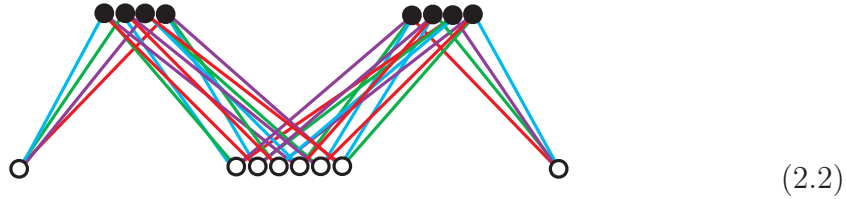
This can be obtained from (2.1) by moving vertices while maintaining the inter-vertex edge connections. This second presentation of the Base Adinkra can be obtained by selecting one bosonic vertex of the Base Adinkra, call it ϕ, and putting it on the left side of the diagram. We then gather together those vertices in the Base Adinkra that are one edge away from ϕ, and these are placed at the same height as they were in the Base Adinkra (one level above ϕ) but slightly to the right of ϕ. We then take the vertices that are two edges away from ϕ, and these six vertices are placed slightly to the right of that, at the correct height (the same height as ϕ), and so on.

The meaning of this arrangement comes about when we consider that each vertex in the Base Adinkra can be obtained by applying a finite antisymmetrized sequence of supersymmetry generators Q_I to ϕ. Thus, we can label the vertices that are one edge away from ϕ using a single index I ranging from 1 to N. Likewise, the vertices that are two edges away from ϕ can be labelled with two indices that are antisymmetrized, and so on. In general, vertices that are p edges away from ϕ are labelled as antisymmetric p-tensors with indices ranging from 1 to N.

This is not merely suggestive formalism. Once we fix the field ϕ, the supersymmetry generators map ϕ to other fields. The group $\mathrm{SO}(N)$ of R-symmetries, which acts naturally on the N supersymmetry generators, thus also acts on the fields. The orbits are precisely the clumps of vertices in the above diagram, and form a representation of $\mathrm{SO}(N)$ that is isomorphic to the corresponding exterior tensor power of the standard representation of $\mathrm{SO}(N)$.

We can abbreviate (2.2) as

(2.3)

Here the vertex multiplicity is indicated by a numeral, and the edges have been coalesced. (In general, a black edge encodes the bundled action of N supersymmetries in a manner which is well defined.)[6]

---

[6]See (4.7) below for an algebraic clarification of this point.

More generally, for any N, the Base Adinkra characteristically admits an accordion-like presentation, for example as shown here:

$$(2.4)$$
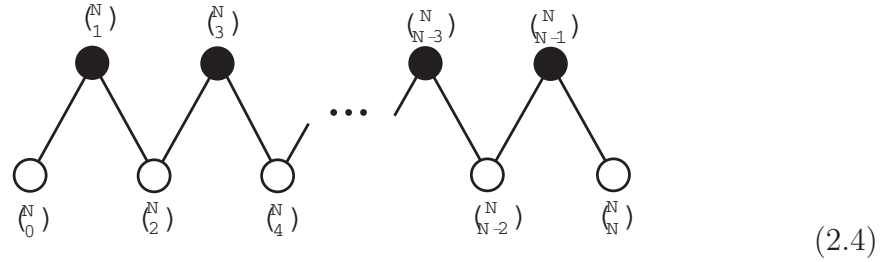
where the indicated vertex multiplicities are the binomial coefficients $\binom{N}{p}$, where $p = 0, 1, \ldots, N$ sequentially labels the compound vertices starting from the left. This reflects the fact that these vertices coincide with rank p antisymmetric tensor representations of $\mathrm{SO}(N)$, each of which describes $\binom{N}{p}$ degrees of freedom. For the sake of brevity, we refer to such representations as "p-forms". Thus, the left-most bosonic vertex (white circle) is a zero-form, and the other vertices are collected into a chain, such that the second compound vertex is a one-form fermion, the third compound vertex is a two-form boson, and so forth. The Adinkra shown in (2.4) is the general Base Adinkra for cases in which N is even; in cases where N is odd, the chain terminates on a fermionic vertex (black circle) rather than on a bosonic vertex (white circle),


$$(2.5)$$
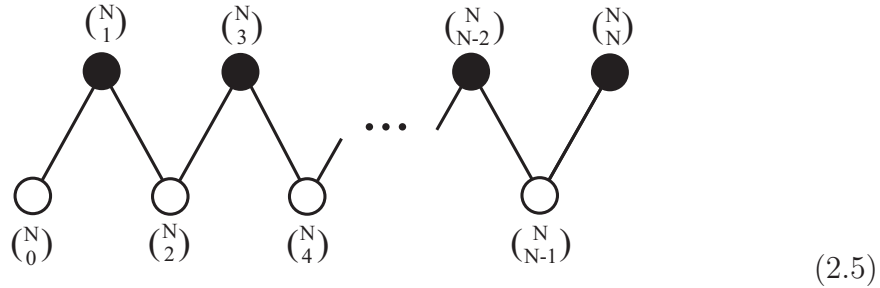
This Adinkra codifies particular transformation rules, which are exhibited below in Subsection 4.2.

2.2   The Dual Base Adinkra

We have explained how the components of the Base Adinkra can be organized into p-form representations of $\mathrm{SO}(N)$, such that the bosons are even-forms and the fermions are odd-forms. Alternatively, the vertices of the Base Adinkra can be organized differently such that the fermions are even-forms and the bosons are odd-forms. To see this, we start with our original rendering of the Base Adinkra, shown in (2.1), but this time re-organize the same vertices into

the following configuration,

$$(2.6)$$
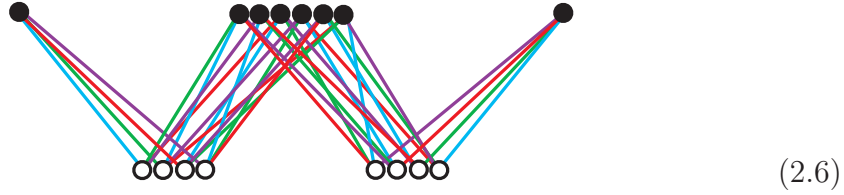
This Adinkra is equivalent to (2.1) and also to (2.2), and these can be transformed into each other merely by grouping the vertices together in different ways. In particular, (2.6) is obtained from (2.1) by starting with a 0-form fermion, and collecting vertices according to their distance from this fermion. We can abbreviate (2.6) as

$$(2.7)$$

The difference between this Adinkra and (2.3) lies in the way the $SO(N)$ structure has been imposed on the skeletal Adinkra (2.1). A similar alternative grouping can be applied to the Base Adinkra for any N, resulting in the following Dual Base Adinkra for even N:

$$(2.8)$$
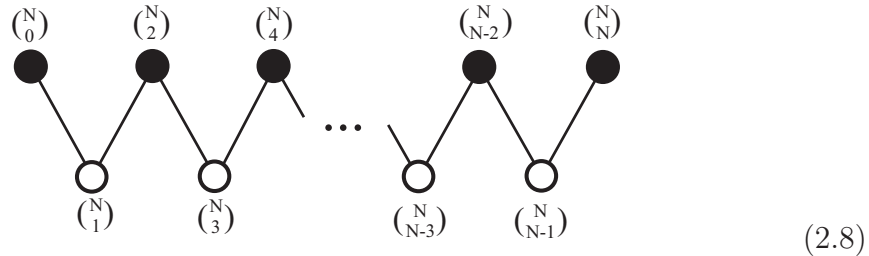
The difference between (2.8) and (2.4) lies in the manner in which the vertices have been grouped into representations of $SO(N)$. An algebraic description of this concept is described in Subsection 4.3.

2.3 The Conjugate Base Adinkra and its dual

Every supermultiplet has a counterpart obtained by toggling the statistics of all of its component fields, replacing each boson with a fermion, and vice-versa; this operation is known as a Klein flip[7] (see also [4]). The Klein flipped analog

---

[7]In the mathematical supersymmetry literature [10,11], this is called parity reversal and is denoted Π.

of the Clifford algebra superfield is called the Conjugate Clifford superfield; the corresponding Adinkra, called the Conjugate Base Adinkra, is (for N even)

$$
\begin{array}{cccccc}
\binom{N}{0} & \binom{N}{2} & \binom{N}{4} & & \binom{N}{N-2} & \binom{N}{N} \\
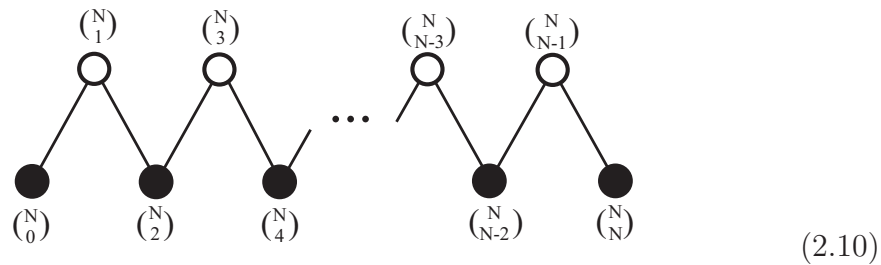\circ & \circ & \circ & \cdots & \circ & \circ \\
& \bullet & \bullet & & \bullet & \bullet \\
& \binom{N}{1} & \binom{N}{3} & & \binom{N}{N-3} & \binom{N}{N-1}
\end{array}
\tag{2.9}
$$
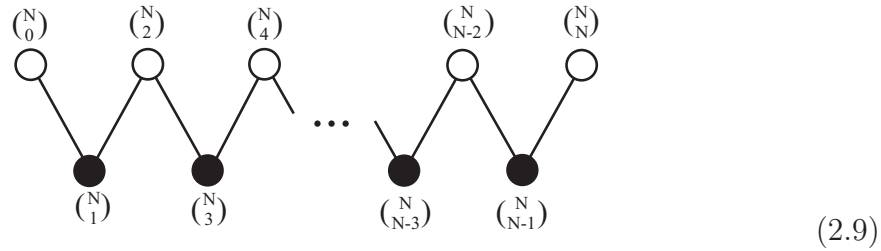
This is obtained from (2.8) by replacing all boson vertices with fermion vertices and vice-versa.[8]

Finally, there is an alternate way to group the vertices of the Conjugate Base Adinkra into $\mathrm{SO}(N)$ tensors; in a manner similar to that described above, we can re-group the compound vertices in (2.9) so that the bosons are odd-forms while the fermions are even-forms (for even N):

$$
\begin{array}{cccccc}
\binom{N}{1} & \binom{N}{3} & & \binom{N}{N-3} & \binom{N}{N-1} \\
\circ & \circ & \cdots & \circ & \circ \\
\bullet & \bullet & \bullet & & \bullet & \bullet \\
\binom{N}{0} & \binom{N}{2} & \binom{N}{4} & & \binom{N}{N-2} & \binom{N}{N}
\end{array}
\tag{2.10}
$$

This Adinkra may also be obtained as the Klein flip of (2.4). The Base Adinkra (2.4), its dual (2.8), its conjugate (2.9), and the Conjugate Base Adinkra (2.10) are depicted together in Figure 1. Each of these constructions can be used as a launching point for describing more general supermultiplets.

2.4   The Scalar Adinkra

The Base Adinkra is central concept in the representation of one-dimensional N-extended supersymmetry. Many of the irreducible representations can be obtained from the Base Adinkra by applying various operations. One such operation was mentioned above in Subsection 2.1: by imposing consistent vertex identifications we can project a given Adinkra onto sub-Adinkras corresponding to smaller representations. For example, we can make pairwise identifications of those vertices in (2.1) mapped into each other by a left-right folding

---

[8]By using (2.4) as an intermediary, we can describe another operative way to connect (2.8) with (2.9). According to this alternate scheme, we first transform (2.8) into (2.4) by merely re-grouping vertices in the manner described above. Then, via a sequence of vertex raises, we can map (2.4) into (2.9).
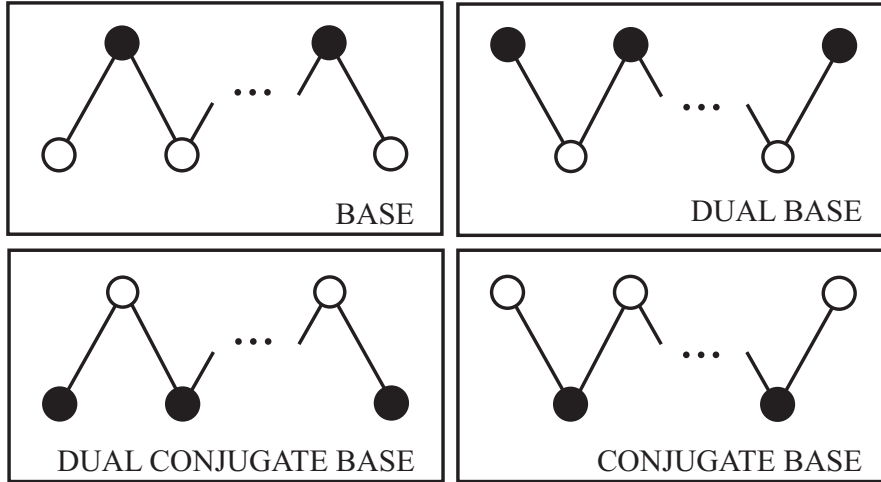
Figure 1: The Base Adinkra describes a zig-zag chain of $SO(N)$ tensor fields starting with a lower component zero-form boson. The Dual Base, the Conjugate Base, and the Dual Conjugate Base Adinkras describe analogous constructions distinguished by whether the zero-form is a boson or a fermion or by whether it is the bosons or fermions which have lower height. We display here the Adinkras for even N; the ones for odd N are analogous, starting with the Base Adinkra (2.5).

operation. What results is the following Adinkra:
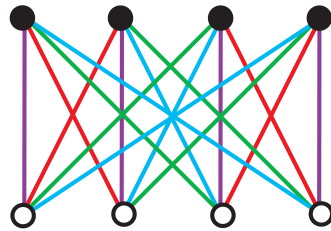
$$(2.11)$$

which corresponds to the irreducible $N = 4$ Scalar supermultiplet. A geometric way to understand this projection is to identify the underlying graph of the $N = 4$ Base Adinkra with the vertices and edges of a 4-dimensional hypercube (or tesseract). We then take a quotient of this hypercube, identifying antipodal vertices and edges (see [3]). In general, the relationship between the Clifford algebra superfield and the Scalar superfield, for each value of N, can be understood in terms of quotients of cubical Adinkras.

2.5 Node raising and other supermultiplets

The generalized Base Adinkras, shown in Figure 1, and the Scalar Adinkra, shown in (2.11), share the feature that their vertices span only two differ-

ent height assignments, or equivalently that the corresponding supermulti-
plets have component fields of only two engineering dimensions. To construct
Adinkras corresponding to supermultiplets with fields of more than two engi-
neering dimensions, we can start with these Adinkras and operate on them by
vertex raising operations. To raise a vertex in an Adinkra, we take a source ver-
tex and increase its height assignment by two. At the level of supermultiplets,
a vertex raising operation replaces a component field with a new component
field given by the τ derivative of the original component field. For example,
if a given vertex corresponds to the field $\phi(\tau)$, then we can define a new field
via $\tilde{\phi} := \partial_\tau \phi$. If ϕ corresponds to a source vertex, then the supersymmetry
transformations continue to involve only local superspace operators, and we
thereby obtain a new supermultiplet. Since the operator ∂_τ carries one unit
of engineering dimension, it follows that $\tilde{\phi}(\tau)$ describes a higher component.
In [1] we discussed these operations at length, explaining relationship between
vertex raising and superspace derivation.

For example, suppose we start with the Base Adinkra, as drawn in (2.4).
If we raise the fifth bosonic vertex, counting from the left, what results is the
following new Adinkra:
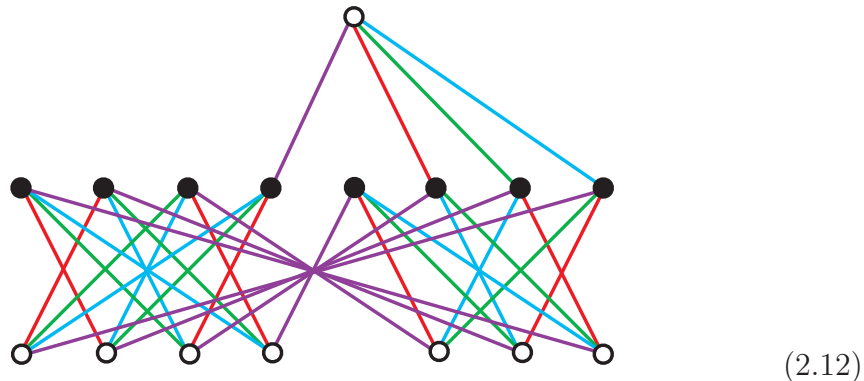
$$(2.12)$$

This Adinkra describes a supermultiplet which is distinct from the Clifford
algebra superfield, as evidenced by the fact that three different height assign-
ments are represented.

