EQUIVALENCES OF FAMILIES OF STACKY TORIC CALABI-YAU HYPERSURFACES

CHARLES F. DORAN, DAVID FAVERO, AND TYLER L. KELLY

(Communicated by Lev Borisov)

ABSTRACT. Given the same anti-canonical linear system on two distinct toric varieties, we provide a derived equivalence between partial crepant resolutions of the corresponding stacky hypersurfaces. The applications include: a derived unification of toric mirror constructions, calculations of Picard lattices for linear systems of quartics in $\mathbf{P}^3$, and a birational reduction of Reid's list to 81 families.

1. INTRODUCTION

Mirror symmetry predicts a correspondence between algebraic and symplectic geometry. This is frequently described by looking at pairs of Calabi-Yau varieties, $X, X^\vee$, whose complex and Kähler deformations are interchanged. This summary is a bit misleading: given X, there is not even a single agreed upon construction in mathematics of the "mirror" $X^\vee$. In this article, we restrict ourselves to the toric setting where several extremely explicit constructions have been put forward [GP90, BH93, Bat94, BB96, Cla17b, Kra09, ACG16].

A priori, these differing toric mirrors look quite different. They are non-isomorphic as stacks, and are hypersurfaces in distinct toric Fano varieties. Let us give a brief example.

Example 1.1. Consider the Calabi-Yau threefolds

$$X_1 := \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbf{P}^4,$$
$$X_2 := \{x_1^5 + x_2^4 x_3 + x_3^5 + x_4^4 x_5 + x_4 x_5^4 = 0\} \subset \mathbf{P}^4.$$

While the Batyrev mirrors to both X_1 and X_2 both are hypersurfaces in $\mathbf{P}^4/(\mathbb{Z}/5\mathbb{Z})^3$, their Berglund-Hübsch-Krawitz mirrors are

$$X_1^\vee = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbf{P}^4/(\mathbb{Z}/5\mathbb{Z})^3,$$
$$X_2^\vee = \{x_1^5 + x_2^4 + x_2 x_4^5 + x_4^4 x_5 + x_4 x_5^4 = 0\} \subset \mathbf{P}^4(4,5,3,4,4).$$

Note the Berglund-Hübsch-Krawitz mirrors are not even in the same toric variety and are seemingly unrelated. However, mirror symmetry intuits that the complex deformation from X_1 to X_2 corresponds to a birational transformation between $X_1^\vee$ and $X_2^\vee$. Furthermore, the homological mirror symmetry conjecture predicts that $X_1^\vee$ and $X_2^\vee$ should be derived equivalent, as X_1 and X_2 are symplectomorphic. Together, these predictions can be seen as a way of unifying the constructions above through category theory and birational geometry. With this ansatz, we will prove the following.

Theorem 1.2. *Consider two Calabi-Yau hypersurfaces X_1 and X_2 in the same toric Fano variety and suppose that one uses one of the following mirror constructions to provide mirrors $X_1^\vee$ and $X_2^\vee$ to X_1 and X_2, respectively:*

    (1) *Batyrev mirror symmetry* [Bat94]*,*
    (2) *Berglund-Hübsch-Krawitz mirror symmetry* [BH93, Kra09]*,*
    (3) *Artebani-Comparin-Guilbot mirror symmetry* [ACG16]*, and*
    (4) *Clarke mirror symmetry* [Cla17b]*.*

Then $X_1^\vee$ is birational to $X_2^\vee$ and we have an equivalence of derived categories on the level of stacks $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{X}_1^\vee) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{X}_2^\vee)$.

This question of comparing toric mirror constructions via birational and derived equivalence has much recent history. In the case of Berglund-Hübsch-Krawitz mirror duality, this question was answered on the level of birational equivalence in [Bor13, Sho14, Kel13, Cla14]. Derived equivalence of the Berglund-Hübsch-Krawitz mirrors to hypersurfaces in the same Gorenstein toric Fano variety was proven in [FK16a] from an examination of the secondary fan associated to the anti-canonical linear system. The analogous question for Batyrev-Borisov mirrors was solved for both birational equivalence [Li16, Cla17a] and derived equivalence [FK17].

In this paper, we recover and generalize these results for hypersurfaces by showing that intersecting polytopes associated to linear systems on toric varieties leads to both birational identifications and derived equivalences among Calabi-Yau Deligne-Mumford stacks. We then specialize our results to applications for mirror constructions.

