

Algebro-Geometric Isomonodromic Deformations Linking Hauptmoduls: Variation of the Mirror Map

Charles F. Doran

ABSTRACT. We introduce a class of parameter-dependent q -series arising from the moduli of certain pencils of elliptic curves and rank 19 lattice polarized K3 surfaces. These series serve to “link” the various Hauptmoduls which arise as modular mirror maps (as characterized in [Dor2, Dor3]), and can in principle be computed from the functional invariants of the pencils. Each corresponds to an *algebraic* solution to an isomonodromic deformation equation which parametrizes the Hurwitz space for those invariants. The discussion is made more concrete in the context of the Painlevé VI equation.

The author’s talk at the workshop, based on a portion of his Harvard thesis [Dor1], gave an algebro-geometric characterization of when the mirror map of certain pencils of elliptic curves or K3 surfaces is a Hauptmodul. This work [Dor2], and a significant generalization of it to multiparameter K3 surface families and pencils of Calabi-Yau threefolds [Dor3], has since appeared in print.

The goal of the present paper is to provide motivation, coming from study of the variation of the mirror map in families, for another story: Algebro-geometric deformations of ordinary differential equations “linking” the uniformizing differential equations for various Hauptmoduls. The deformation theory used is the theory of isomonodromic (monodromy-preserving) deformation of Fuchsian ordinary differential equations on the one hand and the theory of Hurwitz spaces parametrizing covers of curves with specified ramification on the other.

As a mathematically comprehensive exposition unmotivated by mirror map considerations will appear elsewhere [Dor4], here we shall show how these results naturally emerge from the study of mirror maps for certain families of pencils of elliptic curves and K3 surfaces.

REMARK. As one might expect, much use is made of notions already introduced in [Dor2]. For the convenience of the reader, we will frequently make specific citations to that paper.

2000 *Mathematics Subject Classification.* Primary: 14J32, 14D05; Secondary: 34M55, 11F03.

This is the final form of the paper.

1. Mirror Maps of Calabi–Yau Pencils

Calabi–Yau manifolds possess unique, up to scaling, holomorphic forms of top dimension. Thus the period mapping for a pencil of Calabi–Yau manifolds is given by a Fuchsian ordinary differential equation on the base of the pencil—the *Picard–Fuchs differential equation* for the periods. Suppose given a regular singular point of this equation with *maximal unipotent monodromy*. Then the *mirror map* about this point is the q -series $s(q)$ in the diagram

$$\begin{array}{ccc} CY_s & \xrightarrow{\quad \text{H} \quad} & \mathbb{P}^1 \\ \downarrow & \tau(s) & \swarrow s(q) \\ \mathbb{P}^1 & \xleftarrow{\quad q := e^{2\pi i \tau} \quad} & \end{array}$$

For the purposes of this paper we will consider only mirror maps for elliptic curve and rank 19 lattice polarized K3 surface pencils. In the former case, the corresponding Picard–Fuchs equations have order two, in the latter order three. For a detailed analysis of these cases see [Dor2, §2, §4]. For numerous examples of elliptic curve and rank 19 lattice polarized K3 surface pencil mirror maps see the papers of Lian–Yau referenced therein and [Nash] for a particularly complete K3 example.

Typically in the literature one chooses coordinates on \mathbb{P}^1 so that $s(q)$ is locally holomorphic. We'll ignore that convention here to save on notation and assume instead that the point of maximal unipotent monodromy is at $\infty \in \mathbb{P}^1$. Since the mirror map depends only on the period mapping up to overall multiplication by a complex scalar, it is characterized by the *projective normal form* of the Picard–Fuchs differential equation [Dor2, Lemma 2.17]. For convenience we will always consider this normalized equation.

2. Variation of the Mirror Map in Families of Pencils

We want now to consider how the mirror map q -series varies in *families* of Calabi–Yau pencils. The condition that the mirror map be well defined requires that we restrict our attention to families where the singular fiber corresponding to the maximal unipotent monodromy point is at $\infty \in \mathbb{P}^1$. In such a situation it is natural to let the positions of the other singular fibers vary (though we might as well fix one at 1 and one at 0 using the $SL(2, \mathbb{C})$ action on the base \mathbb{P}^1). In the case of Weierstrass pencils of elliptic curves, i.e., elliptic surfaces over \mathbb{P}^1 with a section, it is known that there is a coarse moduli space parametrizing such families of pencils [Sei, Bes].