If we raise *en masse* a collection of vertices which have been coalesced into
an SO(N) p-form, then the resulting Adinkra respects the SO(N) structure
in the sense that the components of each SO(N) tensor continue to share a
common engineering dimension. For instance, if we start with the $N = 4$
Base Adinkra as shown in (2.3), we can raise the multiplicity-six compound

two-form vertex to obtain

$$(2.13)$$
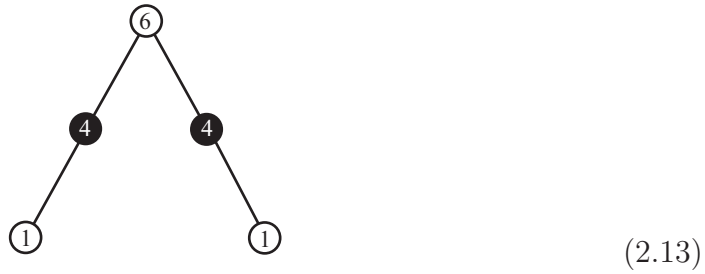
As another possibility, we could start again with the Base Adinkra (2.3) and raise the singlet four-form vertex, to obtain

$$(2.14)$$
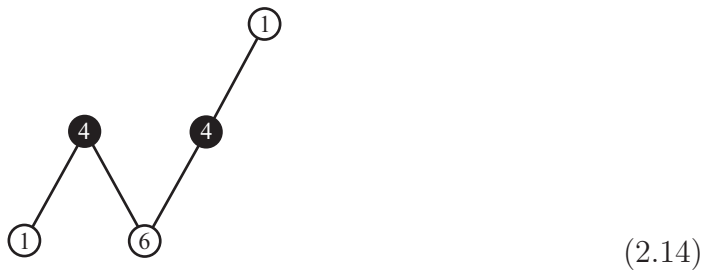
There are many other possibilities. A subset of the possible vertex raising operations maintains the height-equivalence of all components of each $SO(N)$ tensor, while the complementary set breaks this height-equivalence feature. As an example of a raising operation in the latter class, we could start with (2.3), and then raise the multiplicity-one four-form vertex and also one of the vertices out of the six in the multiplicity-six compound vertex, as follows:

$$(2.15)$$
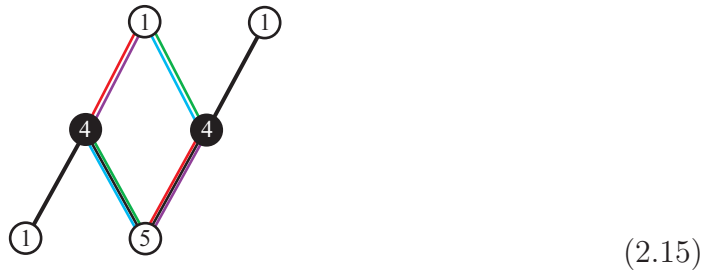
In this Adinkra, the black edges correspond, as above, to a bundling of all four supersymmetries. However, the bicolored edges indicate vertex interrelationships involving only two of the four supersymmetries. The edges with combined black and colored edges describe a bundling of all four supersymmetries, but some of the implicit connectivity, namely that associated with the bicolored edges, is missing from this bundling. Here we see that the six components of the bosonic two-form do not share a common height assignment. Thus, the two-form does not have a collectively unambiguous engineering dimension. In

this case we say that the supermultiplet has a *skew R*-charge, as opposed to a conventional *R*-charge.

Recall that the discussion of *p*-forms in Adinkras depends on choosing a starting vertex to be the 0-form, and note that the skewness of the *R*-charge may depend on this choice of 0-form vertex. Indeed, the *R*-symmetries act on the supersymmetry generators, not the supermultiplet, unless we fix a choice of a vertex. We will therefore define a supermultiplet or Adinkra to have a conventional *R*-charge if there exists some choice of 0-form vertex so that the vertices for each *p*-form all have the same engineering dimension, and a supermultiplet is said to be skew otherwise.

In the case of an Adinkra with a conventional *R*-charge, the generator **d** introduced in Subsection 2.1 commutes with the generator of the $SO(N)$ *R*-charge. For any Adinkra with a skew *R*-charge, these generators do not commute.

Also note that the presence of a skew *R*-charge does not preclude the existence of an invariant functional, built using the components of such a supermultiplet, which is both supersymmetric and $SO(N)$-invariant. Multiplets having a conventional *R*-charge form a class which is distinct from those in which the $SO(N)$-structure is skew, and may prove interesting to model building. However, in the balance of this paper we consider only supermultiplets having conventional *R*-charge.

3   Top Adinkras and Salam-Strathdee Superfields

The Base Adinkra and its kin described in the previous section comprise an extreme class of supermultiplets in the sense that the component fields span a minimal number of distinct engineering dimensions, namely two. Another extreme class of supermultiplets are those described by a connected Adinkra involving 2^N total component fields (vertices) spanning a *maximal* number of distinct height assignments. This class of supermultiplets can be obtained from the generalized Base Adinkras by raising vertices until the chains depicted in Figure 1 are fully extended rather than maximally compressed. For instance, if we start with the Base Adinkra (2.4), we can lift vertices while maintaining

R-charge until we obtain the following Adinkra:

$$\begin{array}{c} \bigcirc \ \binom{N}{N} \\ \\ \bullet \ \binom{N}{N\text{-}1} \\ \vdots \\ \bigcirc \ \binom{N}{2} \\ \\ \bullet \ \binom{N}{1} \\ \\ \bigcirc \ \binom{N}{0} \end{array} \tag{3.1}$$

This is the unique fully-extended Adinkra having a zero-form boson as its lowest component, and is called the Top Adinkra. This supermultiplet spans $N + 1$ different height assignments, and corresponds directly with the scalar Salam-Strathdee superfield, $\Phi(\tau, \theta^I)$, where θ^I are the fermionic superspace coordinates. The lowest vertex in (3.1) corresponds to the lowest component of the superfield, $\Phi|$, *i.e.*, that component which survives projection to the $\theta^I \to 0$ submanifold of the superspace, sometimes called the *body* of the superfield. The next highest vertices in (3.1) correspond to the body of the derivative superfield, $D_I \Phi|$. In general, the *p*-form vertices in a Top Adinkra are proportional to $D_{[I_1} \cdots D_{I_p]} \Phi|$.[9] The component transformation rules associated with the generators of supersymmetry transformations on superspace are identical with those codified by the Adinkra, as spelled out in [1,2,3].

Starting with the Top Adinkra (3.1), one can raise the lowermost vertex to

---

[9]A superspace derivative is defined as $D_I := \partial_I + i\,\theta_I\,\partial_\tau$, whereby $\{D_I, D_J\} = 2\,i\,\partial_\tau$. It follows that the product of two superspace derivatives can be decomposed as $D_I\,D_J = D_{[I}\,D_{J]} + i\,\delta_{IJ}\,\partial_\tau$. Because of this, a complete set of differential operators on superspace is generated by the antisymmetric operator products $D_{[I_1} \cdots D_{I_n]}$, and by the time derivatives ∂_τ^p following the action of these.

obtain the following distinct Adinkra:

$$\text{(3.2)}$$

This new Adinkra corresponds to the superderivative superfield $D_I\Phi$, modulo the zero-mode of the singlet field labeled "$\binom{N}{0}$". In this way, the superspace derivative operation is mirrored on the Top Adinkra by a vertex raising. To be more precise, the singlet vertex "$\binom{N}{0}$" in the Adinkra (3.2) describes the τ derivative of the corresponding singlet field in Φ, its lowest component. This begs an interesting and relevant question: Does there exist a superspace description of the supermultiplet described by (3.2) in which the Adinkra vertices correlate one-to-one, without derivatives, to the components of some unconstrained superfields? As it turns out, such a construction does exist, is related to the result of Theorem 7.6 of Ref. [1], and we describe it in detail below. This construction requires not one, but two unconstrained superfields, Φ_1 and Φ_2, called "prepotentials" and which are associated to the two local maxima (sinks) of the Adinkra (3.2): those labeled "$\binom{N}{N}$" and "$\binom{N}{0}$". A particular linear combination of superspace derivatives of these contains precisely the supermultiplet described by (3.2). This linear combination comprises a superfield subject to a constraint. There are twice as many component fields collectively described by Φ_1 and Φ_2 as there are described by (3.2). The excess component fields correspond to *gauge* degrees of freedom and do not appear in (3.2).

Next, starting with (3.2) we can raise the one-form vertices to obtain:

$$\begin{matrix}
\binom{N}{N}\,\bigcirc \\
\vdots \\
\binom{N}{4}\,\bigcirc \\
\binom{N}{3}\,\bullet \qquad \bullet\,\binom{N}{1} \\
\binom{N}{2}\,\bigcirc \qquad \bigcirc\,\binom{N}{0}
\end{matrix} \tag{3.3}$$
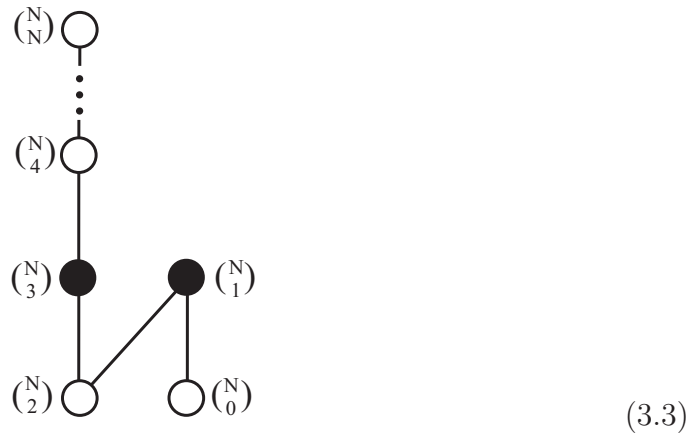
This Adinkra does not correspond directly to a single unconstrained super-field. What we mean by this is that the particular component transformation rules encoded by (3.3) do not coincide with the transformation rules associated with the components of any particular unconstrained superfield for which the superfield components and the Adinkra vertices are in one-to-one correspondence. This does not mean that the supermultiplet described by (3.3) does not have a superspace interpretation. Indeed, the source-sink reversal of Theorem 7.6 of Ref. [1] guarantees that a corresponding superfield exists, and provides a general algorithm for its determination. In this case, for example, the supermultiplet in question can be described by an unconstrained scalar superfield along with a set of N unconstrained fermionic superfields, respectively corresponding to the vertex labeled "$\binom{N}{N}$" and the multiple vertex labeled "$\binom{N}{1}$". These $N+1$ total unconstrained prepotential superfields once again correspond to the $N+1$ local maxima (sinks) in the Adinkra (3.3), and they involve a total of $(N+1) \cdot 2^N$ total components, significantly more than the 2^N vertices appearing in (3.3). The extra degrees of freedom appearing in the superspace description are associated with gauge degrees of freedom; there exist linear combinations of the prepotential superfields and derivatives thereof which involve only those degrees of freedom corresponding to the Adinkra vertices in (3.3). The superfield corresponding to such a linear combination is subject to constraints. It has been a historically interesting question to attempt to classify the possible realizable superfield constraints which give rise to irreducible supermultiplets. The paradigm we are espousing speaks pointedly to this endeavor. We expand on these ideas, and include relevant algebraic details, later in this paper.

Starting with (3.3), one can raise the singlet vertex labeled "$\binom{N}{0}$" to obtain:

$$(3.4)$$
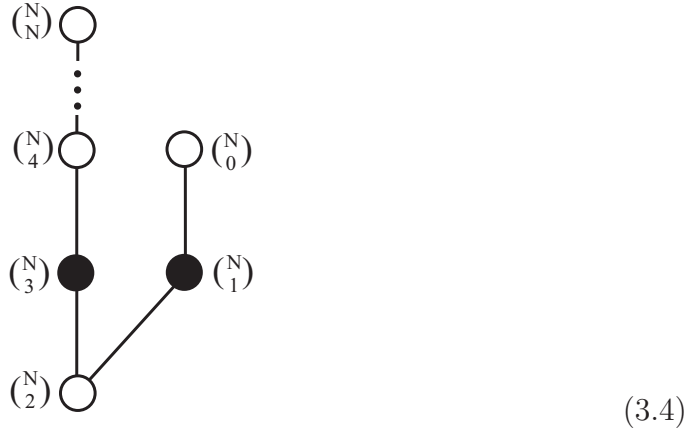
This Adinkra corresponds to the superderivative superfield $D_{[I}D_{J]}\Phi$ modulo several modes associated with the lifted vertices. More precisely, the one-form vertex in (3.4) describes the τ derivative of the corresponding one-form fermion in Φ and the singlet vertex in (3.4) describes the second derivative ∂_τ^2 of the corresponding singlet boson in Φ, its lowest component.

   The Adinkras (3.2) and (3.4) share a feature not exhibited by (3.3) or by the majority of Adinkras: namely, these Adinkras have exactly two sinks and exactly one compound source, meaning that the source vertices combine into a particular $SO(N)$ p-form. These Adinkras are obtained from the Top Adinkra by lifting its lowest vertex upward, dragging other vertices behind, as if one were raising a chain. Adinkras with this feature correspond to the antisymmetric product of superderivatives acting on an unconstrained superfield Φ, which itself corresponds to the Top Adinkra. This concept is illustrated by Figure 2 in the particular case of $N = 4$ supersymmetry. The dots which appear on some vertices in Figure 2 indicate the relationship between these vertices and the vertices in the leftmost, Top, Adinkra. For example, the topmost vertex in the rightmost Adinkra, the one corresponding to $D_{[I}D_J D_K D_{L]}\Phi$, has a blue numeral and has four dots. These dots indicate that this vertex describes the fourth derivative ∂_τ^4 of the field corresponding to the lowermost vertex in the Top Adinkra. (This is the unique vertex in the Top Adinkra having a blue numeral.) Notice that the top derivative $D^N = \frac{1}{4!}\,\varepsilon^{I_1 \cdots I_N}\,D_{I_1}\cdots D_{I_N}$ completely swivels the Top Adinkra about its hook, so that its source becomes a sink, albeit differentiated, and its sink becomes a source.

   The Top Adinkra, which corresponds to an unconstrained superfield Φ, and its elemental derivatives,

$$\Xi_p^q = D_{[I_1}\cdots D_{I_p]}\,\partial_\tau^q\,\Phi\,, \tag{3.5}$$

describe building blocks from which more general superfields may be constructed by forming linear combinations. The question of which superfield
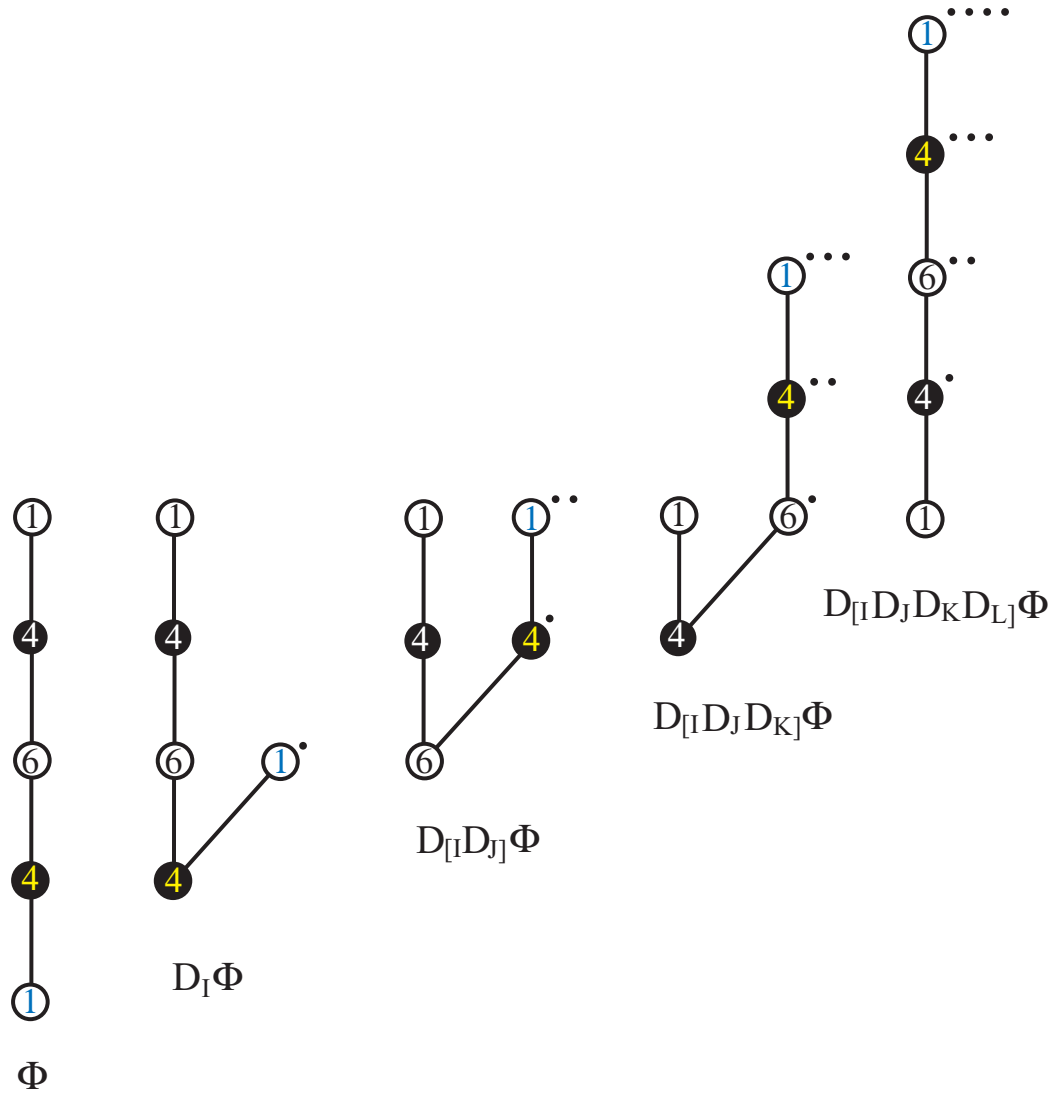
Figure 2: The $N = 4$ Top Adinkra, corresponding to the unconstrained super-field Φ, and a sequence of related Adinkras obtained as antisymmetric products of superspace derivatives acting on Φ. We have drawn distinctions between the two multiplicity-four fermion vertices and the two singlet vertices by using different coloring on the multiplicity labels appearing on these vertices.

constraints correspond to which irreducible supermultiplets can be re-phrased as a question of which linear combinations of the basic building blocks Ξ_p^q correspond to the irreducible supermultiplets.