To set up our technical tool, let us introduce a bit of notation. Let N and M be dual lattices and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Let $\Sigma \subseteq N_\mathbb{R}$ be a complete fan and set ∇_Σ to be the convex hull of all minimal generators $u_\rho \in N$ of the rays ρ in $\Sigma(1)$. Recall that for a given polytope $\nabla \subset N_\mathbb{R}$, we define its polar polytope ∇^* as the set

$$\nabla^* := \{m \in M_\mathbb{R} | \langle m, n \rangle \geq -1 \text{ for all } n \in \nabla\}.$$

As usual, on a toric variety X_Σ, we can use the rays $\rho \in \Sigma(1)$ to enumerate the variables x_ρ. Then each lattice point m in ∇_Σ^* corresponds to a monomial x^m via the correspondence

$$m \longleftrightarrow x^m := \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m, u_\rho \rangle + 1}.$$

Moreover, $m \in \nabla_\Sigma^*$ if and only if x^m lies in the anti-canonical linear system for the toric variety X_Σ. Hence, a set of lattice points $\Xi \subseteq \nabla_\Sigma^* \cap M$ determines an anti-canonical linear system $\mathcal{F}_\Xi$ given by the space spanned by the corresponding monomials. Rephrasing, a linear combination of elements in Ξ give a polynomial $\sum c_m x^m$ which corresponds to a hypersurface $Z_{w,\Sigma}$ in X_Σ or a stacky hypersurface $\mathcal{Z}_{w,\Sigma}$ in the Cox stack $\mathcal{X}_\Sigma$ (see Definition 2.2).

Definition 1.3. We say that a finite set of lattice points $\Xi \subseteq \nabla_\Sigma^* \cap M$ is good if $\mathrm{Conv}(\mathrm{Conv}\,\Xi^* \cap N)$ has a unique interior lattice point.

Theorem 1.4 (Theorem 2.7). *Let $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{R}$ be complete fans and let*

$$\Xi \subseteq \nabla_{\Sigma_1}^* \cap \nabla_{\Sigma_2}^* \cap M$$

be a good collection of lattice points inside the intersection of the two corresponding anti-canonical linear systems. Assume that the associated toric varieties $X_{\Sigma_1}, X_{\Sigma_2}$ are projective and that the polynomial corresponding to $w \in \mathcal{F}_\Xi$ is irreducible. Then there exist partial resolutions $\widetilde{\mathcal{Z}_{w,\Sigma_1}}, \widetilde{\mathcal{Z}_{w,\Sigma_2}}$ of the corresponding hypersurfaces $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ in the toric stacks $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ and a derived equivalence

$$\mathrm{D}^b(\mathrm{coh}\,\widetilde{\mathcal{Z}_{w,\Sigma_1}}) \cong \mathrm{D}^b(\mathrm{coh}\,\widetilde{\mathcal{Z}_{w,\Sigma_2}}).$$

Furthermore, Z_{w,Σ_1} and Z_{w,Σ_2} are birational.

In Subsection 3.1, we describe when the resolutions are unnecessary and/or when the family is birational and derived equivalent to a hypersurface in a Fano Gorenstein toric variety. In Subsection 3.2, we examine Reid's list of the 95 weighted-projective 3-spaces where the generic anti-canonical hypersurface has at most Gorenstein singularities. While only fourteen of the weighted-projective spaces correspond to reflexive polytopes, i.e., are Gorenstein toric Fano varieties, each family of K3 surfaces in a weighted-projective 3-space on the list corresponds to the generic family of K3 surfaces on some Gorenstein toric Fano 3-fold. Some of these Gorenstein toric Fano 3-folds are repeated, yielding 81 distinct families after resolving singularities. In Subsection 3.3, we illustrate how our theorem can be used to realize many families of K3 surfaces as quartic linear systems in $\mathbf{P}^3$. This recovers the normal form for an $(E_8 \oplus E_7 \oplus H)$-polarized family of K3 surfaces in $\mathbf{P}^3$ discovered independently by Clingher and Doran [CD12] and by Vinberg [Vin13]. Our method, for example, provides 429 ways to realize 52 of the families on Reid's list as families of hypersurfaces in $\mathbf{P}^3$ up to birational equivalence. Finally, in Subsection 3.4, we provide a homological unification of mirror constructions, proving Theorem 1.2.

2. Derived equivalences of hypersurfaces in linear systems

2.1. **Background.** We work over a field κ of characteristic 0. Let N be a lattice. Consider a fan $\Sigma \subseteq N_\mathbb{R}$ with n rays. We can construct a new fan in $\mathbb{R}^{\Sigma(1)}$ given by

$$\mathrm{Cox}(\Sigma) := \{\mathrm{Cone}(e_\rho|\ \rho \in \sigma)|\ \sigma \in \Sigma\}.$$

Enumerating the rays, this fan is a subfan of the standard fan for $\mathbb{A}_\kappa^n := \mathbb{A}^n$:

$$\Sigma_t := \{\mathrm{Cone}(e_i|\ i \in I)|\ I \subseteq \{1, ..., n\}\}.$$

Hence, the associated toric variety $X_{\mathrm{Cox}(\Sigma)}$ over κ is an open subset of $\mathbb{A}^n$.

Definition 2.1. We call $X_{\mathrm{Cox}(\Sigma)}$ the Cox open set associated to Σ.

This quasi-affine variety also comes with the action of a subgroup of the standard torus $\mathbb{G}_m^n$ restricted from its scaling action on $\mathbb{A}^n$. This is described as follows. Let M be the dual lattice to N and consider the right exact sequence given by the divisor class map

$$(2.1) \qquad M \xrightarrow{\text{div}_\Sigma} \mathbb{Z}^n \xrightarrow{\pi} \text{Cl}(X_\Sigma) \to 0$$

$$m \mapsto \sum_{\rho \in \Sigma(1)} \langle u_\rho, m \rangle e_\rho,$$

where $\mathbb{Z}^n$ represents the group of torus invariant divisors with basis e_ρ and $\text{Cl}(X_\Sigma)$ is the divisor class group of X_Σ.