After fixing three singular fibers to lie over $\{0, 1, \infty\}$, the simplest examples in the elliptic curve case come from rational elliptic surface families with four singular fibers. An exhaustive classification (by Kodaira singular fiber types) and explicit Weierstrass presentation of rational elliptic surfaces with four singular fibers has been completed by Hertfurther [Her]. Twelve of these define one parameter families; these are listed in Table 1.

Of these twelve, all but five have a singular fiber of I_0^* type (which implies independence of the mirror maps from the variational parameter; see Section 3).

For the five families which remain we provide their Weierstrass presentations in Table 2.

TABLE 1. Singular fiber types of the twelve families.

I_1, I_1, I_4, I_0^*	I_1, I_1, II, IV^*
I_2, I_2, I_2, I_0^*	I_1, I_1, II, I_0^*
I_1, I_3, II, I_0^*	I_1, I_1, I_2, I_0^*
I_1, I_2, III, I_0^*	I_1, I_1, I_1, I_3
I_1, I_1, IV, I_0^*	I_1, I_1, I_1, III^*
I_1, I_1, III, I_0^*	I_1, I_1, I_2, IV^*
I_2, I_2, II, I_0^*	

TABLE 2. The five families with varying mirror maps.

Family	Sing. fiber types	Weierstrass presentation
1	I_1, I_1, II, IV^*	$g_2(a, s) = 3(s-1)(s-a^2)^3$ $g_3(a, s) = (s-1)(s-a^2)^4(s+a)$
2	I_1, I_1, I_2, I_0^*	$g_2(a, s) = 12s^2(s^2+as+1)$ $g_3(a, s) = 4s^3(2s^3+3as^2+3as+2)$
3	I_1, I_1, I_1, I_0^*	$g_2(a, s) = 12s^2(s^2+2as+1)$ $g_3(a, s) = 4s^3(2s^3+3(a^2+1)s^2+6as+2)$
4	I_1, I_1, I_1, III^*	$g_2(a, s) = 3s^3(s+a)$ $g_3(a, s) = s^5(s+1)$
5	I_1, I_1, I_2, IV^*	$g_2(a, s) = 3s^3(s+2a)$ $g_3(a, s) = s^4(s^2+3as+1)$

3. Linking Hauptmoduln With Mirror Map q -Series

Why do the families with an I_0^* singular fiber yield mirror maps that are independent of variation of the pencils? To answer this we must take a closer look at the mirror map of a pencil of elliptic curves. The mirror map is a local inverse, about a maximal unipotent monodromy point, to the projectivized period mapping $\tau(s)$:

$$\begin{array}{ccc} E_s & \longrightarrow & \mathcal{E} \xrightarrow{\quad \tau(s) \quad} \mathbb{H} \\ & \downarrow & \nearrow J(\tau) \\ \mathbb{P}^1 & \xrightarrow{\quad \mathcal{J}(s) \quad} & \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}^* \cong \mathbb{P}_J^1 \end{array}$$

As such it is given by the pullback of the elliptic modular function $J(\tau)$ by the rational function

$$(3.1) \quad \mathcal{J}(s) = \frac{g_2(s)^3}{g_2(s)^3 - 27g_3(s)^2},$$

Kodaira's *functional invariant*, associated with the pencil. It is easy to check that the functional invariant of any pencil in each of the seven families with I_0^* singular fibers remains constant across that family, so the mirror map will similarly fail to vary with the pencils in those cases. Equivalently one can check that the positions of the I_0^* singular fibers no longer correspond to regular singular points of the projective normalized Picard–Fuchs equations (essentially because the local monodromy about an I_0^* fiber = $-\text{Id}$). An analogous “triangle picture” exists for pencils of rank 19 lattice polarized K3 surfaces, with Kodaira's functional invariant replaced by a *generalized functional invariant*. The former is easy to compute using the Weierstrass presentation and Eq. (3.1), but given an explicit pencil of rank 19 lattice polarized K3 surfaces, the generalized functional invariant may be quite difficult to compute [Dor2, §5.4].

In [Dor2, Theorems 4.10 and 5.20] and [Dor3] we use these (generalized) functional invariants to give an algebro-geometric characterization of pencils with modular mirror maps. The condition in each case amounts to specifying that

1. all ramification of the functional invariant lies over orbifold points of the corresponding modular curve, i.e. there are no *excess* ramification points, 2. for points corresponding to singular fibers in the pencil, the ramification indices of the functional invariant properly divide the orders of the corresponding orbifold points, and,
3. all other ramification points have ramification indices a positive integer multiple of the orders of the corresponding orbifold points.