Note that the basic building blocks can be visualized in terms of sets, such as those pictured in Figure 2 plus versions of such diagrams raised by global differentiation, by which we mean similar diagrams obtained by adding a common number q of τ derivatives to each vertex. Figure 2 enumerates the set $\{\Xi_p^0\}$ in the case $N = 4$, for the cases $p = 0, 1, 2, 3, 4$. Additional diagrams $\Xi_p^{q \neq 0}$ are obtained by differentiating all vertices q times. Each τ derivative lifts the entire diagram by one engineering dimension, which corresponds to two height units since the height is twice the engineering dimension. For example, the relationship between the diagrams $\Xi_1^0 = D_I \Phi$ and $\Xi_1^1 = D_I \dot{\Phi}$ is seen as follows:

$$\Xi_1^0 \qquad \Xi_1^1 \tag{3.6}$$
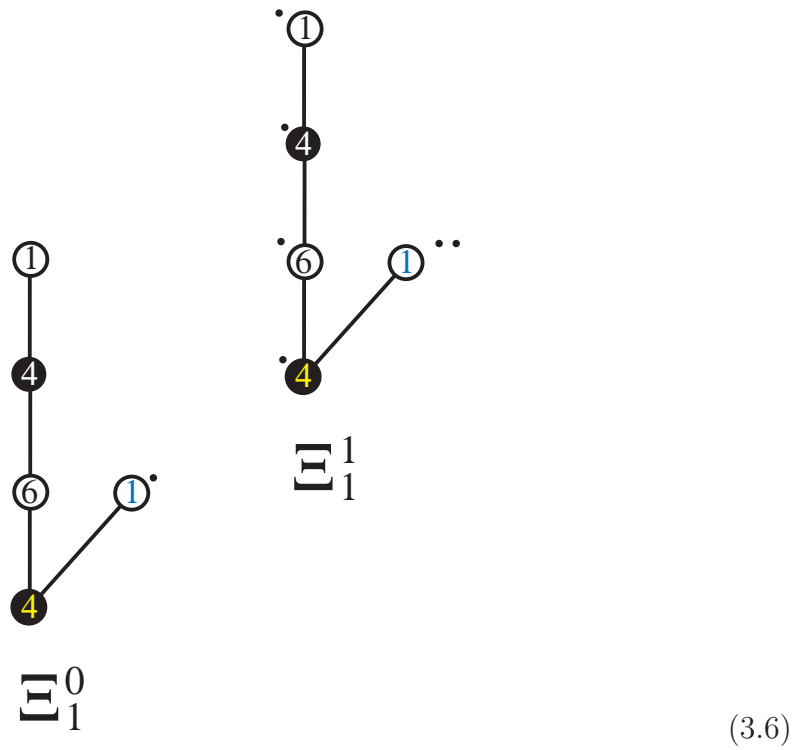
The fact that the vertices of Ξ_1^1 have two height units greater than their counterparts in Ξ_1^0 is manifested by the raised placement of the second diagram relative to the first.

4 Garden Algebras, and Clifford Algebra Superfields

Adinkra diagrams provide a concise and elegant way to represent supermultiplets. This is loosely analogous to the way Feynman diagrams represent integrals appearing in field theory calculations. But Adinkra diagrams have their own magic; these can be manipulated in a way which mirrors various algebraic tasks associated with superfields or associated with component field calculations. Some of these have been described above. The core algebraic underpinning of Adinkras lies, however, in the realm of $\mathcal{GR}(\mathrm{d}, N)$ algebras, introduced in [5] and [6], which have emerged as vitally important for supersymmetry representation theory. In this section we review these algebras and their relevance to one-dimensional supersymmetry. We use these to describe the algebraic counterpart to the pictorial presentation in Section 2.

A supersymmetry transformation $\delta_Q(\epsilon)$ is parameterized by ϵ^I, where $I = 1, ..., N$ is an $\mathrm{SO}(N)$ vector index. Two supersymmetry transformations commute into a time translation according to

$$[\, \delta_Q(\epsilon_1)\, ,\, \delta_Q(\epsilon_2)\,] = -2\, i\, \epsilon_1^I\, \epsilon_2^I\, \partial_\tau\, . \tag{4.1}$$

The representations of (4.1) can be classified using the so-called "garden algebra" $\mathcal{GR}(\mathrm{d}, N)$, generated by two sets $(\, L_I\,)_i{}^{\hat{j}}$ and $(\, R_I\,)_{\hat{i}}{}^j$ of N $d \times d$ matrices, known as "garden matrices", subject to the relations

$$(\, L_I\, R_J + L_J\, R_I\,)_i{}^j = -2\, \delta_{IJ}\, \delta_i{}^j$$

$$(\, R_I\, L_J + R_J\, L_I\,)_{\hat{i}}{}^{\hat{j}} = -2\, \delta_{IJ}\, \delta_{\hat{i}}{}^{\hat{j}}$$

$$L_I = -R_I^T\, . \tag{4.2}$$

We note that garden algebras are closely related to Clifford algebras. Indeed, a choice of garden matrices generating $\mathcal{GR}(\mathrm{d}, N)$ contains the same mathematical information as a representation of the Clifford algebra $\mathrm{Cl}(N)$ on a $d + d$-dimensional super vector space, with the Clifford generators acting by odd, skew-adjoint operators.

For fixed values of N, there are multiple values of d for which these matrices exist. But there is one d_N that is the least integer for which $d \times d$ garden matrices exist. The value of this integer for every N is tabulated in [3]. There are different sorts of indices adorning $(\, L_I\,)_i{}^{\hat{j}}$ and $(\, R_I\,)_{\hat{i}}{}^j$. The non-hatted indices $i, j, ...$ span a vector space $\mathcal{V}_L \cong \mathbb{R}^d$ while the hatted indices $\hat{i}, \hat{j}, ...$ span another vector space $\mathcal{V}_R \cong \mathbb{R}^d$. These indices adorn the matrices L_I and R_I, the first index labels the row (thus, the range) and the second index labels the column (thus, the domain). Thus, the L_I describe linear maps from $\mathcal{V}_R$ to $\mathcal{V}_L$, and the R_I describe linear maps from $\mathcal{V}_L$ to $\mathcal{V}_R$. The compositions of these

will then be maps on $\mathcal{V}_L \oplus \mathcal{V}_R$ that can be furthermore classified according to their domain and range as follows:

$$\{\,\mathcal{M}_L\,\} : \mathcal{V}_R \to \mathcal{V}_L \qquad \{\,\mathcal{U}_L\,\} : \mathcal{V}_L \to \mathcal{V}_L$$

$$\{\,\mathcal{M}_R\,\} : \mathcal{V}_L \to \mathcal{V}_R \qquad \{\,\mathcal{U}_R\,\} : \mathcal{V}_R \to \mathcal{V}_R\,. \tag{4.3}$$

This formalism produces a visualization of these concepts in a coordinate-independent manner. We use Venn diagrams to represent the sets $\mathcal{V}_L$ and $\mathcal{V}_R$. The set of linear operators that act between and on these sets may be represented by a set of directed arrows as shown in the Placement-putting Graph [7], shown in Figure 3

In this paper the action of a matrix is defined in terms of left multiplication. Thus, $(\,L_I\,)_i{}^{\hat{j}} \in \mathcal{M}_L$ and $(\,R_I\,)_{\hat{i}}{}^j \in \mathcal{M}_R$. The "normal part of the enveloping algebra", denoted $\wedge\mathcal{EGR}(\mathrm{d}, N)$, is generated by the wedge products involving L_I and R_I,

$$(\,f_I\,)_i{}^{\hat{j}} = (\,L_I\,)_i{}^{\hat{j}} \qquad\qquad (\,\tilde{f}_I\,)_{\hat{i}}{}^j = (\,R_I\,)_{\hat{i}}{}^j$$

$$(\,f_{IJ}\,)_i{}^j = (\,L_{[I}\,R_{J]}\,)_i{}^j \qquad\qquad (\,\tilde{f}_{IJ}\,)_{\hat{i}}{}^{\hat{j}} = (\,R_{[I}\,L_{J]}\,)_{\hat{i}}{}^{\hat{j}}$$

$$(\,f_{IJK}\,)_i{}^{\hat{j}} = (\,L_{[I}\,R_J\,L_{K]}\,)_i{}^{\hat{j}} \qquad (\,\tilde{f}_{IJK}\,)_{\hat{i}}{}^j = (\,R_{[I}\,L_J\,R_{K]}\,)_{\hat{i}}{}^j\,, \tag{4.4}$$

and so forth. The subsets of the vector spaces $\mathcal{U}_{L,R}$ and $\mathcal{M}_{L,R}$, defined in (4.3), which are also in $\wedge\mathcal{EGR}(\mathrm{d}, N)$, are called, respectively, $\mathcal{U}_{L,R}^{(n)}$ and $\mathcal{M}_{L,R}^{(n)}$, where the superscript (n) indicates the "normal part".

4.1 The Placement-Putting of Adinkra Nodes

The component fields of one-dimensional supermultiplets naturally admit a structural organization associated with the vector spaces described above. For example, the components of a Clifford algebra superfield, or equivalently the vertices of a Base Adinkra, are valued in subsets of $\wedge\mathcal{EGR}(\mathrm{d}, N)$, such as $\mathcal{U}_L^{(n)}$ and $\mathcal{M}_R^{(n)}$, whereas the components of Scalar supermultiplets, or equivalently the vertices of a Scalar Adinkra, are valued in $\mathcal{V}_R$ and $\mathcal{V}_L$. In fact, the garden algebras represent special cases of real Clifford algebras, such that component fields, or equivalently collections of Adinkra vertices, that are valued in $\mathcal{V}_L$ or $\mathcal{V}_R$ transform as some spinor representation(s) of an associated SO(N) symmetry. Similarly, fields or vertices valued in $\wedge\mathcal{EGR}(\mathrm{d}, N)$ transform as some p-form representation(s) of an SO(N) symmetry. Thus, the vertices of Adinkras span representations whose generators act on the vertices by multiplication with appropriate matrices. We refer to the distinct SO(N) representations carried by vertices, as their "R-charge".
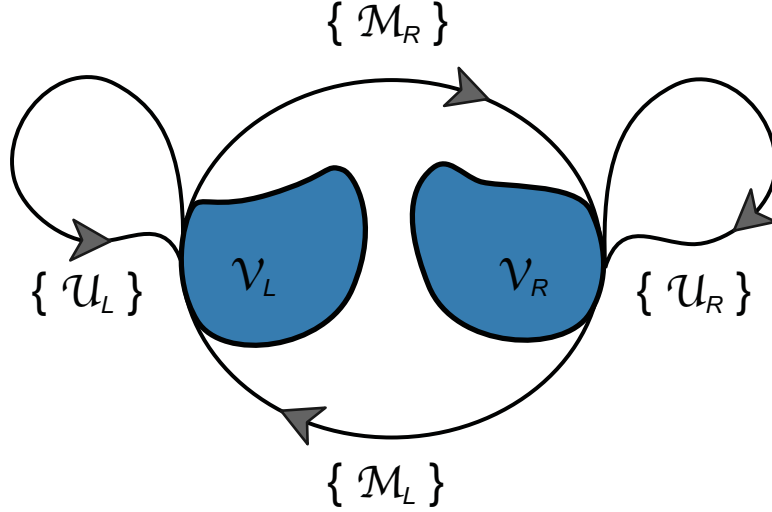
Figure 3: The PpG diagram.

4.2 The Clifford Algebra Superfield

A fundamental representation of the N-extended superalgebra is given by the Clifford algebra superfield. This involves 2^{N-1} bosons $\Phi_i{}^j \in \mathcal{U}_L^{(n)}$ and 2^{N-1} fermions $\Psi_{\hat{i}}{}^j \in \mathcal{M}_R^{(n)}$, subject to the following transformation rules:

$$\delta\,\Phi_i{}^j = -i\,\epsilon^I\,(\,L_I\,)_i{}^{\hat{k}}\,\Psi_{\hat{k}}{}^j$$

$$\delta\,\Psi_{\hat{i}}{}^j = \epsilon^I\,(\,R_I\,)_{\hat{i}}{}^k\,\partial_\tau\Phi_k{}^j\,. \tag{4.5}$$

If we make a particular choice for the garden matrices $(\,L_I\,)_i{}^{\hat{j}}$ and $(\,R_I\,)_{\hat{i}}{}^j$, then it is straightforward to translate the transformation rules (4.5) into the equivalent Adinkra, using the techniques developed in [1] and [3]. The resulting Adinkra is shown in (2.4). We can expand the component fields in (4.5) using the bases (4.4) as follows:

$$\Phi_i{}^j = \sum_{p=0}^{\lfloor N/2 \rfloor} (\,f^{I_1\cdots I_{2\,p}}\,)_i{}^j\,\phi_{I_1\cdots I_{2\,p}}$$

$$\Psi_{\hat{i}}{}^j = \sum_{p=1}^{\lfloor N/2 \rfloor} (\,\tilde{f}^{I_1\cdots I_{2\,p-1}}\,)_{\hat{i}}{}^j\,\psi_{I_1\cdots I_{2\,p-1}}\,, \tag{4.6}$$

where $\lfloor\,\cdot\,\rfloor$ selects the integer part of its argument. In this way we can replace the matrix fields $\Phi_i{}^j$ and $\Psi_{\hat{i}}{}^j$ with p-forms on SO(N). The transformation

rules (4.5) imply the following corresponding rules for the p-form fields:

$$\delta \, \phi^{[p_{\text{even}}]} \; = \; -i \, \epsilon^{[I_1} \, \psi^{I_2 \cdots I_p]} + (p+1) \, i \, \epsilon_J \, \psi^{I_1 \cdots I_p \, J}$$

$$\delta \, \psi^{[p_{\text{odd}}]} \; = \; -\epsilon^{[I_1} \, \dot{\phi}^{I_2 \cdots I_p]} + (p+1) \, \epsilon_J \, \dot{\phi}^{I_1 \cdots I_p \, J} \,. \tag{4.7}$$

Thus, the Clifford algebra superfield involves 2^{N-1} bosons which assemble as even-forms on SO(N) and 2^{N-1} fermions which assemble as odd-forms on SO(N). Equation (4.7) is equivalent to equation (4.5). The equivalence may be proved using algebraic identities satisfied by the garden matrices, which follow as corollaries of the garden algebra. The Adinkra counterpart to (4.7) is (2.4).

Notice that $\Phi_i{}^j \in \mathcal{U}_L^{(n)}$, which is spanned by even-forms of the sort defined in (4.4), and $\Psi_i{}^j \in \mathcal{M}_R$, which is spanned by odd-forms. It is for this reason that the Clifford algebra superfield corresponds to the Base Adinkra, shown in Figure 1, rather than to the Dual Base Adinkra. The latter construction involves even-form fermions rather than even-form bosons. The algebraic counterpart to the Dual Base Adinkra is described in the following subsection.

4.3 The Dual Clifford Superfield

An alternative way to formulate a Clifford algebra superfield involves 2^{N-1} bosons $\Phi_{\hat{\imath}}{}^j \in \mathcal{M}_R^{(n)}$ and 2^{N-1} fermions $\Psi_i{}^j \in \mathcal{U}_L^{(n)}$, subject to the following transformation rules:

$$\delta \, \Phi_{\hat{\imath}}{}^j \; = \; -i \, \epsilon^I \, (\, R_I \,)_{\hat{\imath}}{}^k \, \Psi_k{}^j$$

$$\delta \, \Psi_i{}^j \; = \; \epsilon^I \, (\, L_I \,)_i{}^{\hat{k}} \, \partial_\tau \Phi_{\hat{k}}{}^j \,. \tag{4.8}$$

The difference between (4.8) and (4.5) lies in the way the component degrees of freedom are embedded in $\wedge \mathcal{EGR}(\mathrm{d}, N)$, as reflected by the placement of the hatted and unhatted indices on the component fields themselves and on whether it is the matrix $(\, L_I \,)_i{}^{\hat{\jmath}}$ or $(\, R_I \,)_{\hat{\imath}}{}^j$ which appears in the transformation rule for the bosons or fermions.

We can expand the component fields in (4.8) using the bases (4.4) as follows:

$$\Phi_{\hat{\imath}}{}^j \; = \; \sum_{p=1}^{\lfloor N/2 \rfloor} (\, \tilde{f}^{I_1 \cdots I_{2\,p-1}} \,)_{\hat{\imath}}{}^j \, \phi_{I_1 \cdots I_{2\,p-1}}$$

$$\Psi_i{}^j \; = \; \sum_{p=0}^{\lfloor N/2 \rfloor} (\, f^{I_1 \cdots I_{2\,p}} \,)_i{}^j \, \psi_{I_1 \cdots I_{2\,p}} \,, \tag{4.9}$$

where $\lfloor \cdot \rfloor$ selects the integer part of its argument. In this way we can replace the matrix fields $\Phi_i{}^j$ and $\Psi_{\hat{\imath}}{}^j$ with p-forms on SO(N). The transformation

rules (4.8) imply the following corresponding rules for the p-form fields:

$$\delta\,\phi^{[p_{\mathrm{odd}}]} = -i\,\epsilon^{[I_1}\,\psi^{I_2\cdots I_p]} + (p+1)\,i\,\epsilon_J\,\psi^{I_1\cdots I_p\,J}$$

$$\delta\,\psi^{[p_{\mathrm{even}}]} = -\epsilon^{[I_1}\,\dot\phi^{I_2\cdots I_p]} + (p+1)\,\epsilon_J\,\dot\phi^{I_1\cdots I_p\,J}\,. \tag{4.10}$$

Thus, the Dual Clifford algebra superfield involves 2^{N-1} bosons which assemble as odd-forms on SO(N) and 2^{N-1} fermions which assemble as even-forms on SO(N). Equation (4.10) is equivalent to equation (4.8). The Adinkra counterpart to (4.10) is shown in (2.8).

4.4   The Conjugate Clifford Superfield, and its dual

Two additional fundamental supermultiplets are the Klein flipped versions of the Clifford algebra superfield and its dual, described in Subsections 4.2 and 4.3, respectively.