By applying $\text{Hom}(-, \mathbb{G}_m)$, we get a left exact sequence

$$0 \longrightarrow \text{Hom}(\text{Cl}(X_\Sigma), \mathbb{G}_m) \xrightarrow{\widehat{\pi}} \mathbb{G}_m^n \xrightarrow{\widehat{\text{div}_\Sigma}} \mathbb{G}_m^{\dim M}.$$

We set

$$S_\Sigma := \text{Hom}(\text{Cl}(X_\Sigma), \mathbb{G}_m) \subseteq \mathbb{G}_m^n.$$

Definition 2.2. The Cox stack associated to Σ is the global quotient stack

$$\mathcal{X}_\Sigma := [X_{\text{Cox}(\Sigma)}/S_\Sigma].$$

We can also associate a polytope to our fan

$$(2.2) \qquad \nabla_\Sigma := \text{Conv}(u_\rho | \rho \in \Sigma(1)),$$

where u_ρ is the minimal generator for a ray $\rho \in \Sigma(1)$. Recall that any polytope $\nabla \subset N_\mathbb{R}$ has a polar polytope ∇^* in $M_\mathbb{R}$:

$$\nabla^* := \{m \in M_\mathbb{R} | \langle m, n \rangle \geq -1 \text{ for all } n \in \nabla\}.$$

For any finite subset of lattice points

$$\Xi := \{m_1, ..., m_t\} \subseteq \nabla_\Sigma^* \cap M,$$

we get a linear system $\mathcal{F}_\Xi$ of anti-canonical functions on X_Σ defined by

$$w := \sum_{m \in \Xi} c_m x^m.$$

Any such element can be realized both as a hypersurface $Z_{w,\Sigma}$ in the toric variety X_Σ and as a hypersurface $\mathcal{Z}_{w,\Sigma}$ in the stack $\mathcal{X}_\Sigma$. The set of all lattice points of ∇_Σ^* gives the complete anti-canonical linear system $\mathcal{F}_{\nabla_\Sigma^* \cap M}$.

Let us recall the following theorem about derived equivalences amongst such hypersurfaces which comes from varying GIT quotients for the action of S_Σ on $\mathbb{A}^n$.

Theorem 2.3. *Let Σ_1, Σ_2 be simplicial fans with full-dimensional convex support such that*

$$\nabla_{\Sigma_1} = \nabla_{\Sigma_2}.$$

Assume that X_{Σ_1} and X_{Σ_2} are quasi-projective varieties with nef anti-canonical bundles. Then for any function $w \in \mathcal{F}_\Xi$, as above, there is an equivalence of categories,

$$\text{D}^\text{b}(\text{coh } \mathcal{Z}_{w,\Sigma_1}) \cong \text{D}^\text{b}(\text{coh } \mathcal{Z}_{w,\Sigma_2}).$$

Proof. If $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are smooth, this is a toric case of Corollary 4.5 of [Kaw05]. For toric varieties, this can be made entirely explicit using Theorem 2 and Theorem 3 of [HW12]. The result also follows from Theorem 5.2.1 of [BFK12] (version 2 on arXiv) or Cor 4.8 and Prop 5.5 of [H-L15]. For an explicit statement of the theorem in this language, specialize Corollary 5.15(3) of [FK16b] to hypersurfaces. $\square$

Proposition 2.4. *Let* $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{R}$ *be fans such that* $\nabla_{\Sigma_1} = \nabla_{\Sigma_2}$ *and*

$$w = \sum_{m \in \nabla_{\Sigma_1}^* \cap M} c_m x^m$$

be irreducible. Then, the varieties $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}$ *are birational.*

Proof. By definition, these two hypersurfaces are defined by the same function on the open dense torus. $\square$

2.2. **Blowups and resolutions.** In this section, we compare partial crepant resolutions of toric hypersurfaces by reducing to the situation of Theorem 2.3 and give a proof of the main result Theorem 1.4. First, we give the definition of a star subdivision (called a generalized star subdivision in [CLS11]).

Definition 2.5. Given a fan $\Sigma \subseteq N_\mathbb{R}$ and a primitive element $v \in |\Sigma| \cap N$, we define the star subdivision of Σ at v to be the fan $\Sigma^*(v)$ consisting of the following cones:

- σ, where $v \notin \sigma \in \Sigma$.
- $\text{Cone}(\tau, v)$, where $v \notin \tau \in \Sigma$ and $\{v\} \cup \tau \subseteq \sigma \in \Sigma$.

Recall that by doing a star subdivision, one obtains an induced projective toric morphism $X_{\Sigma^*(v)} \to X_\Sigma$ (Proposition 11.1.6 of [CLS11]) and that a toric resolution of singularities of X_Σ can always be obtained through a sequence of star subdivisions (Theorem 11.1.9 of [CLS11]). We now take a certain sequence of star subdivisions which doesn't necessarily resolve X_Σ but produces a desired structure we use later.