As expected, the first condition is generically violated by the functional invariants of the five families of pencils in Table 2. However, these conditions are satisfied for certain values of the deformation parameter a . Taking the simplest case of the first family, we find the functional invariant

$$(3.2) \quad J(a, s) = -\frac{1}{(a+1)^2} \frac{(s-1)(s-a^2)}{s}$$

with ramification at $s = \{1, a^2\}$ over $0, s = -a$ over 1, $s = \{0, \infty\}$ over ∞ , and

$$s = a \text{ over } \left(\frac{a-1}{a+1}\right) \in \mathbb{P}^1.$$

If we let $a = 0$, then $s_0(q) = 1 - J(q)$, a Hauptmodul for $\text{PSL}(2, \mathbb{Z})$.

Even when the family of functional invariants is known, it seems quite difficult in general to explicitly determine the algebraically varying family of q -series linking Hauptmoduls. We can, however, be more specific in this simplest nontrivial example; the corresponding family of mirror maps is

$$\begin{aligned} s_a(q) = & -\frac{(a+1)^2}{q} - \frac{(743(a+1)^2 + 2a)}{(743(a+1)^2 + 2a)} \\ & - \frac{(196384(a+1)^4 - a^2)}{(a+1)^2} q - \frac{(21493760(a+1)^6 + a^2(743(a+1)^2 + 2a))}{(a+1)^4} q^2 \\ & - \frac{(864299970(a+1)^8) - a^2(355165(a+1)^4) - a^3(2972(a+1)^2) - 5a^4}{(a+1)^6} q^3 \\ & - \dots \end{aligned}$$

If we consider instead, say, the third family, then after normalizing the positions of two of the I_1 singular fibers at $s = 0$ and ∞ , and the I_3^* singular fiber at $s = 1$, one

can check that the family of mirror maps $s_a(q)$ algebraically “links” Hauptmoduls $s_{-2}(q)$ for $\Gamma_0(4)$ and $s_{-5/3}(q)$ for $\Gamma_0(3)$.

REMARK 3.1. Torsion-free genus zero congruence subgroups have recently been classified, and the modular relations between their Hauptmoduls and $J(q)$ determined [SeMc]. In light of the resulting list of functional invariants for the corresponding Weierstrass fibrations (with semistable Kodaira fibers and modular mirror maps), one should certainly attempt to explicitly link these Hauptmoduls through elliptic surface moduli.

4. Geometric Isomonodromic Deformations From Moduli

In light of the difficulty of presenting these families, especially in the case of rank 19 lattice polarized K3 surface pencils, we seek a way to describe the algebraic dependence of a family while still allowing one to recover the mirror map.

It turns out that the excess ramification points of the functional invariants of these pencils have a particularly nice interpretation in the context of their projective normalized Picard–Fuchs differential equations. Excess ramification points correspond to *apparent singularities* of the differential equations, i.e., regular singular points about which the local monodromy generator is $= \text{Id}$. As Kodaira's singular fiber types pin down the local monodromies (up to global conjugacy) we know that the corresponding family of Picard–Fuchs equations will be *isomonodromic*, i.e. monodromy-preserving, and consequently that it defines an algebraic solution to an isomonodromic deformation equation. In fact this algebraic isomonodromy interpretation is true much more generally, stemming from the integrability and algebricity of the Gauss–Manin connection, and we call such solutions *geometric* [Dor1, Dor4].

Forgoing here a complete discussion of the theory of isomonodromy of Fuchsian ordinary differential equations (see [Dor4] for an ample summary, or [IKSY] for a comprehensive treatment), we will be quite specific in the simplest case, in which the above five families yield explicit algebraic solutions of the Painlevé VI equation. The Painlevé VI equation is a nonlinear ordinary differential equation PVI($\alpha, \beta, \gamma, \delta$) specified by four complex parameters. Solutions of the differential equation correspond to monodromy-preserving families of second order Fuchsian ordinary differential equations with five regular singular points, one of which is apparent, where three of the non-apparent singularities have been normalized to lie at $\{0, 1, \infty\} \subset \mathbb{P}^1$. If we denote the position of the fourth non-apparent singularity by t and the position of the apparent singularity by λ , then the Painlevé VI equation reads:

$$\begin{aligned} \frac{d^2\lambda}{dt^2} = & \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \left(\frac{d\lambda}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} \\ & + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{\lambda^2} + \gamma \frac{t-1}{(\lambda-1)^2} + \delta \frac{t(t-1)}{(\lambda-t)^2} \right). \end{aligned}$$