The Conjugate Clifford superfield involves 2^{N-1} fermions $\tilde\Psi^{\hat i}{}_j \in \mathcal{M}_R^{(n)}$ and 2^{N-1} bosons $\tilde\Phi^i{}_j \in \mathcal{U}_L^{(n)}$, subject to the following transformation rules:

$$\delta\,\Psi^{\hat i}{}_j = \epsilon^I\,(\,R_I\,)^{\hat i}{}_k\,\Phi^k{}_j$$

$$\delta\,\Phi^i{}_j = -i\,\epsilon^I\,(\,L_I\,)^i{}_{\hat k}\,\partial_\tau\,\Psi^{\hat k}{}_j\,. \tag{4.11}$$

In this case the fermions describe the lower components and decompose as odd-forms on SO(N), while the bosons describe the higher components and decompose as even-forms on SO(N). This supermultiplet is represented by the Conjugate Base Adinkra, which is shown in (2.9).

The Dual Conjugate Clifford superfield involves 2^{N-1} fermions $\tilde\Psi^i{}_j \in \mathcal{U}_L^{(n)}$ and 2^{N-1} bosons $\tilde\Phi^{\hat i}{}_j \in \mathcal{M}_R^{(n)}$, subject to the following transformation rules:

$$\delta\,\Psi^i{}_j = \epsilon^I\,(\,L_I\,)^i{}_k\,\Phi^k{}_j$$

$$\delta\,\Phi^{\hat i}{}_j = -i\,\epsilon^I\,(\,R_I\,)^{\hat i}{}_{\hat k}\,\partial_\tau\,\Psi^{\hat k}{}_j\,. \tag{4.12}$$

In this case the fermions describe the lower components and decompose as even-forms on SO(N), while the bosons describe the higher components and decompose as odd-forms on SO(N). This supermultiplet is represented by the Dual Conjugate Base Adinkra, which is shown in (2.10).

5   Scalar Multiplets

Generalized Clifford algebra superfields, which correspond to generalized Base Adinkras, are reducible for $N \geq 4$. Similarly, unconstrained Salam-Strathdee

superfields, which correspond to Top Adinkras, are also reducible for $N \geq 4$. On the other hand, the Scalar supermultiplets are irreducible for all N. A Scalar supermultiplet involves d bosonic fields $\phi_i \in \mathcal{V}_L \cong \mathbb{R}^d$ and d fermionic fields $\psi_{\hat{\imath}} \in \mathcal{V}_R \cong \mathbb{R}^d$, where $d = d_N$ is the minimum value for which $d \times d$ garden matrices exist. The supersymmetry transformation rules are given by

$$\delta_Q(\epsilon)\, \phi_i = -i\, \epsilon^I\, (\, L_I\,)_i{}^{\hat{\jmath}}\, \psi_{\hat{\jmath}}$$

$$\delta_Q(\epsilon)\, \psi_{\hat{\imath}} = \epsilon^I\, (\, R_I\,)_{\hat{\imath}}{}^{j}\, \partial_\tau\, \phi_j \,. \tag{5.1}$$

A Scalar supermultiplet is represented by an Adinkra having d fermionic vertices all at height zero and d bosonic vertices all at height minus one.[10] For example, in the case $N = 4$, we have $d_N = 4$, and the Scalar Adinkra is the following complete bipartite graph with $4 + 4$ vertices:
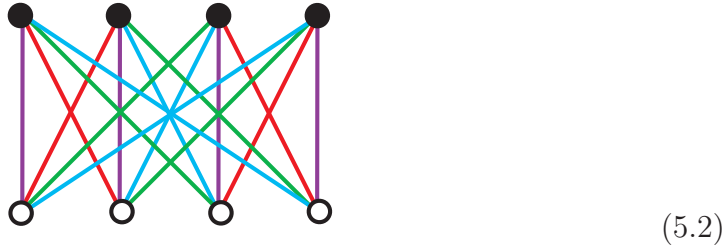
$$\tag{5.2}$$

We now wish to examine the following question: Does there exist a superspace description of a general-N Scalar Adinkra for which the vertices correlate one-to-one with the components of some unconstrained *prepotential* Salam-Strathdee superfields? (As we saw in Subsection 2.4, the Scalar supermultiplet can be constructed from the Clifford algebra superfield via a projection determined by the symmetries of the Base Adinkra. However, this is not immediately helpful, as we have not yet given a superspace description of the general-N Clifford algebra superfield in terms of Salam-Strathdee superfields.) In addition, we ask whether we can construct a supersymmetric action functional in terms of a superspace integral built from these prepotentials for which the propagating fields correspond precisely to the Scalar supermultiplet? It should be kept in mind that the definitions of superfields we have been using so far in this discussion are *totally* independent of the Salam-Strathdee superfield formalism. So the answer has not been presumed in our discussion to this point.

---

[10]By making this choice, we are normalizing the height of the Scalar supermultiplet by choosing canonical dimensions for its component fields: A propagating scalar field ϕ has a canonical kinetic action given by $S_\phi = \int d\tau\, \frac{1}{2}\, \dot{\phi}^2$; since the engineering dimension of τ is minus one, it follows that S is dimensionless only if ϕ has dimension minus one-half. Since the height parameter is twice the engineering dimension, it follows that the canonical height of a one-dimensional scalar is minus one. Similar reasoning may be applied to the canonical fermion action $S_\psi = \int d\tau \psi\dot{\psi}$ to conclude that canonical propagating fermions have zero height.

The answer to this question has been known for some years. It was explicitly stated, for example in a 1982 work by Gates and Siegel [12]: "Conversely, the highest-dimension component field appearing in an action is the highest θ-component of a superfield appearing in this action." As it turns out, a given Adinkra can be described using one prepotential superfield for *each* of its hooks, *i.e.*, its sink vertices. The statistics and the $\mathcal{GR}(\mathrm{d}, N)$ or $\mathrm{SO}(N)$ structure of these prepotentials are dictated by statistics and the $\mathcal{GR}(\mathrm{d}, N)$ or $\mathrm{SO}(N)$ structures of the sink vertices on the target Adinkra, *i.e.*, the Adinkra we wish to describe using the prepotentials. Since the prepotentials are unconstrained, these correspond to Top Adinkras, each of which has exactly one hook. The statistics for each prepotential are chosen such that the hook of the corresponding Top Adinkra correlates with one hook of the target Adinkra. A fermionic hook therefore corresponds to a bosonic prepotential in cases where N is odd and to a fermionic prepotential in cases where N is even. Similarly, a bosonic hook corresponds to a fermionic prepotential in cases where N is odd and to a bosonic prepotential in cases where N is even. These conclusions follow because the statistics of a Salam-Strathdee superfield coincide with the statistics of the source vertex (the lowest vertex) on the associated Top Adinkra and because a Top Adinkra spans $N+1$ different height assignments. Therefore the statistics of the Top Adinkra source vertex coincides with the statistics of its hook in cases where N is even and differs from the statistics of its hook in cases where N is odd.

According to this claim, the $N = 4$ Scalar supermultiplet requires four real fermionic superfields, $\mathcal{F}_{\hat{i}}$, as prepotentials, where the index $\hat{i}$ spans $\mathcal{V}_R \cong \mathbb{R}^4$. This is determined by (5.2), where we see that the $N = 4$ Scalar Adinkra has four real fermionic scalar hooks; these span $\mathcal{V}_R$ since the four fermion fields corresponding to these vertices are $\psi_{\hat{i}} \in \mathcal{V}_R$. For the case of general-N Scalar supermultiplets, similar reasoning implies that d real superfields $\mathcal{S}_{\hat{i}}$ should suffice. We prove below, in Section 7, that such a prepotential construction does properly describe any Scalar supermultiplet, and we also show how these unconstrained superfields can be used to build supersymmetric action functionals. The reader might wonder because d unconstrained real fermionic scalar superfields involve a total of $d \cdot 2^{N-1} + d \cdot 2^{N-1}$ component fields whereas a Scalar supermultiplet has only $2^{N-1} + 2^{N-1}$ component fields, an apparent mismatch. The resolution is that the excess correspond to gauge degrees of freedom; there is a particular linear combination of the building blocks $D_{[I_1} \cdots D_{I_p]} \partial_\tau^q \mathcal{S}_{\hat{i}}$, where $\mathcal{S}_{\hat{i}}$ are the prepotentials, which describes precisely the degrees of freedom in the Scalar supermultiplet.

The fact that this works in the general case might be surprising to some readers. We think it is helpful, therefore, to describe in detail the simplest case—the case $N = 2$—in order to illustrate clearly how the gauge structure appears in the prepotentials. (In the case $N = 1$ the unconstrained superfield

and the Scalar supermultiplet are identical.) Accordingly, the following section focuses on the case $N = 2$. The general case is described in Section 7.

Our basic strategy, which ultimately gives rise to the solution described above, is predicated on the following thoughts: Scalar supermultiplets span two height assignments, whereas Top Adinkras span $N+1$ height assignments. Thus, if we wish to describe a Scalar supermultiplet using a prepotential or a set of prepotentials, we will have to operate on these. We explained at length above how superspace differential operators lift vertices and can reduce the span of an Adinkra. A complete set of differential operators on superspace are given by $D_{[I_1} \cdots D_{I_p]} \partial_\tau^q$, where p and q are integers constrained by $0 \leq p \leq N$ and $q \geq 0$. Our strategy will be to consider linear combinations

$$\Gamma(\{a\})_{m;I_1\cdots I_r} := \sum_{n=1}^{\lambda} \sum_{s=0}^{\infty} \sum_{p=0}^{N} (\, a_s^p \,)_{m;I_1\cdots I_r}^{n;J_1\cdots J_p} D_{[J_1} \cdots D_{J_p]} \partial_\tau^s \mathcal{S}_n \,, \qquad (5.3)$$

where $\mathcal{S}_n$ are a set of $\lambda \in \mathbb{N}$ unconstrained prepotential superfields and the complex tensor coefficients $(\, a_s^p \,)_{m;I_1\cdots I_r}^{n;J_1\cdots J_p}$ remain to be determined,[11] and where the indices m and n take values in a vector space $\mathcal{W}$, for which $\dim \mathcal{W} = \lambda$, to be determined. (For the Scalar supermultiplets, it turns out that $\mathcal{W} = \mathcal{V}_R$ and $\lambda = d$.) The statistics of the prepotentials may vary. In other words, some number b of the $\mathcal{S}_n$ could be bosonic while the remaining $\lambda - b$ prepotentials would be fermionic. (Clearly, $0 \leq b \leq \lambda$.)

Our tasks are: 1) to determine whether a set of prepotentials (specified by a choice of λ and a choice of b) and a set of complex coefficients $(\, a_s^p \,)_{m;I_1\cdots I_r}^{n;J_1\cdots J_p}$ exist such that $\Gamma(\{a\})$ includes precisely the field content of a given target supermultiplet, in this case a Scalar supermultiplet, and 2) to use the prepotential superfields to build a supersymmetric action functional depending only on the degrees of freedom corresponding to the target supermultiplet.

6 $N = 2$ **Prepotentials**

In this section we address the question posed in the previous section, regarding the existence of a suitable prepotential for Scalar supermultiplets, in the restricted context of $N = 2$ supersymmetry. We do not presuppose the particular solution described in the previous section but instead arrive at this solution via methodical reasoning.

---

[11]Note: Two superfields may be added only if these have the same statistics, and describe the same SO(N) representation, and have the same engineering dimension. The third of these restrictions dramatically limits the possibilities. The summands in (5.3) have engineering dimension $\eta = \frac{1}{2} p + s$, since the D_I have dimension one-half. This number should be the same for each summand in order for $\Gamma(\{a\})$ to have definite engineering dimension. Thus, there is only one value of p for each value of s for which the coefficients can be nonvanishing.
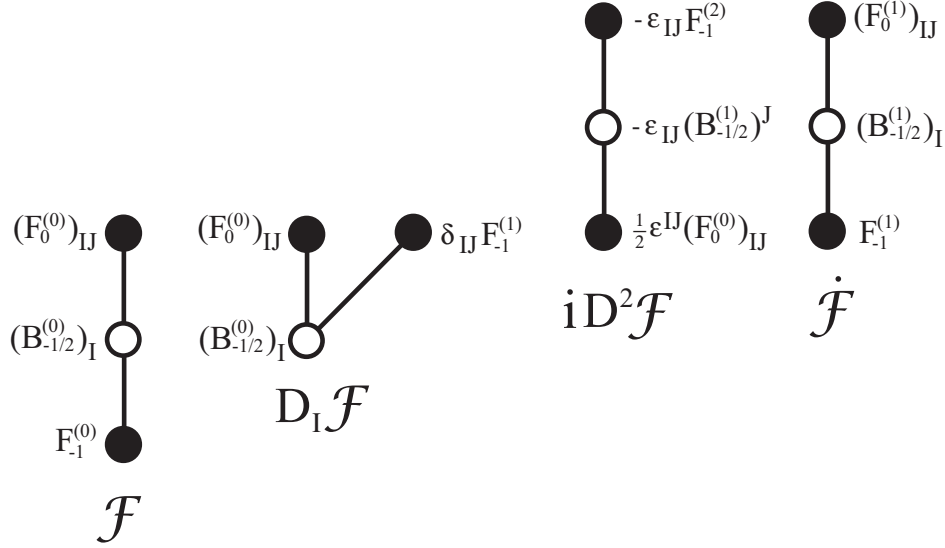
Figure 4: The Adinkras for the fermionic $N = 2$ superfield $\mathcal{F}$, and derivatives thereof. The vertical placement of the vertices in this graphic correlates faithfully with the vertex height assignments. This illustrates again how the application of a superderivative lifts vertices, how application of a top derivative, D^2 in this case, swivels the Top Adinkra around its hook by 180°, and how the operator ∂_τ lifts the $\mathcal{F}$ Adinkra up without swiveling. (Thus, τ derivatives lift entire Adinkras while superderivatives lift vertices.)

We adopt a notational convention in which the name of a component field provides information regarding the statistics (boson or fermion), the engineering dimension, and the number of derivatives appearing on that object. In particular, we use $B_\delta^{(m)}$ to refer to the mth τ derivative of a bosonic component field having engineering dimension δ and $F_\delta^{(m)}$ to refer to the mth τ derivative of a fermionic component field having engineering dimension δ. Thus, a boson field having engineering dimension minus one-half would be named $B_{-1/2}^{(0)}$, whereas the fourth derivative of this field would be named $B_{-1/2}^{(4)}$. The engineering dimension of an object can be read off of the labels, since $[B_\delta^{(m)}] = (\delta + m)$.

To begin, we consider the simplest possibility, and involve only one real prepotential, *i.e.*, we make an ansatz $\lambda = 1$. We develop the case where this superpotential is fermionic (so that $b = 0$).[12] Using the notational convention introduced in the previous paragraph, an unconstrained fermionic $N = 2$ superfield is given by

---

[12]As it turns out, it is not possible to build a local superspace action having canonical kinetic terms in the case of even-N Scalar supermultiplets using bosonic prepotentials. The reasons for this are explained near the end of this section.

$$\mathcal{F} = F_{-1}^{(0)} + \theta^I \, (\, B_{-1/2}^{(0)} \,)_I + \tfrac{1}{2!} \, i \, \theta^I \, \theta^J \, (\, F_0^{(0)} \,)_{IJ} \,. \qquad (6.1)$$

We can enumerate the possible terms in a "Gamma expansion", defined in (5.3), by computing various derivatives of this superfield. Since $N = 2$, there are only three possible terms for which $q = 0$. The first involves (6.1) itself. The second involves the first superderivative,

$$D_I \, \mathcal{F} \; = \; (\, B_{-1/2}^{(0)} \,)_I + i \, \theta^J \left(\, (\, F_0^{(0)} \,)_{IJ} + \delta_{IJ} \, F_{-1}^{(1)} \, \right)$$
$$+ \tfrac{1}{2!} \, i \, \varepsilon_{JK} \, \theta^J \, \theta^K \left(\, \varepsilon_{IL} \, (\, B_{-1/2}^{(1)} \,)^L \, \right) , \qquad (6.2)$$

and the third involves the second superderivative,

$$i \, D_{[I} \, D_{J]} \, \mathcal{F} \; = \; (\, F_0^{(0)} \,)_{IJ} + \theta^K \left(\, -2 \, \delta_{K[I} \, (\, B_{-1/2}^{(1)} \,)_{J]} \, \right)$$
$$+ \tfrac{1}{2!} \, i \, \varepsilon_{KL} \, \theta^K \, \theta^L \left(\, -\varepsilon_{IJ} \, F_{-1}^{(2)} \, \right) . \qquad (6.3)$$

We have included a conventional factor of i in (6.3) because the operator $i \, D_{[I} D_{J]}$ preserves the phase of $\mathcal{F}$. We will abbreviate $\tfrac{1}{2} \varepsilon^{IJ} D_I D_J$ by writing D^2. The Adinkras corresponding to the superfields $\mathcal{F}$, $D_I \, \mathcal{F}$, and $i \, D^2 \, \mathcal{F}$ are shown in Figure 4, where the precise correspondence between the Adinkra vertices and the superfield components are indicated. Every other possible term in the Gamma expansion corresponds to a τ derivative of one of these three terms, *i.e.*, $\mathcal{F}^{(q)}$, $D_I \, \mathcal{F}^{(q)}$, or $i \, D^2 \, \mathcal{F}^{(q)}$, where $X^{(q)} := \partial_\tau^q X$. Figure 4 also shows the Adinkra corresponding to $\mathcal{F}^{(1)} = \dot{\mathcal{F}}$; note that it has the same graphical form as the Adinkra for $i \, D^2 \, \mathcal{F}$, but with different labels.