Proposition 2.6. *Let* $\Sigma \subseteq N_\mathbb{R}$ *be a complete fan and* $\Delta \subseteq \nabla_\Sigma^*$ *be a full dimensional lattice polytope. Then, there is a simplicial fan* Σ^{star} *with* $\nabla_{\Sigma^{\text{star}}} = \text{Conv}(\Delta^* \cap N)$ *obtained from a sequence of star subdivisions of* Σ.

Proof. First take the simplicial refinement of Σ (this is obtained by a sequence of star subdivisions; see Proposition 11.1.7 of [CLS11]). This does not change the rays of Σ hence does not alter ∇_Σ. Now, star subdivides the refinement in all the new lattice points in $\text{Conv}(\Delta^* \cap N)$ and we call the resultant fan Σ^{star}. $\square$

Theorem 2.7. *Let* $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{R}$ *be complete fans and let* $\Xi \subseteq \nabla_{\Sigma_1}^* \cap \nabla_{\Sigma_2}^* \cap M$ *be a good collection of lattice points inside the intersection of the two corresponding anti-canonical linear systems. Assume that the associated toric varieties* $X_{\Sigma_1}, X_{\Sigma_2}$ *are projective and that the polynomial corresponding to* $w \in \mathcal{F}_\Xi$ *is irreducible. Then there exist partial resolutions* $\widetilde{\mathcal{Z}_{w,\Sigma_1}}, \widetilde{\mathcal{Z}_{w,\Sigma_2}}$ *of the corresponding hypersurfaces* $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ *in the toric stacks* $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ *and a derived equivalence*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\widetilde{\mathcal{Z}_{w,\Sigma_1}}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\widetilde{\mathcal{Z}_{w,\Sigma_2}}).$$

Furthermore, Z_{w,Σ_1} *and* Z_{w,Σ_2} *are birational.*

Proof. Notice that $\nabla_{\Sigma_1}, \nabla_{\Sigma_2} \subseteq \mathrm{Conv}(\Xi)^*$. Furthermore, since they are lattice polytopes, $\nabla_{\Sigma_1}, \nabla_{\Sigma_2} \subseteq \mathrm{Conv}(\mathrm{Conv}(\Xi)^* \cap N)$.

By Proposition 2.6, we may blowup $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ to obtain $\mathcal{X}_{\Sigma_1^{\mathrm{star}}}, \mathcal{X}_{\Sigma_2^{\mathrm{star}}}$ with

$$\mathrm{Conv}(\mathrm{Conv}(\Xi)^* \cap N) = \nabla_{\Sigma_1^{\mathrm{star}}} = \nabla_{\Sigma_2^{\mathrm{star}}}.$$

Since Ξ is good (Definition 1.3), it follows that $\nabla_{\Sigma_1^{\mathrm{star}}}, \nabla_{\Sigma_2^{\mathrm{star}}}$ have a unique interior lattice point. Hence, the anti-canonical divisors on $X_{\Sigma_1^{\mathrm{star}}}, X_{\Sigma_2^{\mathrm{star}}}$ are nef. The statement then follows from Theorem 2.3 and Proposition 2.4. $\square$

Remark 2.8. In general, the stacks $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are not isomorphic, e.g., the inertia stacks may differ (see, e.g., Section 5.3 of [Kel13]).

Remark 2.9. If $\widetilde{\mathcal{Z}_{w,\Sigma_1}}$ and $\widetilde{\mathcal{Z}_{w,\Sigma_2}}$ are connected, then they have trivial canonical bundle by the adjunction formula. Hence, the partial resolutions are crepant.

Corollary 2.10. *Let $\Sigma_1, \Sigma_2 \subseteq N_{\mathbb{R}}$ be complete simplicial fans and $\Xi \subseteq (\nabla_{\Sigma_1}^* \cap \nabla_{\Sigma_2}^*) \cap M$ be a good subset. Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are projective. Suppose that $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are smooth connected hypersurfaces in $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ defined by the same function,*

$$w = \sum_{m \in \Xi} c_m x^m.$$

Then

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{w,\Sigma_1}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{w,\Sigma_2}).$$

Proof. Using Theorem 2.7, we know that $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \widetilde{\mathcal{Z}_{w,\Sigma_1}}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \widetilde{\mathcal{Z}_{w,\Sigma_2}})$, where $\widetilde{\mathcal{Z}_{w,\Sigma_i}}$ is the zero locus in $\mathcal{X}_{\Sigma_i^{\mathrm{star}}}$ as defined in Proposition 2.6. It suffices to prove that under these additional hypotheses that $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \widetilde{\mathcal{Z}_{w,\Sigma_i}}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_{w,\Sigma_i})$. Our strategy to prove this result is to compare the spaces $\tilde{\mathcal{V}} := \mathrm{tot}(-K_{X_{\Sigma_i^{\mathrm{star}}}})$ and $\mathcal{V} := \mathrm{tot}(-K_{X_{\Sigma_i}})$ using geometric invariant theory. This may not be done directly, but can be done after partially compactifying $\mathcal{V}$. The desired equivalence will then be obtained by constructing said partial compactification $X_{\bar{\Sigma}}$ of $\mathcal{V}$ which can instead be compared with $\tilde{\mathcal{V}}$ using Corollary 4.7 of [FK16a].