The eigenvalues of the monodromies about the singularities of the Fuchsian equations being deformed determine the values of α, β, γ , and δ . Now by checking monodromies and plugging in the expressions for the positions of the extra apparent singularities and freely moving fourth regular singular point for each of the five families, we find the full list of geometric solutions to the Painlevé VI equation given in Table 3. It can be shown that this list, coming

from rational Weierstrass fibrations, includes all possible solutions from Weierstrass fibrations over \mathbb{P}^1 [Dor1, Dor4].

We have now effectively separated the “algebraic variation” of the mirror map from the Fuchsian equations whose solutions determine it. These two quantities are still connected though, as the algebraic equations so obtained solve the corresponding isomonodromic deformation equation.

5. Algebraic Isomonodromic Deformations From Hurwitz Curves

In fact, the methods used in the previous section suggest a generalization beyond even variation of mirror maps. All we really need to compute our algebraic Painlevé VI solutions is a family of rational functions mapping to a genus zero orbifold uniformized curve with certain very special conditions on the ramifications of each general member. Moduli spaces for the problem of placing such conditions on the ramification of rational functions are called *Hurwitz spaces*. What we are saying is that the geometric Painlevé VI solutions described in Table 3 come from specially parametrized Hurwitz curves. So we ask: Under what conditions on ramification and on a choice of orbifold \mathbb{P}^1 can we obtain such a solution to a Painlevé VI equation by “pulling back” the uniformizing differential equation on the orbifold? Let \mathcal{L} be a rank 2 regular local system on

$$\mathbb{P}^1 \setminus B = \mathbb{P}^1 \setminus \{p_1, \dots, p_m, q_1, \dots, q_s\}$$

with local monodromies of orders $a_1, \dots, a_m, \infty, \dots, \infty$. Assume that the associated rank two Fuchsian ordinary differential equation on \mathbb{P}^1 has no singular points outside of $B = \{p_1, \dots, p_m, q_1, \dots, q_s\}$. Define the *topological type* of a covering of $\mathbb{P}^1 \setminus B$ to be the set of data

$$(n; r_1, \dots, r_{m+s}; l)$$

where n is the number of sheets in the cover, l is the ramification index of the unique (an assumption) ramification point not over B , and r_i are partitions of n describing the monodromy of the cover. Let

- $\beta_i = \#$ unramified points over p_i .
- $\gamma_i = \beta_i + \gamma_i = \#$ points over p_i .
- $\mu_i = \#$ points lying over p_i with ramification a multiple of a_i by a positive integer, and
- $d_k = \#$ points over q_k .

Consider a cover such that

$$\sum_{i=1}^m (r_i - \mu_i) + \sum_{k=1}^s d_k = 4$$

and

$$\sum_{i=1}^m \mu_i = (m+s-2)n + l - 3$$

defining a one parameter family of genus zero covers

$$\pi_t: \mathbb{P}_z^1 \rightarrow \mathbb{P}_x^1,$$

where $t \in \mathbb{P}_z^1 \setminus \{0, 1, \infty\}$, $\pi_t(1), \pi_t(\infty) \in \{p_1, \dots, p_m, q_1, \dots, q_s\}$

and $\pi_t(\lambda) \in \mathbb{P}^1 \setminus B$. Then our question is answered by

THEOREM 5.1 ([Dor4]). *The pulled back family of projective normalized local systems $\overline{\pi^*(\mathcal{L})}$ forms an isomonodromic family with five regular singular points, exactly one apparent (at $\lambda(t)$), and hence determines an algebraic solution $\lambda(t)$ to the Painlevé VI equation $\text{PVI}(\alpha, \beta, \gamma, \delta)$. The particular values of $\alpha, \beta, \gamma, \delta$ are determined as usual by the traces of the local monodromies about 0, 1, ∞ , t , i.e., here they are specified by the local monodromies of \mathcal{L} about $p_1, \dots, p_m, q_1, \dots, q_s$, and the ramification data of (any) π_t , $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.*