There are very few possibilities for forming linear combinations of the superfields described so far. The only way to obtain an SO(2) singlet superfield as a sum is by adding

$$\partial_\tau \, \mathcal{F} = F_{-1}^{(1)} + \theta^I \, (\, B_{-1/2}^{(1)} \,)_I + \tfrac{1}{2!} \, i \, \theta^I \, \theta^J \, (\, F_0^{(1)} \,)_{IJ} \qquad (6.4)$$

to some multiple of

$$i \, D^2 \, \mathcal{F} \; = \; \tfrac{1}{2} \, \varepsilon^{IJ} \, (\, F_0^{(0)} \,)_{IJ} + \theta^I \left(\, -\varepsilon_{IJ} \, (\, B_{-1/2}^{(1)} \,)^J \, \right)$$
$$+ \tfrac{1}{2!} \, i \, \theta^I \, \theta^J \left(\, -\varepsilon_{IJ} \, F_{-1}^{(2)} \, \right) , \qquad (6.5)$$

or by adding together total derivatives ∂_τ^q of both of these. Consider first the sum of (6.4) and (6.5), using a relative coefficient of unity. This yields

$$(\, i \, D^2 + \partial_\tau \,) \, \mathcal{F} \; = \; F_{-1}^{(1)} + \tfrac{1}{2} \, \varepsilon^{IJ} \, (\, F_0^{(0)} \,)_{IJ}$$
$$+ \theta^I \left(\, (\, B_{-1/2}^{(1)} \,)_I - \varepsilon_{IJ} \, (\, B_{-1/2}^{(1)} \,)^J \, \right)$$
$$- \tfrac{1}{2!} \, i \, \varepsilon_{IJ} \, \theta^I \, \theta^J \left(\, F_{-1}^{(2)} - \tfrac{1}{2} \, \varepsilon^{KL} \, (\, F_0^{(1)} \,)_{KL} \, \right) . \qquad (6.6)$$

This operation preserves the overall phase of $\mathcal{F}$. For instance, if $\mathcal{F}$ is a real superfield, satisfying $\mathcal{F} = \mathcal{F}^\dagger$, then $(i\,D^2 + \partial_\tau)\,\mathcal{F}$ is a new real superfield built by rearranging the components of $\mathcal{F}$. So the Adinkra corresponding to (6.6) is a Top Adinkra, not a Scalar Adinkra. A similar conclusion follows if we consider any combination $(a\,i\,D^2 + \partial_\tau)\,\mathcal{F}$, where a is any real number. Thus, this does not provide us with what we are looking for, *i.e.*, a superfield corresponding to a two-height Adinkra. It might seem odd to expect that by adding together three-height Adinkras we could obtain a two-height Adinkra. But this is possible if the addition serves to project out some the vertices, as we show presently.

We have determined that a linear combination of $\partial_\tau\,\mathcal{F}$ and $i\,D^2\,\mathcal{F}$ might correspond to a two-height Adinkra only if the relative coefficient is not real. It follows that we must allow $\mathcal{F}$ to be a complex superfield.[13] Thus, we have revised our original ansatz and are now considering the case $\lambda = 2$. Now, we can form a new linear combination of $\partial_\tau\,\mathcal{F}$ and $i\,D^2\mathcal{F}$, this time including a relative factor of i between the two terms,

$$\Psi := (-D^2 + \partial_\tau)\,\mathcal{F}. \tag{6.7}$$

The component expansion for this superfield is

$$\begin{aligned}
\Psi = {}& F^{(1)}_{-1} + \tfrac{1}{2}\,i\,\varepsilon^{IJ}\,(F^{(0)}_0)_{IJ} \\[4pt]
& + \theta^I\left((B^{(1)}_{-1/2})_I - i\,\varepsilon_{IJ}\,(B^{(1)}_{-1/2})^J\right) \\[4pt]
& + \tfrac{1}{2!}\,\varepsilon_{IJ}\,\theta^I\,\theta^J\left(F^{(2)}_{-1} + \tfrac{1}{2}\,i\,\varepsilon^{KL}\,(F^{(1)}_0)_{KL}\right).
\end{aligned} \tag{6.8}$$

This combination has a remarkable feature: the highest component of Ψ is the τ derivative of its lowest component. Thus, the highest component is completely determined by data included at a lower level in the superfield. In fact, the combinations which appear at the lowest two levels, namely

$$(\widehat{B}^{(0)}_{-1/2})_I := (\delta_{IJ} - i\,\varepsilon_{IJ})\,(B^{(0)}_{-1/2})^J$$

$$\widehat{F}^{(0)}_0 := F^{(1)}_{-1} + \tfrac{1}{2}\,i\,\varepsilon^{IJ}\,(F^{(0)}_0)_{IJ}, \tag{6.9}$$

describe a new supermultiplet spanning two engineering dimensions $d = -1/2$ and $d = 0$. Using the definitions (6.9) we can rewrite (6.8) as

$$\Psi = \widehat{F}^{(0)}_0 + \theta^I\,(\widehat{B}^{(1)}_{-1/2})_I + \tfrac{1}{2}\,i\,\varepsilon_{IJ}\,\theta^I\,\theta^J\,\widehat{F}^{(1)}_0. \tag{6.10}$$

---

[13]Equivalently, we could add another real prepotential so that we have two of these, say $\mathcal{F}_1$ and $\mathcal{F}_2$. But these could be complexified by writing $\mathcal{F} = \mathcal{F}_1 + i\,\mathcal{F}_2$, which ultimately amounts to the same thing.

Here $\widehat{F}_0^{(0)}$ is a complex scalar, and thus has two real degrees of freedom, and $(\widehat{B}_{-1/2}^{(0)})_I$, though it appears at first to have four real degrees of freedom, really has two, for the following reason.

If we define $(P_\pm)_{IJ} := \delta_{IJ} \pm i\epsilon_{IJ}$, then $(P_\pm)_I{}^J$ are a pair of complementary projection operators, meaning $P_+{}^2 = P_+$, $P_-{}^2 = P_-$, $P_+P_- = P_-P_+ = 0$, and $P_+ + P_- = 1$. The operator P_+ projects to the set of self-dual one-forms and P_- to the anti-self-dual one-forms. Every one-form ω can be written as $P_+\omega + P_-\omega$, where the first term is self-dual and the second term is anti-self-dual. The expression in (6.9) explicitly shows that $\widehat{B}_{-1/2}^{(0)}$ is anti-self-dual, so the range of possibilities here is halved. In other words, $\widehat{B}_{-1/2}^{(0)}$ satisfies the constraint $P_+\widehat{B}_{-1/2}^{(0)} = 0$. Thus the components of (6.9) involve two bosonic degrees of freedom and two fermionic degrees of freedom, or exactly half of those in $\mathcal{F}$.

Another way to think of this situation is in terms of gauge equivalences. It is straightforward to see that $\widehat{F}_0^{(0)}$ and $(\widehat{B}_{-1/2}^{(1)})_I$, and therefore the entire superfield Ψ, are invariant under the following gauge transformation:

$$\delta(F_0^{(0)})_{IJ} = \dot{\beta}_{IJ}$$

$$\delta(B_{-1/2}^{(0)})_I = (\delta_{IJ} + i\,\varepsilon_{IJ})\,a^J$$

$$\delta F_{-1}^{(0)} = -\tfrac{1}{2}\,i\,\varepsilon^{IJ}\,\beta_{IJ}\,, \tag{6.11}$$

where β_{IJ} is a complex two-form, describing two fermionic degrees of freedom, and a^J is a self-dual complex one-form describing two bosonic degrees of freedom. We can use this freedom to make a gauge choice

$$F_{-1}^{(0)} = 0$$

$$(\delta_{IJ} + i\,\varepsilon_{IJ})\,(B_{-1/2}^{(0)})^J = 0\,. \tag{6.12}$$

But, the degrees of freedom removed from $\mathcal{F}$ by making this choice are also removed by the operation described by (6.7). An Adinkrammatic depiction of (6.7) is given in Figure 5.

The superfield Ψ describes the general solution to the following constraint:

$$(\delta_{IJ} + i\,\varepsilon_{IJ})\,D^J\,\Psi = 0\,. \tag{6.13}$$

Equivalently, Ψ describes the projection (6.7) of an unconstrained superfield $\mathcal{F}$. Figure 6 describes the Adinkrammatics associated with the projection (6.7). But the most important way to regard this supermultiplet, for the purposes of our discussion, is via its relationship to the $N = 2$ Scalar supermultiplet. The complex, anti-self-dual boson $(\widehat{B}_{-1/2}^{(0)})_I$ and the complex fermion $\widehat{F}_0^{(0)}$ describe the $2 + 2$ components corresponding to ϕ_i and $\psi_{\hat{i}}$, as defined in (5.1),

Figure with Adinkra diagrams:

First diagram (two-height, two columns of nodes):
- Top nodes (2): $-i\varepsilon_{IJ}\,F^{(2)}_{-1}$ and $(F^{(1)}_0)_{IJ}$
- Middle nodes (4): $-i\varepsilon_{IJ}(B^{(1)}_{-1/2})^J$ $\;+\;$ (4) $(B^{(1)}_{-1/2})^J$ $\;=\;$ (2) $(\delta_{IJ}-i\varepsilon_{IJ})(B^{(1)}_{-1/2})^J$
- Bottom nodes (2): $\frac{1}{2}i\varepsilon^{IJ}(F^{(0)}_0)_{IJ}$, (2) $F^{(1)}_{-1}$, (2) $F^{(1)}_{-1}+\frac{1}{2}i\varepsilon^{IJ}(F^{(0)}_0)_{IJ}$

$$-\mathrm{D}^2\mathcal{F} \quad + \quad \dot{\mathcal{F}} \quad = \quad \Psi$$

Figure 5: When adding together the Adinkras corresponding to the super-fields $-D^2\mathcal{F}$ and $\partial_\tau\mathcal{F}$, we take an appropriate linear combination of these superfields which induces a projection onto a sub-Adinkra corresponding to the superfield Ψ. The fermionic degrees of freedom $F^{(1)}_{-1} - \frac{1}{2}\varepsilon^{IJ}(F^{(0)}_0)_{IJ}$ are removed by this process, as are the bosonic degrees of freedom $(\delta_{IJ} + i\varepsilon_{IJ})(B^{(0)}_{-1/2})^J$, owing to the presence of the gauge symmetry shown in (6.11).

respectively. The precise correspondence is described immediately below in subsection 6.1. The superfield Ψ, and its Adinkra, shown in Figure 7, correspond to a vertex-raised version of this Scalar supermultiplet/Adinkra, since it is the derivative $(\widehat{B}^{(1)}_{-1/2})_I$ which contributes to these, rather than the elemental field $(\widehat{B}^{(0)}_{-1/2})_I$. A more fundamental superfield which contains precisely $(\widehat{B}^{(0)}_{-1/2})_I$ and $\widehat{F}^{(0)}_0$ is called Φ_I, and is described in subsection 6.2. The fact that this describes the $N = 2$ Scalar supermultiplet is easy to prove, since there is only one way to draw a two-height $N = 2$ Adinkra where the lower components are bosons.

Note that equation (6.13) may be construed as a chirality constraint, since we could regard the operator $(\delta_{IJ} + i\varepsilon_{IJ})D^J$ as a complex superspace derivative. Thus, the superfield Ψ is an example of a "chiral" superfield. Note that in the context of $D = 4$ $N = 1$ supersymmetry (which is equivalent to $D = 1$ $N = 4$ supersymmetry) similar superfields, along with a differential condition analogous to (6.13) also exist.

The $SO(2)$-invariant Levi-Civita tensor ε_{IJ} defines a complex structure on the "target space". However, additional algebraic structures suggested by the vector spaces defined in the PpG diagram (Figure 3) prove instrumental to the ability to generalize the developments of this section to the context of higher-N supersymmetry. In the following subsection we describe some of this additional structure in the case $N = 2$.
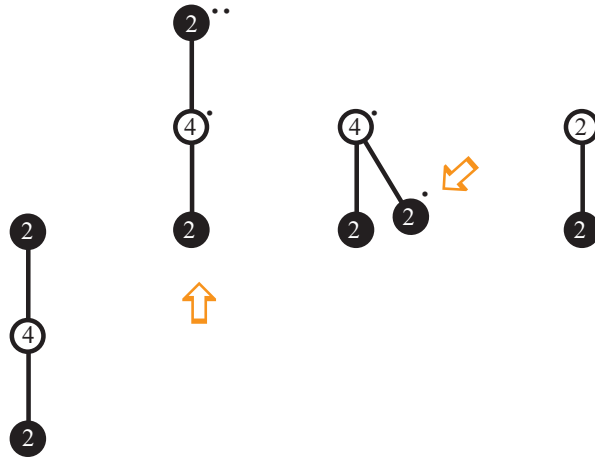
Figure 6: Schematically, the $N = 2$ operator $(-D^2 + \partial_\tau)$ implements the indicated operations on the Top Adinkra. First, it raises the Top Adinkra two height units by a combination of swiveling and lifting. Then it lowers the topmost vertex, by folding the raised Adinkra in half, removing half of the degrees of freedom in the process. What results is the gauge-invariant Ψ Adinkra.
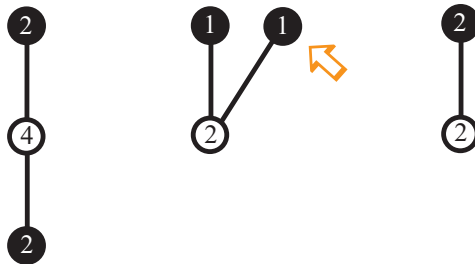
Figure 7: The $N = 2$ operator $\frac{1}{2}(\delta_{IJ} - i\,\varepsilon_{IJ})\,D^J$ implements the indicated operations on the Top Adinkra. If lifts the lowermost vertex, and removes half the degrees of freedom. What results is the gauge-invariant Φ Adinkra. The Ψ Adinkra, shown in Figure 6 is obtained by raising the lower vertex in the Φ Adinkra, which is equivalent to swiveling Φ Adinkra 180° about its hook.

6.1 $\mathcal{GR}(2,2)$ Structure

The component transformation rules can be determined from (6.1) using $\delta_Q(\epsilon) = -i\,\epsilon^I\,Q_I$, where ϵ^I is an $SO(2)$ doublet of real supersymmetry parameters and $Q_I = i\,\partial_I + \theta_I\,\partial_\tau$ is the local superspace supersymmetry generator. Accordingly, the superfield $\mathcal{F}$ transforms as

$$\delta_Q(\epsilon)\,\mathcal{F} = \epsilon^I\,(\,\partial_I - i\,\theta_I\,\partial_\tau\,)\,\mathcal{F}\,. \tag{6.14}$$

Via explicit computation using (6.1), this tells us

$$\delta_Q\,F^{(0)}_{-1} \;=\; \epsilon^I\,(\,B^{(0)}_{-1/2}\,)_I$$

$$\delta_Q\,(\,B^{(0)}_{-1/2}\,)_I \;=\; i\,\epsilon^J\,(\,F^{(0)}_0\,)_{IJ} + i\,\epsilon_I\,F^{(1)}_{-1}$$

$$\delta_Q\,(\,F^{(0)}_0\,)_{IJ} \;=\; -2\,\epsilon_{[I}\,(\,B^{(1)}_{-1/2}\,)_{J]}\,. \tag{6.15}$$

We can use (6.15) to determine the corresponding transformations of the "gauge-invariant" degrees of freedom defined in (6.9), with the result given by

$$\delta_Q\,(\,\widehat{B}^{(0)}_{-1/2}\,)_I \;=\; i\,(\,\epsilon_I - i\,\varepsilon_{IJ}\,\epsilon^J\,)\,\widehat{F}^{(0)}_0$$

$$\delta_Q\,\widehat{F}^{(0)}_0 \;=\; (\,\epsilon_I + i\,\varepsilon_{IJ}\,\epsilon^J\,)\,\partial_\tau\,(\,\widehat{B}^{(0)}_{-1/2}\,)^I\,. \tag{6.16}$$

This illustrates explicitly that the gauge-invariant fields do properly comprise a supersymmetry representation in and of themselves. Now consider the following independent real combinations, which suggestively package the $2+2$ invariant degrees of freedom,

$$\phi_1 \;:=\; \mathrm{Re}\,(\,\widehat{B}^{(0)}_{-1/2}\,)_2 - \mathrm{Im}\,(\,\widehat{B}^{(0)}_{-1/2}\,)_1$$

$$\phi_2 \;:=\; \mathrm{Im}\,(\,\widehat{B}^{(0)}_{-1/2}\,)_2 + \mathrm{Re}\,(\,\widehat{B}^{(0)}_{-1/2}\,)_1$$

$$\psi_{\hat{1}} \;:=\; \mathrm{Re}\,\widehat{F}^{(0)}_0$$

$$\psi_{\hat{2}} \;:=\; \mathrm{Im}\,\widehat{F}^{(0)}_0\,. \tag{6.17}$$

In terms of these, the transformation rules (6.16) become

$$\delta_Q\,\phi_1 \;=\; -i\,\epsilon^1\,\psi_{\hat{2}} + i\,\epsilon^2\,\psi_{\hat{1}}$$

$$\delta_Q\,\phi_2 \;=\; i\,\epsilon^1\,\psi_{\hat{1}} + i\,\epsilon^2\,\psi_{\hat{2}}$$

$$\delta_Q\,\psi_{\hat{1}} \;=\; \epsilon^1\,\dot{\phi}_2 + \epsilon^2\,\dot{\phi}_1$$

$$\delta_Q\,\psi_{\hat{2}} \;=\; -\epsilon^1\,\dot{\phi}_1 + \epsilon^2\,\dot{\phi}_2\,. \tag{6.18}$$

Using the index conventions described in section 4, the "physical" degrees of freedom, ϕ_i and $\psi_{\hat{i}}$, may be assigned values in $\mathcal{V}_L$ and $\mathcal{V}_R$, respectively, thereby exposing a natural $\mathcal{GR}(2,2)$ structure associated with this supermultiplet. This is made all the more explicit if we make the following particular basis choice for the $N=2$ garden matrices,

$$
L_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

$$
L_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.19}
$$

Using these, the transformation rules (6.18) can be written more concisely as

$$
\delta_Q(\epsilon)\,\phi_i = -i\,\epsilon^I\,(\,L_I\,)_i{}^{\hat{j}}\,\psi_{\hat{j}}
$$

$$
\delta_Q(\epsilon)\,\psi_{\hat{i}} = \epsilon^I\,(\,R_I\,)_{\hat{i}}{}^j\,\partial_\tau\,\phi_j\,, \tag{6.20}
$$

which, as we recognize from (5.1), precisely describes a Scalar supermultiplet.