We now argue that there exists a semiprojective toric variety $\mathcal{X}_{\bar{\Sigma}}$ such that $\mathcal{V}$ is open (torically) in $\mathcal{X}_{\bar{\Sigma}}$ and the support of $\bar{\Sigma}$ is equal to the support of the fan for $\tilde{\mathcal{V}}$. This can be done iteratively. Starting with the fan $\Sigma \subseteq N_{\mathbb{R}} \times \mathbb{R}$ for $\mathcal{V}$, we can construct a new fan Γ by taking an external star subdivision with respect to the lattice point $(n, 1) \in N \times \mathbb{Z}$, where n is a minimal generator of a ray in $\Sigma_i^{\mathrm{star}}(1)$ but not in $\Sigma_i(1)$. Since Ξ is good, the fans for $\mathcal{V}$ have full convex support, thus $\mathcal{V}$ is semiprojective. Hence, Σ is obtained from a regular triangulation in the hyperplane $N_{\mathbb{R}} \times \{1\}$, i.e., there exist weights ω_i for all minimal generators $(u_{\rho_i}, 1) \in N_{\mathbb{R}} \times \{1\}$ so that every simplex of Σ is in $N_{\mathbb{R}} \times \{1\}$ in the lower hull, as described on page 740 of [CLS11]. By choosing a weight ω sufficiently large for the point $(n, 1)$ along with the weights ω_i, one can show that the simplices defined via the star subdivision along $(n, 1)$ create a regular triangulation of $\mathrm{Conv}((n, 1), (u_{\rho_i}, 1))$. Since this triangulation is regular, X_{Γ} is semiprojective. We iterate this process until we arrive at $X_{\bar{\Sigma}}$.

The global section w corresponds to a global function W on X_{Γ} and $\mathcal{V}$. As $\Xi \subseteq \nabla_{\Sigma_i}^*$, we have that $\Xi \times \{1\} \subseteq |\bar{\Sigma}|^{\vee}$. For any fan Ψ with support $|\bar{\Sigma}|$, we have a function W corresponding to w by taking $W = \sum_{m \in \Xi} c_m \prod_{\rho \in \Psi(1)} x_{\rho}^{\langle (m,1), u_{\rho} \rangle}$. For

the line bundles $\mathcal{V}$ and $\tilde{\mathcal{V}}$, there is one new ray in their fans compared to X_{Σ_i} and $X_{\Sigma_i^{\mathrm{star}}}$, which we denote by u corresponding to the ray generated by $(0,1)$. For these line bundles, W is simply uw.

We are now squarely in the context of Section 4 of [FK16a]. Namely, we have two fans: $\bar{\Sigma}$ and the fan Σ_i^{star} for $\tilde{\mathcal{V}}$. Both fans have ray generators in $N_{\mathbb{R}} \times \{1\}$, have the same support, and correspond to semiprojective toric varieties ($\tilde{\mathcal{V}}$ is also semiprojective since Ξ is good). Furthermore, $\mathcal{V}$ is open in $X_{\bar{\Sigma}}$. Hence, the result will now follow from Corollary 4.7 of [FK16a] assuming the conditions are satisfied. Phrased geometrically, Corollary 4.7 of [FK16a] requires that the critical locus of W does not intersect the boundary $X_{\bar{\Sigma}} \backslash \mathcal{V}$. This is shown as follows.

First, the assumption that $\mathcal{Z}_{w,\Sigma_i}$ is smooth implies that the critical locus $\partial W|_{\mathcal{V}}$ of W in $\mathcal{V}$ is contained in the zero section of the bundle. Hence, it is projective as Z_{w,Σ_i} is projective. Therefore, $\partial W|_{\mathcal{V}}$ is closed in $X_{\bar{\Sigma}}$. Furthermore,

$$\partial W|_{X_{\bar{\Sigma}}} \cap \mathcal{V} = \partial W|_{\mathcal{V}}.$$

It follows that $X_{\bar{\Sigma}} \cap \mathcal{V}$ and $X_{\bar{\Sigma}} \cap \partial W|_{\mathcal{V}}$ are open subsets which separate $\partial W|_{X_{\bar{\Sigma}}}$ into two connected components. Hence, $(X_{\bar{\Sigma}} \backslash \mathcal{V}) \cap \partial W|_{X_{\bar{\Sigma}}}$ must be empty if $\partial w|_{X_{\bar{\Sigma}}}$ is connected.