To make this still more concrete, we’ll give the complete answer for orbifold \mathbb{P}^1 ’s given by quotients of the upper half plane by arithmetic triangle groups. Here, as with the case of the J -line (orbifold for the arithmetic triangle group $\text{PSL}(2, \mathbb{Z})$), we have $m = 2, s = 1$. But now instead of orbifold orders 2, 3, and ∞ , we have various other triples. The classification of all 85 types is due to Takeuchi [Tak]. Using his list of triples and Theorem 5.1 we find

COROLLARY 5.2 ([Dor4]). *The following is the complete list of topological types corresponding to algebraic Painlevé VI solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

$$\begin{aligned} &(2; [2], [1, 1], [1, 1]; 2) \\ &(3; [2, 1], [3], [1, 1, 1]; 2) \\ &(4; [2, 2], [3, 1], [2, 1, 1]; 2) \\ &(4; [2, 2], [4], [1, 1, 1, 1]; 2) \\ &(6; [2, 2, 2], [3, 3], [2, 2, 1, 1]; 2) \\ &(6; [2, 2, 2], [3, 3], [3, 1, 1, 1]; 2) \\ &(10; [2, \dots, 2], [3, 3, 3, 1], [7, 1, 1, 1]; 2) \\ &(12; [2, \dots, 2], [3, 3, 3, 3], [7, 2, 1, 1]; 2) \\ &(12; [2, \dots, 2], [3, 3, 3, 3], [8, 1, 1, 1]; 2) \\ &(18; [2, \dots, 2], [3, \dots, 3], [7, 7, 1, 1, 1]; 2) \end{aligned}$$

Here * represents any of the possible entries as listed in [Tak, Table (1)].

Note that applying this to $(2, 3, \infty)$ we recover a list of the topological types of the five families.

REMARK 5.3. It is reasonable to ask if one can pass from the data of the topological type of covers defining a Hurwitz curve to an explicit algebraic parametrization, i.e., for our application, to an explicit algebraic solution to Painlevé VI. In fact, a method of Couveignes may be applied to perform such computations [Cou].

TABLE 3. Equations for the geometric PVI solutions from the five families.

Sol'n.	Polynomial equation ($\lambda = \text{apparent}, t = \text{free}$)	PVI($\alpha, \beta, \gamma, \delta$)
1A	$\lambda^2 - t$	(0, 0, $\frac{1}{18}, \frac{4}{9}$) ($\frac{1}{18}, -\frac{1}{18}, 0, \frac{1}{2}$)
1B	$\lambda^2 - 2\lambda + t$	($\frac{1}{18}, 0, \frac{1}{18}, \frac{1}{2}$) (0, $-\frac{1}{18}, 0, \frac{4}{9}$)
1C	$\lambda^2 - 2\lambda t + t$	($\frac{1}{18}, 0, 0, \frac{4}{9}$) (0, $-\frac{1}{18}, \frac{1}{18}, \frac{1}{2}$)
2A	$\lambda^2 - t$	(0, 0, 0, $\frac{1}{2}$)
2B	$\lambda^2 - 2\lambda + t$	(0, 0, 0, $\frac{1}{2}$)
2C	$\lambda^2 - 2\lambda t + t$	(0, 0, 0, $\frac{1}{2}$)
3A	$\lambda^4 - 6\lambda^2 t + 4\lambda t + 4\lambda^2 t^2 - 3t^2$	(0, 0, 0, $\frac{1}{2}$)
3B	$3\lambda^4 - 4\lambda^3 - 4\lambda^3 t + 6\lambda^2 t - t^2$	(0, 0, 0, $\frac{1}{2}$)
3C	$\lambda^4 - 4t\lambda^3 + 6t\lambda^2 - 4t^2\lambda + t^2$	(0, 0, 0, $\frac{1}{2}$)
3D	$\lambda^4 - 4t\lambda^3 + 6t\lambda^2 - 4t\lambda + t^2$	(0, 0, 0, $\frac{1}{2}$)
4A	$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t$	(0, $-\frac{1}{18}, 0, \frac{1}{2}$)
4B	$\lambda^4 - 2t\lambda^3 + 2t^2\lambda - t^3$	(0, 0, $\frac{1}{18}, \frac{1}{2}$)
4C	$\lambda^4(t^2 - t + 1) - 2\lambda^3(t + 1) + 6t^2\lambda^2 - 2\lambda t^2(t + 1) + t^3$	($\frac{1}{18}, 0, 0, \frac{1}{2}$)
4D	$\lambda^4 - 2\lambda^3 + 2t\lambda - t$	(0, 0, 0, $\frac{4}{9}$)
5A	$-2\lambda^3 + 3t\lambda^2 + 3\lambda^2 - 6t\lambda + t^2 + t$	(0, $-\frac{1}{18}, 0, \frac{1}{2}$)
5B	$\lambda^3 - 3\lambda^2 + 3t\lambda - 2t^2 + t$	(0, $-\frac{1}{18}, 0, \frac{1}{2}$)
5C	$\lambda^3 - 3t\lambda^2 + 3t\lambda + t^2 - 2t$	(0, $-\frac{1}{18}, 0, \frac{1}{2}$)
5D	$2\lambda^3 - 3t\lambda^2 + t^2$	(0, 0, $\frac{1}{18}, \frac{1}{2}$)
5E	$\lambda^3 - 3t\lambda + t^2$	(0, 0, $\frac{1}{18}, \frac{1}{2}$)
5F	$\lambda^3 - 3t\lambda^2 + 3t\lambda - t^2$	(0, 0, $\frac{1}{18}, \frac{1}{2}$)
5G	$\lambda^3(2 - t) - 3t\lambda^2 + 3t^2\lambda - t^2$	($\frac{1}{18}, 0, 0, \frac{1}{2}$)
5H	$\lambda^3(t + 1) - 6t\lambda^2 + 3t(t + 1)\lambda - 2t^2$	($\frac{1}{18}, 0, 0, \frac{1}{2}$)
5I	$(1 - 2t)\lambda^3 + 3t\lambda^2 - 3t\lambda + t^2$	($\frac{1}{18}, 0, 0, \frac{1}{2}$)
5J	$\lambda^3 - 3t\lambda^2 + 3t\lambda - t$	(0, 0, 0, $\frac{4}{9}$)
5K	$\lambda^3 - 3t\lambda + 2t$	(0, 0, 0, $\frac{4}{9}$)
5L	$2\lambda^3 - 3t\lambda^2 + t$	(0, 0, 0, $\frac{4}{9}$)