6.2   An $N=2$ Invariant Action

A manifestly supersymmetric action can be written as follows:

$$
S = \int d\tau\,d^2\theta \left(\tfrac{1}{2}\,i\,\mathcal{F}^\dagger\,\dot{\Psi} + \text{h.c.} \right). \tag{6.21}
$$

Using the component expansion for $\mathcal{F}$, given in (6.1) and the component expansion for Ψ, given in (6.8), then performing the theta integration, we determine the action in terms of the component fields,

$$
S = \int d\tau \left(-\tfrac{1}{2}\,(\,\widehat{B}^{(1)}_{-1/2}\,)^\dagger_I (\,\widehat{B}^{(1)}_{-1/2}\,)^I - i\,\tfrac{1}{2}\,(\,\widehat{F}^{(0)\,\dagger}_0\,\widehat{F}^{(1)}_0 - \widehat{F}^{(1)\,\dagger}_0\,\widehat{F}^{(0)}_0\,) \right), \tag{6.22}
$$

where $(\,\widehat{B}_{-1/2}\,)_I$ and $\widehat{F}_0$ are the gauge-invariant combinations given in (6.9). This demonstrates that (6.21) provides for a canonical kinetic action for the component fields and also that this result is invariant under the gauge transformation (6.11). Thus, (6.21) depends only on the 2+2 gauge-invariant degrees of freedom defined in (6.9) rather than on the full $4+4$ degrees of freedom described by $\mathcal{F}$. By way of contrast, the prepotential superfield $\mathcal{F}$ includes the physical as well as the spurious, gauge degrees of freedom.

We could repeat the above analysis using bosonic prepotentials rather than fermionic prepotentials. However, it proves impossible in that case to use gauge invariant superfield combinations to build a bosonic action analogous to (6.22) which involves canonical kinetic terms for the component fields. The diligent

reader can verify that the same arguments and computations would then apply, but the bosonic and fermionic components would be switched. The resulting action (6.22) would then be trivial. Altering (6.21) turns out not to help, and this fact can be seen from dimensional arguments.

We have seen that the lowest component of Ψ corresponds to the gauge-invariant physical fermions. We notice that the gauge-invariant physical bosons are the lowest components of the anti self-dual part of $D_I \mathcal{F}$. Accordingly, we define

$$\Phi_I := \tfrac{1}{2} \left(\delta_{IJ} - i \, \varepsilon_{IJ} \right) D^J \mathcal{F} \,. \tag{6.23}$$

We then notice that

$$\Psi = -\varepsilon^{IJ} D_I \Phi_J \,. \tag{6.24}$$

as is readily verified using (6.7) along with the identity $D_I D_J = D_{[I} D_{J]} + i \, \delta_{IJ} \partial_\tau$. Equations (6.23) and (6.24) describe the gauge-invariant superfield analogs of the physical component fields described by the prepotential $\mathcal{F}$. The map from $\mathcal{F}$ to Φ_I is described Adinkrammatically in Figure 7. The Φ_I Adinkra contains precisely the physical degrees of freedom in this supermultiplet. The Ψ Adinkra, by way of contrast, is missing the zero mode of the physical bosons.

7   Scalar Prepotentials for General N

The strategy employed in the previous section ought to generalize to cases where $N > 2$. To determine how, it is helpful to reflect on how we managed to succeed in that case. This is made transparent by looking at Figure 5, which we can re-write in more streamlined form as follows:

$$\tag{7.1}$$

where the first term on the left-hand side is the Adinkra corresponding to $D^2 \mathcal{F}^{(0)}$, the second term is the Adinkra corresponding to $\mathcal{F}^{(1)}$, ignoring for the moment numerical coefficients. We have distinguished one fermionic vertex by adding a yellow spot, which manifests the de facto orientation of the vertex chain. This is helpful for purposes of enabling inter-term vertex comparisons. (The spotted vertex is the source of the $\mathcal{F}^{(0)}$ Adinkra; the fact that this vertex appears at the top of the $D^2 \mathcal{F}$ Adinkra rather than at the bottom reflects

the fact that the operator D^2 swivels the $\mathcal{F}$ Adinkra $180°$ about its hook.)
Now we see that the vertex-wise addition of the Adinkras on the left-hand
side of (7.1) exhibits a promising feature: the same two terms appear in the
vertex sum at the lowest level as appear in the vertex sum at the highest level,
albeit with one extra dot at the higher level. This suggests that one might be
able to choose the relative numerical coefficient between the two superfields
corresponding to these Adinkras in just such a way that their sum includes
precisely the same information at two different levels of the superfield. This
would then render the highest vertex of the Adinkra sum superfluous. This
would also necessarily implement a gauge projection, since the sum of two fields
containing two degrees of freedom can carry only two degrees of freedom itself.
Furthermore, since supersymmetry automatically ensures a balance between
fermionic and bosonic degrees of freedom, the gauge projection on the fermionic
vertices would necessarily be accompanied by a similar gauge projection on the
bosonic vertices. What we showed above, via explicit computation, was that
all of this is, in fact, tractable.

Let's examine how this reasoning might generalize to higher N. In general,
a Top Adinkra, which corresponds to a real prepotential superfield, say $\mathcal{F}^{(0)}$,
spans $N + 1$ distinct height assignments. We can attempt to reduce the span
of this Adinkra by adding $\mathcal{F}^{(N/2)} = \partial_\tau^{N/2} \mathcal{F}^{(0)}$ to a swiveled version of $\mathcal{F}^{(0)}$,
corresponding to $D^N \mathcal{F}^{(0)}$, where $D^N = \frac{1}{N!} \varepsilon^{I_1 \cdots I_N} D_{I_1} \cdots D_{I_N}$. The purpose
of the $N/2$ derivatives on $\mathcal{F}^{(N/2)}$ is to ensure that the engineering dimension
of this term coincides properly with that of $D^N \mathcal{F}^{(0)}$. Based on our previous
analysis, we suspect that this could succeed only if we used instead *at least* a
pair of real superfields $\mathcal{F}_1$ and $\mathcal{F}_2$, or, to be more general, some set of superfields
$\mathcal{F}_n$ where $n = 1, ..., \lambda$, rather than a single real superfield $\mathcal{F}$. But the best we
could realistically hope for in this case would be to find a suitable coefficient
between each $\mathcal{F}_n^{(N/2)}$ term and the corresponding $D^N \mathcal{F}_n^{(0)}$ term so as to reduce
the span from $N + 1$ distinct height assignments to N, whereas if we want to
describe a Scalar supermultiplet, we have to reduce the span all the way to
two.

The problem then is how to include more terms in our Γ expansion (5.3)
in such a way that not only do the engineering dimensions of each term match
up properly, but so too do the SO(N) tensor structures. For example, both
$\mathcal{F}^{(N/2)}$ and $D^N \mathcal{F}^{(0)}$ are SO(N) invariant, but none of the other possible terms
$D_{[I_1} \cdots D_{I_p]} \mathcal{F}^{(N-p)/2}$, where $0 < p < N$, are SO(N) invariant, nor are contrac-
tions of these with the Levi-Civita symbol $\varepsilon^{I_1 \cdots I_N}$. It is here that the garden
algebra saves the day. Objects precisely of the sort defined in (4.4) are indi-
cated for this purpose. These are p-forms on SO(N) that are also operators
on $\wedge \mathcal{GR}(\mathrm{d}, N)$. We can use these objects to add tensorial balance to more
possible terms. But if we do this, it becomes necessary that the prepoten-
tials take values in $\mathcal{V}_R$ or $\mathcal{V}_L$, since elements of $\wedge \mathcal{GR}(\mathrm{d}, N)$ act on elements of

$\mathcal{V}_L \oplus \mathcal{V}_R$. This, of course plays right into our hand, since, ultimately, the component fields of the Scalar supermultiplets should take their values in precisely these vector spaces. In this way, we could, for instance, form an SO(N) singlet superfield by adding up terms of the following sort:

$$\varepsilon^{I_1 \cdots I_N} D_{I_N} \cdots D_{I_2 \, s+2} \, (\, f_{I_2 \, s+1 \cdots I_1} \,)_i{}^{\hat{i}} \, \partial_\tau^s \mathcal{S}_{\hat{i}} \,. \tag{7.2}$$

to define a superfield $\Phi_i := (\mathcal{O}_\phi)_i{}^{\hat{i}} \mathcal{S}_{\hat{i}}$ which would include only the fields of the Scalar supermultiplet. Alternatively, we could add up terms of the following sort:

$$\varepsilon^{I_1 \cdots I_N} D_{I_N} \cdots D_{I_2 \, s+1} \, (\, \tilde{f}_{I_2 \, s \cdots I_1} \,)_{\hat{i}}{}^{\hat{j}} \, \partial_\tau^s \mathcal{S}_{\hat{j}} \,. \tag{7.3}$$

to define another superfield $\Psi_{\hat{i}} := (\mathcal{O}_\psi)_{\hat{i}}{}^{\hat{j}} \mathcal{S}_{\hat{j}}$ which would also include only the fields of the Scalar supermultiplet albeit organized differently than in the field Φ_i. It is natural (and ultimately successful) to assume that there exist superfields Φ_i and $\Psi_{\hat{i}}$, defined precisely in this way, with the properties that the lowest components coincide precisely with the component fields ϕ_i and $\psi_{\hat{i}}$ defined in (5.1).

Based on the above discussion, we presuppose that a Scalar supermultiplet is described by a set of d unconstrained prepotential superfields $\mathcal{S}_{\hat{i}}$. For reasons similar to those described in Section 6, it turns out that when N is an even integer, the prepotential will be fermionic while in cases where N is an odd integer, the prepotential will be bosonic. Accordingly, when N is even we will write $\mathcal{S}_{\hat{i}} := \mathcal{F}_{\hat{i}}$ and when N is odd we will write $\mathcal{S}_{\hat{i}} := \mathcal{B}_{\hat{i}}$. In terms of the prepotentials, we define related superfields by

$$\Phi_i = (\mathcal{O}_\phi)_i{}^{\hat{i}} \mathcal{S}_{\hat{i}}$$

$$\Psi_{\hat{i}} = (\mathcal{O}_\psi)_{\hat{i}}{}^{\hat{j}} \mathcal{S}_{\hat{j}} \,. \tag{7.4}$$

where $(\mathcal{O}_\phi)_i{}^{\hat{i}}$ and $(\mathcal{O}_\psi)_{\hat{i}}{}^{\hat{j}}$ are operators determined such that $\Phi_i | = \phi_i$ and $\Psi_{\hat{i}} | = \psi_{\hat{i}}$, where ϕ_i and $\psi_{\hat{i}}$ are the particular component fields appearing in (5.1). By making this definition, we imply that the supersymmetry transformation rules induced on ϕ_i and $\psi_{\hat{i}}$ by virtue of the fact that these are the lowest components of the superfields defined in (7.4), via the realization of a supersymmetry transformation on superspace, correspond precisely with the component transformation rules given in (5.1). Among other things, this also implies

$$D_I \Phi_i = -i \, (\, L_I \,)_i{}^{\hat{i}} \, \Psi_{\hat{i}}$$

$$D_I \Psi_{\hat{i}} = (\, R_I \,)_{\hat{i}}{}^{i} \, \partial_\tau \, \Phi_i \,. \tag{7.5}$$

Using (7.5), it is straightforward to determine

$$
D_{[I_1} \cdots D_{I_p]} \Psi_{\hat{\imath}} = \begin{cases} (-i)^{p/2} \, (\, \tilde{f}_{I_p \cdots I_1} \,)_{\hat{\imath}}{}^{\hat{\jmath}} \, \partial_\tau^{p/2} \, \Psi_{\hat{\jmath}} & ; \; p \text{ even} \,, \\[2mm] (-i)^{(p-1)/2} \, (\, \tilde{f}_{I_p \cdots I_1} \,)_{\hat{\imath}}{}^{j} \, \partial_\tau^{(p+1)/2} \, \Phi_j & ; \; p \text{ odd} \,. \end{cases}
\tag{7.6}
$$

By substituting the definitions (7.4) into (7.5), and using the fact that the prepotentials $\mathcal{S}_{\hat{\imath}}$ are unconstrained, we obtain the following operator equations as corollaries of (7.5),

$$
-i \, (\, L_I \,)_i{}^{\hat{k}} \, (\, \mathcal{O}_\psi \,)_{\hat{k}}{}^{\hat{\jmath}} = D_I \, (\, \mathcal{O}_\phi \,)_i{}^{\hat{\jmath}}
$$

$$
(\, R_I \,)_{\hat{\imath}}{}^{k} \, \partial_\tau \, (\, \mathcal{O}_\phi \,)_k{}^{\hat{\jmath}} = D_I \, (\, \mathcal{O}_\psi \,)_{\hat{\imath}}{}^{\hat{\jmath}} \,.
\tag{7.7}
$$

Now, if we contract the first of these equations from the left with $(\, R_J \,)_{\hat{\imath}}{}^{i}$ and then symmetrize on the indices I and J, and use the garden algebra (4.2), we determine

$$
(\, \mathcal{O}_\psi \,)_{\hat{\imath}}{}^{\hat{\jmath}} = -i \, \tfrac{1}{N} \, (\, R^I \,)_{\hat{\imath}}{}^{k} \, D_I \, (\, \mathcal{O}_\phi \,)_k{}^{\hat{\jmath}} \,.
\tag{7.8}
$$

Operating with (7.8) on the prepotential $\mathcal{S}_{\hat{\imath}}$ allows us to re-write this consistency condition as

$$
\Psi_{\hat{\imath}} = -i \, \tfrac{1}{N} \, (\, R^I \, D_I \, \Phi \,)_{\hat{\imath}} \,.
\tag{7.9}
$$

Thus, the superfields $\Psi_{\hat{\imath}}$ can be expressed in a simple way, in terms of the superfields Φ_i.

How, then, can we determine the superfield Φ_i, or, equivalently, the operator $(\, \mathcal{O}_\phi \,)_i{}^{\hat{\imath}}$? As it turns out, this problem is part and parcel of the problem of finding an invariant action. That problem, in turn, can be solved by straightforward computation, by postulating that the logical choice for a manifestly supersymmetric action, expressed as an integration over superspace of a particular locally-defined superfield expression built using the $\mathcal{S}_{\hat{\imath}}$, is equivalent to a demonstrably supersymmetric component action built using the scalar component fields ϕ_i and $\psi_{\hat{\imath}}$. Details are given presently.

7.1   Invariant Actions

A manifestly supersymmetric action can be written as

$$
S = \int d\tau \, d^N \theta \, \mathcal{L}
\tag{7.10}
$$

where $\mathcal{L}$ is a locally-defined superfield $\mathcal{L}$agrangian, built using the available superfields $\mathcal{S}_{\hat{\imath}}$, Φ_i and/or $\Psi_{\hat{\imath}}$. We seek a free action, which implies that $\mathcal{L}$ is

bilinear in these fields. Furthermore, our action should also be invariant under $\wedge \mathcal{GR}(\mathrm{d}, N)$, so that indices i and $\hat{\imath}$ must be contracted. There are exactly three possible terms in this regard which also provide for a dimensionless action. The first is proportional to $\Phi^i \partial_\tau^{\frac{4-N}{2}} \Phi_i$, the second is proportional to $\mathcal{S}^{\hat{\imath}} \partial_\tau^{\frac{2+N}{2}} \mathcal{S}_{\hat{\imath}}$, and the third is proportional to $\mathcal{S}^{\hat{\imath}} \partial_\tau \Psi_{\hat{\imath}}$. In each case the appropriate power of ∂_τ is determined by dimensional analysis.[14] The first two possibilities will, in general involve too many τ derivatives to provide for a canonical kinetic component action. Accordingly, as a well-motivated ansatz, we write the following,

$$S = i^{1-\alpha} \cdot i^{\left\lfloor \frac{N}{2} \right\rfloor} \int d\tau \, d^N\theta \left(\tfrac{1}{2} \mathcal{S}^{\hat{\imath}} \, \partial_\tau \, \Psi_{\hat{\imath}} \right). \tag{7.11}$$

where $\alpha = 0$ if N is even and $\alpha = 1$ if N is odd, and where $\Psi_{\hat{\imath}} = (\, \mathcal{O}_\psi \, \mathcal{S} \,)_i$, as described above. The purpose of the N-dependent phase in (7.11) is to ensure that the action is real. Now impose that (7.11) is equivalent to

$$S = \int d\tau \left(\tfrac{1}{2} \dot{\phi}^i \, \dot{\phi}_i - \tfrac{1}{2} i \, \psi^{\hat{\imath}} \, \dot{\psi}_{\hat{\imath}} \right), \tag{7.12}$$

which is demonstrably invariant under (5.1), owing to the relationship $L_I = -R_I^T$.[15] It is straightforward, using standard superspace techniques, and a little bit of algebra, to re-write (7.11) as

$$S = \int d\tau \left(-\tfrac{1}{2} (\, \mathcal{O}_\phi \, \mathcal{S} \,)^i \, \ddot{\Phi}_i - \tfrac{1}{2} i \, (\, \mathcal{O}_\psi \, \mathcal{S} \,)^{\hat{\imath}} \, \dot{\Psi}_{\hat{\imath}} \right) \Big|. \tag{7.13}$$

where $(\, \mathcal{O}_\phi \,)_i{}^{\hat{\jmath}}$ and $(\, \mathcal{O}_\psi \,)_{\hat{\imath}}{}^{\hat{\jmath}}$ are determined as

$$(\, \mathcal{O}_\phi \,)_i{}^{\hat{\imath}} = i^{1-\alpha} \cdot i^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N}$$

$$\times \sum_{s=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} \binom{N}{2\,s+1} D_{I_N} \cdots D_{I_{2\,s+2}} (\, f_{I_{2\,s+1} \cdots I_1} \,)_i{}^{\hat{\imath}} (\, i\, \partial_\tau)^s$$

$$(\, \mathcal{O}_\psi \,)_{\hat{\imath}}{}^{\hat{\jmath}} = i^{2-\alpha} \cdot i^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N}$$

$$\times \sum_{s=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \binom{N}{2\,s} D_{I_N} \cdots D_{I_{2\,s+1}} (\, \tilde{f}_{I_{2\,s} \cdots I_1} \,)_{\hat{\imath}}{}^{\hat{\jmath}} (\, i\, \partial_\tau)^s. \tag{7.14}$$

---

[14]Since $[\, d\tau \,] = -1$ and $[\, d^N\theta \,] = \tfrac{1}{2} N$, it follows that $[\, S \,] = 0$ only if $[\, \mathcal{L} \,] = 1 - \tfrac{1}{2} N$. The engineering dimensions of Φ_i and $\Psi_{\hat{\imath}}$ follow from the requirements $\Phi_i \,| = \phi_i$ and $\Psi_{\hat{\imath}} \,| = \psi_{\hat{\imath}}$, coupled with the fact that the engineering dimension of a propagating boson is $[\, \phi_i \,] = -\tfrac{1}{2}$ and that of a propagating fermion is $[\, \psi_{\hat{\imath}} \,] = 0$, as explained in footnote *10*. Since $(\, \mathcal{O}_\phi \,)_i{}^{\hat{\imath}}$ and $(\, \mathcal{O}_\psi \,)_{\hat{\imath}}{}^{\hat{\jmath}}$ are built using terms of the sort (7.2) and (7.3), respectively, it follows that $[\, \mathcal{S}_{\hat{\imath}} \,] = -\tfrac{1}{2} N$. These facts suffice for determining the appropriate power of ∂_τ appearing in the superfield products involving $\mathcal{S}_{\hat{\imath}}$, $\Psi_{\hat{\imath}}$, and Φ_i.