We now show that $\partial W|_{X_{\bar{\Sigma}}}$ is connected. First, since $Z_{w,\Sigma_i^{\mathrm{star}}}$ is smooth and connected, the critical locus of W is equal to $Z_{w,\Sigma_i^{\mathrm{star}}}$ inside the zero section of $\tilde{\mathcal{V}}$. In particular, it is connected. Now, the torus equivariant birational map from $\tilde{\mathcal{V}}$ to $X_{\bar{\Sigma}}$ admits a factorization (Theorem A of [Wło97])
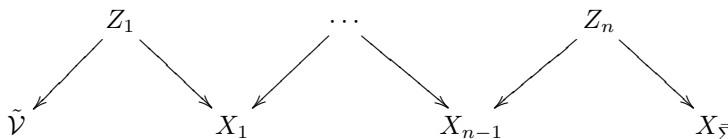
where each map in the diagram is a blowup and all maps are as varieties over $\mathbb{A}^1$ via the global function W on each space. Since ∂W is connected on $\tilde{\mathcal{V}}$ it follows that its preimage is connected in Z_1. Hence, the image is connected in X_1 but this is just ∂W on X_1. Continuing, we get that ∂W is connected in $X_{\bar{\Sigma}}$, as desired. $\square$

3. Applications and examples

3.1. Reflexivity, genericness, and smoothness.

Proposition 3.1 (Bertini, Mullet). *Let X_{Σ} be a projective toric variety. The generic stacky hypersurface in a basepoint free linear system $\mathcal{Z} \subseteq \mathcal{X}_{\Sigma}$ is smooth and connected.*

Proof. This is Proposition 6.7 of [Mul09]. $\square$

Proposition 3.2. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a simplicial fan such that ∇_{Σ} is reflexive. Assume X_{Σ} is projective. Any linear system which contains the vertices of ∇_{Σ} is basepoint free.*

Proof. Consider the special case where Σ is the normal fan of ∇_{Σ}^*. Note that any facet F_{σ} of ∇_{Σ} is the convex hull $\mathrm{Conv}(u_{\rho} \mid \rho \in \sigma(1))$ for some maximal cone σ. Since ∇_{Σ} is reflexive, there is a lattice point $m_{\sigma} \in \nabla_{\Sigma}^*$ so that $\langle m_{\sigma}, f \rangle = -1$ for all $f \in F_{\sigma}$ hence $\langle m_{\sigma}, u_{\rho} \rangle = -1$ for all $\rho \in \sigma(1)$. By Proposition 6.1.1 of [CLS11], since $m_{\sigma} \in \nabla_{\Sigma}^* = P_{-K_{X_{\Sigma}}}$, the anti-canonical divisor is basepoint free. The general

case follows: any simplicial fan Σ' that satisfies our hypotheses corresponds to toric variety $X_{\Sigma'}$ which is a blowup of the toric variety X_Σ in our special case, and we then use the fact that the pullback of a basepoint free linear system is a basepoint free linear system. $\qquad\square$

Corollary 3.3. *Let $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{R}$ be complete fans and let $\Xi \subseteq \nabla^*_{\Sigma_1} \cap \nabla^*_{\Sigma_2} \cap M$ be a finite set. Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are projective and that $\mathrm{Conv}(\Xi)$ is reflexive. Let $\widetilde{\Sigma_{\mathrm{Conv}(\Xi)}}$ be a simplicial resolution of the normal fan to $\mathrm{Conv}(\Xi)$.*

Then for a generic choice of coefficients

$$w := \sum_{m \in \Xi} c_m x^m$$

the following categories are equivalent:

$$\mathrm{D}^b(\mathrm{coh}\, \mathcal{Z}_{w,\widetilde{\Sigma_{\mathrm{Conv}(\Xi)}}}) \cong \mathrm{D}^b(\mathrm{coh}\, \widetilde{\mathcal{Z}_{w,\Sigma_1}}) \cong \mathrm{D}^b(\mathrm{coh}\, \widetilde{\mathcal{Z}_{w,\Sigma_2}}),$$

where $\widetilde{\mathcal{Z}_{w,\Sigma_i}}$ is a smooth stacky crepant resolution of the hypersurface defined by w in the stack $\mathcal{X}_{\Sigma_i}$. Moreover, $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}, Z_{w,\Sigma_{\mathrm{Conv}(\Xi)}}$ are all birational.

Proof. Since $\mathrm{Conv}(\Xi)$ is reflexive, Ξ is good. Hence, we may apply Theorem 2.7 for the comparison between the $\widetilde{\mathcal{Z}_{w,\Sigma_i}}$. The adjunction formula tells us that $\widetilde{\mathcal{Z}_{w,\Sigma_1}}, \widetilde{\mathcal{Z}_{w,\Sigma_2}}$ are smooth Deligne-Mumford stacks with trivial canonical bundle, hence the resolutions are crepant.

To compare with $\mathcal{Z}_{w,\widetilde{\Sigma_{\mathrm{Conv}(\Xi)}}}$, notice that it is smooth and connected by Propositions 3.1 and 3.2. Hence, we may apply Corollary 2.10. $\qquad\square$

We now will give a criteria for when the generic stacky anti-canonical hypersurface is smooth in a finite quotient of weighted-projective space.