References

- [Bes] A. Besser, *Elliptic fibrations of K3 surfaces and QM Kummer surfaces*, Math. Z. 228 (1998), 283–308.
- [Coo] J.-M. Couveignes, *Tools for the computation of families of coverings*, Aspects of Galois Theory, London Math. Soc. Lecture Note Ser., vol. 256, Cambridge Univ. Press, 1999, pp. 38–65; <http://www.ufr-mi.u-bordeaux.fr/~couveign/>.
- [Dor1] C. F. Doran, *Picard-Fuchs uniformization and geometric isomonodromic deformations: Modularity and variation of the mirror map*, Harvard thesis, 1999.
- [Dor2] ———, *Picard-Fuchs uniformization: Modularity of the mirror map and mirror-moonshine*, The Arithmetic and Geometric of Algebraic Cycles (Banff, 1998), Centre de Recherches Mathématiques, CRM Proc. and Lecture Notes, vol. 24, Amer. Math. Soc., Providence, RI, 2000, pp. 257–281.
- [Dor3] ———, *Picard-Fuchs uniformization and modularity of the mirror map*, Comm. Math. Phys. 212 (2000), 625–647.
- [Dor4] ———, *Algebraic and geometric isomonodromic deformations*, J. Diff. Geom. (to appear).
- [Her] S. Hertl, *Elliptic surfaces with four singular fibres*, Math. Ann. 291 (1991), no. 2, 319–342.
- [IKSY] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida, *From Gauss to Painlevé: A modern theory of special functions*, aspects of mathematics, vol. E16, Friedr. Vieweg & Sohn, Braunschweig, 1991.
- [JMcS] J. McKay and A. Sebbar, *J Arithmetic semistable elliptic surfaces and graphs*, this volume.
- [NaSh] N. Narumiya and H. Shiga, *The mirror map for a family of K3 surfaces induced from the simplest 3-dimensional reflexive polytope*, this volume.
- [SeMc] W. Seiler, *Moduli Elliptischer Flächen mit Schmitt*, Karlsruhe dissertation, 1982.
- [Ta] K. Takeuchi, *Commensurability classes of arithmetic triangle groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), no. 1, 201–212.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027.
E-mail address: doran@math.columbia.edu