[15]The requirement that (7.11) be invariant under (5.1) is the underlying motivation for the criterion $L_I = -R_I^T$.

N	1	2	3	4	5	6	7	8
d_N	1	2	4	4	8	8	8	8
g_N	0	4	24	56	240	496	1008	2032

Table 1: The number of gauge degrees of freedom g_N included in the Scalar supermultiplet superfields grows large as N increases.
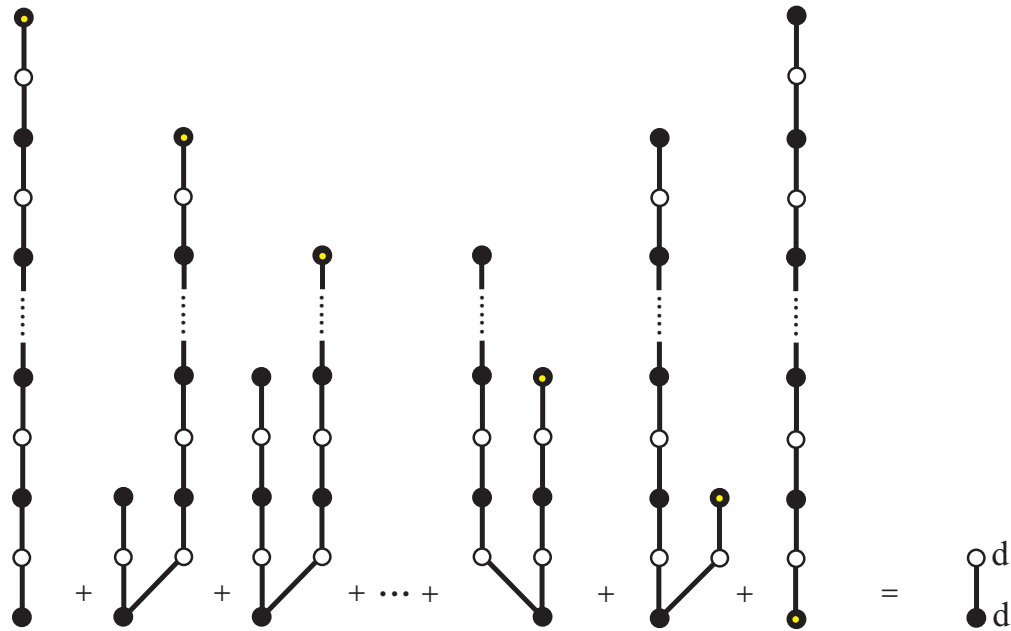
An explicit derivation of this is shown explicitly in Appendix A. It is gratifying to check that the operators in (7.14) properly satisfy the consistency condition (7.8).

A graphical depiction which illustrates how it is that the projectors $(\mathcal{O}_\psi)_{\hat{i}}{}^{\hat{j}}$ and $(\mathcal{O}_\phi)_i{}^{\hat{j}}$ can work their magic is given in Figures 8 and 9, in the case where N is even and the prepotentials $\mathcal{S}_{\hat{i}} := \mathcal{F}_{\hat{i}}$ are fermionic. To (superficially) see how this works, consider Figure 8. The very first and the very last terms in on the left-hand side of this figure correspond (schematically) to $D^N \mathcal{F}^{(0)}$ and $\mathcal{F}^{(N/2)}$. We explained above how the relative coefficient between these terms can be tuned so that the degrees of freedom corresponding to the sum of the uppermost vertices is merely a differentiated version of the same sum appearing in the sum of the lowermost vertices. Therefore, the uppermost vertices are projected out upon summation. As an extra bonus, bosons at the second to the top level are also projected away automatically as a consequence of supersymmetry. The puzzle facing us previously was how could we continue to diminish the span of the summed Adinkra. This is resolved by looking at the second term and the second to last term on the left-hand side of the figure. These correspond to $D^{N-2} \mathcal{F}^{(1)}$ and $D^2 \mathcal{F}^{(\frac{N-2}{2})}$, respectively. These Adinkra terms already have their spans reduced by two as compared to the outside terms considered previously. As a consequence, when these new inside terms are included in the sum they cannot undo any of the projections already accomplished at the highest Adinkra levels. But the second Adinkra and the second to last Adinkra can now have their relative coefficient tuned to as to implement a projection on fermionic and bosonic vertices at lower heights. Continuing this process inward, the coefficients can be tuned so as to implement a zippering action, removing vertices all the way down to the lowest two levels, precisely what is needed to describe the Scalar Adinkra.

7.2   Gauge Transformations

The supersymmetric action (7.11) can be re-written as

$$S = \int d\tau \, d^N \theta \, \tfrac{1}{2} \mathcal{S}^{\hat{i}} \, \widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{j}} \mathcal{S}_{\hat{j}} \,, \tag{7.15}$$

$$\mathrm{D}^{\mathrm{N}}\mathcal{F}^{(0)} \ + \ \mathrm{D}^{\mathrm{N}-2}\mathcal{F}^{(1)} \ + \ \mathrm{D}^{\mathrm{N}-4}\mathcal{F}^{(2)} + \ \cdots \ + \ \mathrm{D}^{4}\mathcal{F}^{(\frac{\mathrm{N}-4}{2})} \ + \ \mathrm{D}^{2}\mathcal{F}^{(\frac{\mathrm{N}-2}{2})} + \ \mathcal{F}^{(\frac{\mathrm{N}}{2})} \quad = \quad \Psi$$

Figure 8: An Adinkrammatic picture of the Ψ projection in the case where N is even. (The odd-N diagram is similar.) This is a generalization of the $N = 2$ version appearing in Figure 5. A yellow dot has been placed in a vertex to distinguish one end of the chain, for purposes of vertex comparison. Most of the vertices in this Figure should have dots, illustrating that they have some number of τ derivatives. But these have been suppressed in this rendering so as to minimize clutter. Similarly, all of the $\wedge \mathcal{GR}(\mathrm{d}, N)$ indices and vertex multiplicities have been suppressed. (In addition to the binomial coefficient vertex multiplicities corresponding to the groupings into p-forms, each of the Adinkras on the left hand side appears with an overall multiplicity of d, corresponding to the d dimensions of $\mathcal{V}_R$.) The purpose of this diagram is to give a course-grained picture of how vertices line up, level-by-level, when the various terms in (7.14) are added together, taking an appropriate linear combination of the corresponding fields.

Figure 9: An Adinkrammatic picture of the Φ projection in the case where N is even. (The odd-N diagram is similar.) A yellow dot has been placed in a vertex to distinguish one end of the chain, for purposes of vertex comparison. Most of the vertices in this Figure should have dots, illustrating that they have some number of τ derivatives. But these have been suppressed in this rendering so as to minimize clutter. Similarly, all of the $\wedge\mathcal{GR}(\mathrm{d}, N)$ indices and vertex multiplicities have been suppressed. (In addition to the binomial coefficient vertex multiplicities corresponding to the groupings into p-forms, each of the Adinkras on the left hand side appears with an overall multiplicity of d, corresponding to the d dimensions of $\mathcal{V}_R$.) The purpose of this diagram is to give a course-grained picture of how vertices line up, level-by-level, when the various terms in (7.14) are added together, taking an appropriate linear combination of the corresponding fields.
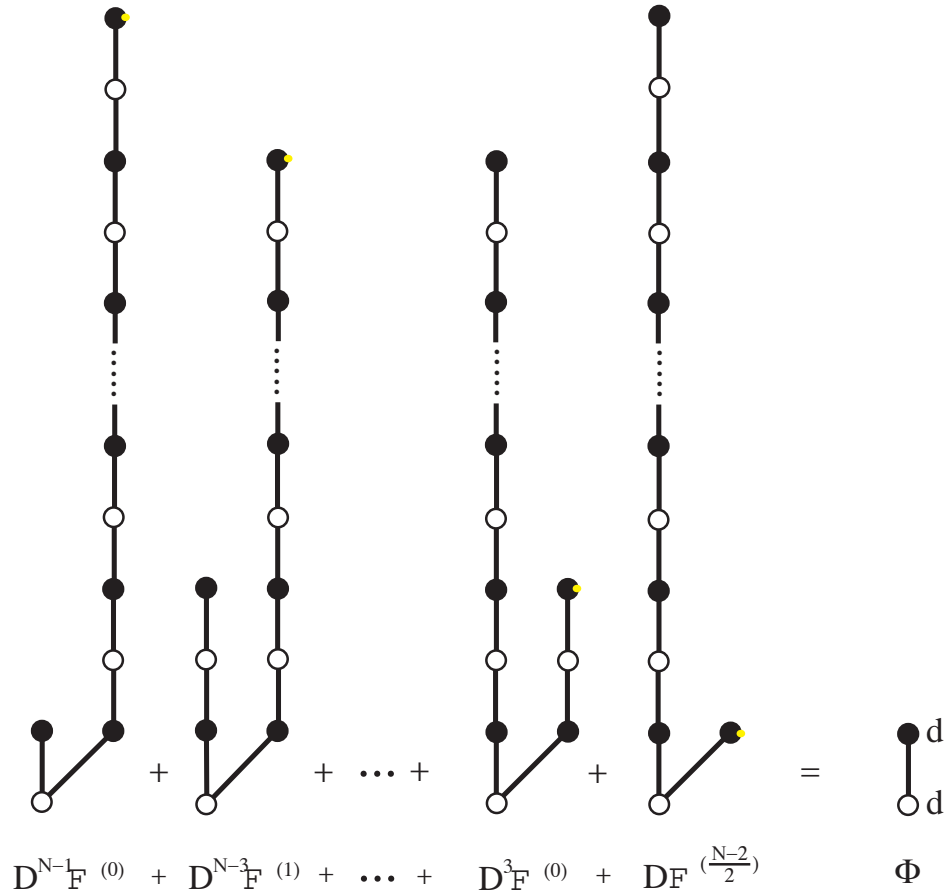
where $\widetilde{\mathcal{K}} = \mathcal{O}_\psi\, \partial_\tau$ or, more specifically,

$$\widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{j}} = i^{1-\alpha} \cdot i^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{1}{N!}\, \varepsilon^{I_1 \cdots I_N}$$

$$\times \sum_{s=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \binom{N}{2s} D_{I_N} \cdots D_{I_{2s+1}} (\,\tilde{f}_{I_{2s} \cdots I_1}\,)_{\hat{i}}{}^{\hat{j}} (\, i\, \partial_\tau\,)^{s+1} \qquad (7.16)$$

is the superspace kinetic operator acting on Salam-Strathdee superfields which are elements of $\mathcal{V}_R$.

A given Scalar supermultiplet includes $d + d$ degrees of freedom. These are explicitly seen in equation (7.12). It is also seen from this expression that there are *no gauge* symmetries associated with this component action. Nevertheless, as we shall now show, the action in (7.11) describes a gauge theory. We have shown how the component fields can be packaged using a set of d unconstrained prepotential superfields $\mathcal{S}_{\hat{i}}$. However, each prepotential includes $2^{N-1} + 2^{N-1}$ degrees of freedom. Thus, each Scalar supermultiplet prepotential construction includes

$$g_N = (\,2^N - 2\,)\, d \qquad (7.17)$$

gauge degrees of freedom which do not appear in the action. The corresponding gauge structure can be described by the transformation $\mathcal{S}_{\hat{i}} \to \mathcal{S}_{\hat{i}} + \delta\, \mathcal{S}_{\hat{i}}$, where

$$\widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{j}} \delta\, \mathcal{S}_{\hat{j}} = 0\,. \qquad (7.18)$$

Thus, there is a portion of $\mathcal{S}_{\hat{i}}$ which is annihilated by the kinetic operators $\widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{j}}$. It is possible to use the gauge freedom parametrized by $\delta\, \mathcal{S}_{\hat{i}}$ to choose a gauge in which the *only* component fields that occur in $\mathcal{S}_{\hat{i}}$ are the d bosons and d fermions in (7.12). This defines the so-called "Wess-Zumino" gauge for the prepotentials. With the realization that $\mathcal{S}_{\hat{i}}$ is a gauge field, it follows that the superfields Φ_i and $\Psi_{\hat{i}}$ defined in (7.4) are field strength superfields (*i.e.* invariants under the transformation $\mathcal{S}_{\hat{i}} \to \mathcal{S}_{\hat{i}} + \delta\, \mathcal{S}_{\hat{i}}$).

These kinetic energy operators also lead to the construction of projection operators (see appendix B for a simple and well-known example of this process). The construction of these projection operators begins by noting that

$$\widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{j}}\, \widetilde{\mathcal{K}}_{\hat{j}}{}^{\hat{k}} = [\partial_\tau]^{\frac{1}{2}N+1}\, \widetilde{\mathcal{K}}_{\hat{i}}{}^{\hat{k}} \qquad (7.19)$$

(and the similarity between (B.2) and (7.19) is obvious). Thus, it is natural

to define a projection operator via

$$
\begin{aligned}
[\,\widetilde{\mathcal{P}}^{(SM)}]_{\hat{\imath}}{}^{\hat{k}} &= [\partial_\tau]^{-\frac{1}{2}N-1}\,\widetilde{\mathcal{K}}_{\hat{\imath}}{}^{\hat{k}} \\
&= i^{1-\alpha}\cdot i^{\left\lfloor\frac{N}{2}\right\rfloor}\,\frac{1}{N!}\,\varepsilon^{I_1\cdots I_N} \\
&\quad \times \sum_{s=0}^{\left\lfloor\frac{N}{2}\right\rfloor} \binom{N}{2\,s}\, D_{I_N}\cdots D_{I_{2\,s+1}}\,(\,\tilde{f}_{I_{2\,s}\cdots I_1}\,)_{\hat{\imath}}{}^{\hat{\jmath}}\,(\,i\,\partial_\tau\,)^{s-\frac{1}{2}N}\ . \quad (7.20)
\end{aligned}
$$

This projection operator permits another characterization of the gauge variation of the prepotential. The gauge variation can be written in the form

$$
\delta\,\mathcal{S}_{\hat{\jmath}} = \{\ \delta_{\hat{\jmath}}{}^{\hat{k}}\ -\ [\widetilde{\mathcal{P}}^{(SM)}]_{\hat{\jmath}}{}^{\hat{k}}\ \}\Lambda_{\hat{k}} \equiv \{\ [\widetilde{\mathcal{P}}^{(nSM)}]_{\hat{\jmath}}{}^{\hat{k}}\ \}\Lambda_{\hat{k}}\ . \qquad (7.21)
$$

where $\Lambda_{\hat{k}}$ is a superfield not subject to any restrictions. The condition in (7.18) is satisfied due to the equations in (7.19) and (7.20).

We thus reach the conclusion that the projection operator defined by (7.16) and (7.20) is associated with the general N version of the Adinkra that appears in (5.2). The general version of this Adinkra has a number of N distinct colored links connecting d_N bosonic vertices and d_N fermionic vertices. All the bosonic vertices have a common height and all the fermionic vertices have a common height.