Definition 3.4. Given a subset $I \subseteq \{0, \ldots, n\}$, we say that a monomial $p \in \kappa[x_0, \ldots, x_n]$ is an *I-root* if $p = \prod_{i \in I} x_i^{r_i}$ for some $r_i \in \mathbb{Z}_{>0}$. We say that p is an *I-pointer* if $p = x_j \prod_{i \in I} x_i^{r_i}$ for some $r_i \in \mathbb{Z}_{>0}$ and $j \notin I$.

Theorem 3.5 (Fletcher). *Let $X_\Sigma = \mathbf{P}^n(a_0, \ldots, a_n)/G$ where G is a finite abelian group acting multiplicatively. The generic stacky member $\mathcal{Z}_{w,\Sigma}$ of an anti-canonical linear system F_Ξ in X_Σ is smooth if for every non-empty $I \subseteq \{0, ..., n\}$ there exists an I-root or an I-pointer in Ξ.*

Proof. Theorem 8.1 of [Fl00] states the same for the generic member without a finite group action. The proof demonstrates that the sum over these monomials is smooth. Since smoothness is an open property of a linear system, the result follows. Furthermore, since smoothness is a property of polynomials in $\mathbb{A}^{n+1} \setminus 0$, it does not depend on whether or not we quotient by $\mathbb{G}_m$ or $\mathbb{G}_m \times G$. $\qquad\square$

Remark 3.6. One can also compare with similar results obtained by Kreuzer and Skarke. See Theorem 1 of [KS92].

Corollary 3.7. *Let $X_{\Sigma_1}, X_{\Sigma_2}$ be quotients weighted-projective spaces by a finite group and let $\Xi \subseteq \nabla^*_{\Sigma_1} \cap \nabla^*_{\Sigma_2} \cap M$ be a good set. Assume that for every non-empty $I \subseteq \{0, ..., n\}$ there exists an I-root or an I-pointer in Ξ for both $X_{\Sigma_1}, X_{\Sigma_2}$. Then for a generic choice of coefficients*

$$w := \sum_{m \in \Xi} c_m x^m$$

there is an equivalence of categories, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{w,\Sigma_1}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{w,\Sigma_2})$. Moreover, $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}$ are birational.

Proof. For a generic choice of coefficients, $\mathcal{Z}_{w,\Sigma_1}$, $\mathcal{Z}_{w,\Sigma_2}$ are smooth by Theorem 3.5. The result follows from Corollary 2.10. □

3.2. **Equivalence in Reid's list of 95 families.** In [KM12], Kobayashi and Mase find birational correspondences between members of Reid's list of all 95 families of K3 surfaces in weighted-projective space whose generic anti-canonical hypersurface has Gorenstein singularities. Here, we will recover these birational correspondences by considering the complete linear systems $\nabla_\Sigma^* \cap M$ for each X_Σ on Reid's list (see equation (2.2)).

Corollary 3.8. *Suppose $X_\Sigma = \mathbf{P}^3(a_1, a_2, a_3, a_4)$ is a weighted-projective space on Reid's list. The polytope $\Delta := \mathrm{Conv}(\nabla_\Sigma^* \cap M)$ is reflexive and $Z_{w,\Sigma}$ and $Z_{w,\widetilde{\Sigma_\Delta}}$ are birational if $w \in \mathcal{F}_{\nabla_\Sigma^* \cap M}$ is irreducible, where $\widetilde{\Sigma_\Delta}$ is a simplicial resolution of the normal fan to Δ. Hence, generically, the minimal resolutions of $Z_{w,\Sigma}$ and $Z_{w,\widetilde{\Sigma_\Delta}}$ are isomorphic K3 surfaces. Finally, generically, the stacks $\mathcal{Z}_{w,\Sigma}$ and $\mathcal{Z}_{w,\widetilde{\Sigma_\Delta}}$ are smooth and there is an equivalence of categories, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{w,\Sigma}) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_{w,\widetilde{\Sigma_\Delta}})$.*

Proof. Proof of reflexivity and smoothness is exhaustive using Macaulay 2 [GS] and Theorem 3.5.[1] The rest follows directly from Corollary 3.3. □

The weighted projective space $X_\Sigma = \mathbf{P}^3(a_1, a_2, a_3, a_4)$ is Gorenstein if and only if a_i divides $\sum a_i$ for all i. This is also equivalent to ∇_Σ being reflexive, in which case $\Delta = \nabla_\Sigma^*$. Only the first 14 weighted-projective spaces on Reid's list are Gorenstein. The remaining 81 non-Gorenstein weighted-projective spaces have generic anti-canonical hypersurfaces which are birational to generic anti-canonical hypersurfaces in a (different) Gorenstein Fano toric variety. After crepantly resolving, the associated stacks are derived equivalent and generically smooth.

There are several weighted-projective 3-spaces that correspond to the same Δ. Using the indexing of reflexive polytopes in [Sage], we organize these cases into the following table:

Reid's families	index(Δ)	Reid's families	index(Δ)
14, 28, 45, 51	4080	43, 48	745
20, 59	3045	46, 65, 80	88
26, 34	1114	50, 82	4147
27, 49	1949	56, 73	2
38, 77	3731	68, 83, 92	221

In particular, all families in the same row are pointwise birational and derived equivalent after resolving.