The operator (7.16) is clearly defined to act upon any Salam-Strathdee superfield element of $\mathcal{V}_R$. On the other hand, a similar kinetic energy operator $\mathcal{K}_{\imath}{}^{\jmath}$ and projection operator $[\mathcal{P}^{(SM)}]_{\hat{\imath}}{}^{\hat{k}}$ defined by

$$
\begin{aligned}
\mathcal{K}_{\imath}{}^{\jmath} &= i^{1-\alpha}\cdot i^{\left\lfloor\frac{N}{2}\right\rfloor}\,\frac{1}{N!}\,\varepsilon^{I_1\cdots I_N} \\
&\quad \times \sum_{s=0}^{\left\lfloor\frac{N}{2}\right\rfloor} \binom{N}{2\,s}\, D_{I_N}\cdots D_{I_{2\,s+1}}\,(\,f_{I_{2\,s}\cdots I_1}\,)_{\imath}{}^{\jmath}\,(\,i\,\partial_\tau\,)^{s+1} \qquad (7.22)
\end{aligned}
$$

$$
\begin{aligned}
[\,\mathcal{P}^{(SM)}]_{\imath}{}^{k} &= i^{1-\alpha}\cdot i^{\left\lfloor\frac{N}{2}\right\rfloor}\,\frac{1}{N!}\,\varepsilon^{I_1\cdots I_N} \\
&\quad \times \sum_{s=0}^{\left\lfloor\frac{N}{2}\right\rfloor} \binom{N}{2\,s}\, D_{I_N}\cdots D_{I_{2\,s+1}}\,(\,f_{I_{2\,s}\cdots I_1}\,)_{\imath}{}^{\jmath}\,(\,i\,\partial_\tau\,)^{s-\frac{1}{2}N}\ . \quad (7.23)
\end{aligned}
$$

can act upon any Salam-Strathdee superfield element of $\mathcal{V}_L$.

8   Conclusions

We have illustrated quite explicitly how the dynamics associated with arbitrary-N Scalar supermultiplets can be described using superspace actions involving unconstrained prepotential superfields. The prepotentials typically include many spurious degrees of freedom upon which their action (7.15) does not depend, and which describe an inherent gauge structure. We have also described a methodology, based on these developments, which should, in principle, enable one to construct prepotential descriptions of any one-dimensional supermultiplet. This methodology associates one prepotential superfield to each hook, *i.e.*, sink, on the Adinkra corresponding to the supermultiplet, where the $\mathcal{GR}(\mathrm{d}, N)$ or $\mathrm{SO}(N)$ assignments of these supermultiplets correlate with the corresponding assignments of the Adinkra hooks.

The discovery of the explicit forms of the operators $[\widetilde{\mathcal{P}}^{(SM)}]_{\hat{j}}{}^{\hat{k}}$ and $[\widetilde{\mathcal{P}}^{(nSM)}]_{\hat{j}}{}^{\hat{k}}$ are new results and among the most important in our longer program of using the mathematical structure of garden algebras and Adinkras to penetrate the still unknown complete structure of irreducible representations of Salam-Strathdee superspace. The former of these operators yields an irreducible representation while the latter does not. It remains a major task to understand completely the structure of the representations which remain after projection with $[\widetilde{\mathcal{P}}^{(nSM)}]_{\hat{j}}{}^{\hat{k}}$. These operators are examples of 1D superprojectors similar to those introduced in [12]. In fact, we may specialize to the case of 1D, $N = 4$, the superprojector $[\widetilde{\mathcal{P}}^{(SM)}]_{\hat{j}}{}^{\hat{k}}$ above. This result may then be compared with the dimensional reduction on a 0-brane of the 4D, $N = 1$ superprojectors given in equation (3.11.18) of [8]. Such an investigation will be undertaken at some future date in order, at least in this special case, to unravel the representations contained in $[\widetilde{\mathcal{P}}^{(nSM)}]_{\hat{j}}{}^{\hat{k}}$. Summarizing this aspect of the present work, we may say that we have presented the first existence proof for extending the concept of superprojectors to 1D arbitrary N-extended Salam-Strathdee superfields. Further exploration of this topic is of vital importance to our future studies.

Our work can also be used to highlight another issue for future study. We have shown by starting from an action of the form of (7.11), it is possible to reach one of the form of (7.15). In the first of these actions, $S^{\hat{i}}$ represents a Top Adinkra while $\Psi_{\hat{i}}$ represents a Base Adinkra. This two objects have other names in the conventional discussion of superfield theories. The former are known as "unconstrained prepotentials" while the latter are called "superfield field strengths". It is a fact, that in every successful quantization of a supersymmetrical theory *in which supersymmetry is manifest in all steps*, there always occur Top Adinkras that allow the passage from the analogs of (7.11) to (7.15). Thus, Top Adinkras are vital in all known manifestly supersymmetrical quantization procedures. This naturally raises a question, "If the Base Adinkra is replaced by some other Adinkra, is it always possible to begin with

the analog (7.11) and arrive at the analog of (7.15)?" If the answer is "No", then such a theory cannot be quantized in a manner that keeps supersymmetry manifest by any known method. We believe that such a question is very relevant to the issue of off-shell central charges, a very old topic in the supersymmetry literature.

> *"The heart's a startin',*
> *And this crown comment, the action so meant*
> *To be used but could not, now spry to foment!"*
> *— Shawn Benedict Jade*

A   Explicit Computation

This Appendix includes an explicit derivation of the projection operators appearing in (7.14), in the specific case where N is even, so that the prepotentials are fermionic. Thus $\mathcal{S}_{\hat{\imath}} := \mathcal{F}_{\hat{\imath}}$. The calculation for odd N is similar.

It is well known that a superspace integration $\int d^N\theta$ is equivalent to taking the $\theta^I \to 0$ limit of the Nth superspace derivative,

$$\int d^N\theta\, \mathcal{L} = (-1)^{\lfloor N/2 \rfloor}\, D^N\, \mathcal{L}\, | \,, \tag{A.1}$$

where $D^N := \frac{1}{N!}\, \varepsilon^{I_1\cdots I_N}\, D_{I_1}\cdots D_{I_N}$. Accordingly, we can re-write (7.11) as $S = \int d\tau\, L$, where the component Lagrangian is given by

$$L = (-1)^{\lfloor N/2 \rfloor}\, i^{1-\alpha}\, i^{\left\lfloor \frac{N}{2} \right\rfloor}\, \tfrac{1}{2}\, D^N\, (\,\mathcal{S}^{\hat{\imath}}\, \dot{\Psi}_{\hat{\imath}}\,)\, | \,. \tag{A.2}$$

Now, if we distribute the N derivatives in D^N by operating to the right, we obtain

$$L = i^{1-\alpha}\, i^{\left\lfloor \frac{N}{2} \right\rfloor}\, \tfrac{1}{2}\, \tfrac{1}{N!}\, \varepsilon^{I_1\cdots I_N}$$
$$\times \sum_{p=0}^{N} (-1)^{p\,(1-\alpha)} \binom{N}{p} (\, D_{I_N}\cdots D_{I_{p+1}}\, \mathcal{S}^{\hat{\imath}}\,) (\, D_{I_p}\cdots D_{I_1}\, \dot{\Psi}_{\hat{\imath}}\,) \Big| \,. \tag{A.3}$$

If we then use the results (7.6), we can re-write this as

$$L = i^{1-\alpha} i^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N}$$

$$\times \left(\sum_{p \text{ odd}} (-1)^{1-\alpha} \binom{N}{p} (D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (-i)^{(p-1)/2} (\tilde{f}_{I_1 \cdots I_p})_{\hat\imath}{}^i \partial_\tau^{(p+1)/2} \ddot\Phi_i \right.$$

$$\left. + \sum_{p \text{ even}} \binom{N}{p} (D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (-i)^{p/2} (\tilde{f}_{I_1 \cdots I_p})_{\hat\imath}{}^{\hat\jmath} \partial_\tau^{p/2} \dot\Psi_{\hat\jmath} \right) \bigg|. \qquad (A.4)$$

Integrating by parts, this becomes

$$L \cong i^{1-\alpha} i^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N} \left((-1)^{1-\alpha} \right.$$

$$\times \sum_{p \text{ odd}} i^{(p-1)/2} \binom{N}{p} (\partial_\tau^{(p-1)/2} D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (\tilde{f}_{I_1 \cdots I_p})_{\hat\imath}{}^i \ddot\Phi_i$$

$$\left. + \sum_{p \text{ even}} i^{p/2} \binom{N}{p} (\partial_\tau^{p/2} D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (\tilde{f}_{I_1 \cdots I_p})_{\hat\imath}{}^{\hat\jmath} \dot\Psi_{\hat\jmath} \right) \bigg|. \qquad (A.5)$$

Using a symmetry property, $(\tilde{f}_{I_1 \cdots I_p})_{\hat\imath}{}^j = -(f_{I_p \cdots I_1})^j{}_{\hat\imath}$, which follows from the definitions (4.4) and the property $L_I = -R_I^T$, this becomes

$$L = i^{1-\alpha} i^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N}$$

$$\times \left(-(-1)^{1-\alpha} \sum_{p \text{ odd}} i^{(p-1)/2} \binom{N}{p} (\partial_\tau^{(p-1)/2} D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (f_{I_p \cdots I_1})^i{}_{\hat\imath} \ddot\Phi_i \right.$$

$$\left. + \sum_{p \text{ even}} i^{p/2} \binom{N}{p} (\partial_\tau^{p/2} D_{I_N} \cdots D_{I_{p+1}} \mathcal{S}^{\hat\imath}) (\tilde{f}_{I_p \cdots I_1})^{\hat\jmath}{}_{\hat\imath} \dot\Psi_{\hat\jmath} \right) \bigg|. \qquad (A.6)$$

Redefining the dummy indices which are summed over, we can re-write this as

$$L = i^{1-\alpha} i^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2} \frac{1}{N!} \varepsilon^{I_1 \cdots I_N}$$

$$\times \left(-(-1)^{1-\alpha} \sum_{s=0}^{\lfloor \frac{N-1}{2} \rfloor} i^s \binom{N}{2s+1} (\partial_\tau^s D_{I_N} \cdots D_{I_{2s+2}} \mathcal{S}^{\hat\imath}) (f_{I_{2s+1} \cdots I_1})^i{}_{\hat\imath} \ddot\Phi_i \right.$$

$$\left. -i \sum_{s=0}^{\lfloor \frac{N}{2} \rfloor} i^{s+1} \binom{N}{2s} (\partial_\tau^s D_{I_N} \cdots D_{I_{2s+1}} \mathcal{S}^{\hat\imath}) (\tilde{f}_{I_{2s} \cdots I_1})^{\hat\jmath}{}_{\hat\imath} \dot\Psi_{\hat\jmath} \right) \bigg|. \qquad (A.7)$$

This form is organized as

$$L = \left(-\tfrac{1}{2} (\mathcal{O}_\phi \mathcal{S})^i \ddot\Phi_i - \tfrac{1}{2} i (\mathcal{O}_\psi \mathcal{S})^{\hat\imath} \dot\Psi_{\hat\imath} \right) |. \qquad (A.8)$$

By comparing (A.7) to (A.8), we can read off the definitions of $(\mathcal{O}_\phi)_i{}^{\hat{j}}$ and $(\mathcal{O}_\psi)_i{}^{\hat{j}}$, with the result given in (7.14).

B   From Maxwell Theory to Projectors

The purpose of this appendix is to demonstrate within the simplest known gauge theory–Maxwell theory–that the presence of the kinetic energy term in the action of necessity leads to the existence of projection operators. In order to illustrate this property, it may be useful to review this process in the more familiar arena of 4D non-supersymmetric Maxwell theory. The usual action can be written in the form

$$\mathcal{S}_{Maxwell} = -\tfrac{1}{4} \int d^4x \; F^{\underline{a}\,\underline{b}} F_{\underline{a}\,\underline{b}} \;=\; \tfrac{1}{2} \int d^4x \; A^{\underline{a}}[\delta_{\underline{a}}{}^{\underline{b}} \partial^{\underline{c}} \partial_{\underline{c}} \;-\; \partial_{\underline{a}} \partial^{\underline{b}}]A_{\underline{b}}$$

$$\equiv \tfrac{1}{2} \int d^4x \; A^{\underline{a}} \mathcal{K}_{\underline{a}}{}^{\underline{b}} A_{\underline{b}} \tag{B.1}$$

and a simple calculation reveals

$$\mathcal{K}_{\underline{a}}{}^{\underline{b}} \mathcal{K}_{\underline{b}}{}^{\underline{c}} \;=\; \partial^{\underline{d}} \partial_{\underline{d}} \mathcal{K}_{\underline{a}}{}^{\underline{c}} \;=\; \Box \, \mathcal{K}_{\underline{a}}{}^{\underline{c}} \quad . \tag{B.2}$$

This implies that a new operator may be defined

$$\mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{b}} \;=\; \frac{1}{\Box} \, \mathcal{K}_{\underline{a}}{}^{\underline{b}} \quad \rightarrow \quad \mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{b}} \mathcal{P}^{(T)}{}_{\underline{b}}{}^{\underline{c}} \;=\; \mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{c}} \quad , \tag{B.3}$$

and it is also well known that there exist solutions $\delta A_{\underline{a}}$ which satisfy $\mathcal{K}_{\underline{a}}{}^{\underline{b}} \delta A_{\underline{b}} = 0$. These solutions can be written as

$$\delta A_{\underline{b}} \;=\; \left[\delta_{\underline{b}}{}^{\underline{c}} \;-\; \mathcal{P}^{(T)}{}_{\underline{b}}{}^{\underline{c}} \right] \Lambda_{\underline{c}} \;\equiv\; \mathcal{P}^{(L)}{}_{\underline{b}}{}^{\underline{c}} \Lambda_{\underline{c}} \quad , \tag{B.4}$$

which can be seen to be solution upon using the second result in (B.3). Upon use of the definition of $\mathcal{P}^{(T)}$, this takes the form

$$\delta A_{\underline{b}} \;=\; \left[\delta_{\underline{b}}{}^{\underline{c}} \;-\; [\delta_{\underline{b}}{}^{\underline{c}} \;-\; \frac{1}{\Box} \, \partial_{\underline{b}} \partial^{\underline{c}}] \right] \Lambda_{\underline{c}} \;=\; \partial_{\underline{b}} \{ \frac{1}{\Box} \, \partial^{\underline{c}} \Lambda_{\underline{c}} \} \quad , \tag{B.5}$$

After making a field definition $\Lambda_{\underline{c}} = \partial_{\underline{c}} \Lambda$, this takes the familiar form of a Maxwell gauge transformation $\delta A_{\underline{b}} = \partial_{\underline{b}} \Lambda$. The operators $\mathcal{P}^{(T)}$ and $\mathcal{P}^{(L)}$ are projection operators since

$$\mathcal{P}^{(L)}{}_{\underline{a}}{}^{\underline{b}} \mathcal{P}^{(L)}{}_{\underline{b}}{}^{\underline{c}} \;=\; \mathcal{P}^{(L)}{}_{\underline{a}}{}^{\underline{c}} \quad , \quad \mathcal{P}^{(L)}{}_{\underline{a}}{}^{\underline{b}} \mathcal{P}^{(T)}{}_{\underline{b}}{}^{\underline{c}} \;=\; 0 \;=\; \mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{b}} \mathcal{P}^{(L)}{}_{\underline{b}}{}^{\underline{c}} \quad ,$$

$$\mathcal{P}^{(L)}{}_{\underline{a}}{}^{\underline{b}} + \mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{b}} \;=\; \delta_{\underline{a}}{}^{\underline{b}} \quad . \tag{B.6}$$

The first projector $\mathcal{P}^{(T)}$ is known as the "transverse" projector and the second $\mathcal{P}^{(L)}$ is known as the "longitudinal" projector. The usual Maxwell action can thus be written as

$$\mathcal{S}_{Maxwell} = \tfrac{1}{2} \int d^4x \; A^{\underline{a}} \, \Box \; \mathcal{P}^{(T)}{}_{\underline{a}}{}^{\underline{b}} A_{\underline{b}} \tag{B.7}$$

Although we picked Maxwell theory to show the relation between the kinetic energy operator in a gauge theory and the presence of projection operators, any theory defined over an ordinary manifold may be chosen as the starting point of a similar discussion. Projection operators are a ubiquitous feature of gauge theory. The only feature that may surprise the reader is that in the context of a supersymmetrical theory, even if none of the component fields in the theory (as those in the Scalar supermultiplet) are gauge fields, none the less, these fields are the components of a *gauge superfield*.

References

[1] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, G. Landweber: *On Graph Theoretic Identifications of Adinkras, Supersymmetry Representations and Superfields*, math-ph/0512016.

[2] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, G. Landweber: *Off-Shell Supersymmetry and Filtered Clifford Supermodules*, math-ph/0603012.

[3] M. Faux and S. J. Gates, Jr.: *Adinkras: A Graphical Technology for Supersymmetric Representation Theory*, Phys. Rev. **D71** (2005), 065002, hep-th/0408004.

[4] S. J. Gates, Jr. and S. V. Ketov: *2D (4,4) Hypermultiplets (II): Field Theory Origins of Duality*, Phys. Lett. **B418** (1998), no. 1–2, 119–124, hep-th/9504077.

[5] S. J. Gates, Jr. and L. Rana: *A Theory of Spinning Particles for Large N-Extended Supersymmetry*, Phys. Lett. **B352** (1995), no. 1–2, 50–58, hep-th/9504025.

[6] S. J. Gates, Jr. and L. Rana: *A Theory of Spinning Particles for Large N-Extended Supersymmetry (II)*, Phys. Lett. **B369** (1996), no. 3–4, 262–268, hep-th/9510051.

[7] S. J. Gates, Jr., W. D. Linch, III and J. Phillips: *When Superspace Is Not Enough*, hep-th/0211034.

[8] S. J. Gates, Jr., M. T. Grisaru, M. Roček, and W. Siegel: *Superspace or One Thousand and One Lessons in Supersymmetry*, The Banjamin/Cummings Pub. Co. Inc., Reading, MA, 1983.

[9] Z. Kuznetsova, M. Rojas, and F. Toppan: *Classification of irreps and invariants of the N-extended Supersymmetric Quantum Mechanics*, JHEP **0603** (2006), 098, hep-th/0511274.

[10] D.S. Freed: *Five lectures on supersymmetry*, American Mathematical Society, Providence, RI, 1999.

[11] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D.K. Kazhdan, J.W. Morgan, D.R. Morrison, and E. Witten, Eds.: *Quantum fields and strings: a course for mathematicians. Vol. 1, 2*, American Mathematical Society, Providence, RI, 1999.

[12] S. J. Gates, Jr. and W. Siegel: *Linearized $N = 2$ Superfield Supergravity*, Nucl. Phys. **B189** (1982) 39.