Remark 3.9. There are, in fact, 104 weighted projective 3-spaces such that the generic member of the complete anti-canonical linear system $\mathcal{Z}_{w,\Sigma}$ is smooth. Indeed, in dimensions 2 and 3 this is equivalent to $\mathrm{Conv}(\Xi)$ being reflexive where F_Ξ is the complete anti-canonical linear system. In higher dimensions this is no longer true. An excellent summary of the intricate relationship between polytopes and generic smoothness can be found in Section 2.4 of [ACG16].

---

[1]The Macaulay2 code is available at www.ualberta.ca/˜favero/code.html.

3.3. **Some normal forms in $\mathbf{P}^3$.** Let X_Σ be a 3-dimensional projective toric variety. If $\Xi \subseteq \nabla^*_\Sigma \cap M$ is a finite set and $\mathrm{Conv}(\Xi)$ is reflexive, then, by Corollary 3.3, the generic member $Z_{w,\Sigma}$ of $\mathcal{F}_\Xi$ is birational to $Z_{w,\Sigma_{\mathrm{Conv}(\Xi)}}$ where $\Sigma_{\mathrm{Conv}(\Xi)}$ is the normal fan to $\mathrm{Conv}(\Xi)$. Therefore, the resolutions are isomorphic. In particular, the Picard lattices of the resolutions are isomorphic.

Recall that, for every weighted-projective space on Reid's list, there is a reflexive polytope Δ associated to it so that the polytope corresponds to the complete anti-canonical linear system. Suppose that $\mathrm{Conv}(\Xi)$ is isomorphic to one of the reflexive polytopes Δ corresponding to a weighted-projective space on Reid's list, i.e., $\mathrm{Conv}(\Xi) = \mathrm{Conv}(\nabla^*_{\Sigma'} \cap M)$ where X'_Σ is a weighted-projective space appearing on Reid's list. By applying Corollary 3.8, we see that $Z_{w,\Sigma}, Z_{w,\Sigma_{\mathrm{Conv}(\Xi)}}$ are also birational to the corresponding hypersurface $Z_{w,\Sigma'}$ in the weighted projective space $X_{\Sigma'}$ on Reid's list. Hence, for the resolutions, $\widetilde{Z_{w,\Sigma}}, \widetilde{Z_{w,\Sigma_{\mathrm{Conv}(\Xi)}}}, \widetilde{Z_{w,\Sigma'}}$ we can describe the Picard lattices

$$\mathrm{Pic}(\widetilde{Z_{w,\Sigma}}) = \mathrm{Pic}(\widetilde{Z_{w,\Sigma_{\mathrm{Conv}(\Xi)}}}) = \mathrm{Pic}(\widetilde{Z_{w,\Sigma'}})$$

by invoking Belcastro's computation of Picard ranks for generic K3 surfaces in each weighted-projective space on Reid's list [Bel02].

Now, consider the special case where Σ is the normal fan to a standard 3-dimensional simplex so that $X_\Sigma = \mathbf{P}^3$. In this case, we consider linear systems $\mathcal{F}_\Xi$ spanned by quartic monomials in $\mathbf{P}^3$. Up to taking convex hull of $\Xi \subseteq \nabla^*_\Sigma \cap M$, there are 429 such choices of linear systems of quartics so that $\mathrm{Conv}(\Xi)$ is isomorphic to some Δ corresponding to a family on Reid's list. These 429 choices cover exactly 52 of the 95 families on Reid's list and give 44 distinct Picard lattices.[2]

In conclusion, using Corollary 3.3, we can find 429 linear systems of quartics in $\mathbf{P}^3$ for which we can describe the Picard lattice of a resolution of the generic member. In total, these linear systems provide 44 distinct Picard lattices.

Remark 3.10. Up to taking convex hull, there are 20260 linear systems of quartics in $\mathbf{P}^3$ such that $\mathrm{Conv}(\Xi)$ is 3-dimensional and reflexive. Exactly 3615 of the 4319 3-dimensional reflexive polytopes occur in such a way.[3] The corresponding Picard lattice is described in [Roh04] but nowhere are they listed in terms of the classification of symmetric unimodular lattices in the $K3$ lattice.

Example 3.11. Consider the set

$$\Xi = \{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}$$

and the corresponding linear system

$$\mathcal{F}_{\mathbf{c}} = c_1 x^2 yz + c_2 xy^2 z + c_3 xyz^2 + c_4 w^4 + c_5 xyzw$$

of quartics in $\mathbf{P}^3$.

Then $\mathrm{Conv}(\Xi) = \Delta_0$ is the reflexive polytope of index 0 in Sage. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_8 \oplus \langle -4 \rangle \oplus H)$ since this appears as Reid family 52 in the weighted-projective space $\mathbf{P}^3(7,8,9,12)$.

Example 3.12. Consider the set

$$\Xi = \{(-1,-1,-1),(-1,-1,0),(0,0,0),(-1,1,0),(2,-1,0),(-1,-1,1)\}$$

---

[2]The Macaulay2 code and full lists can be found at www.ualberta.ca/~favero/code.html.

[3]The Sage code and full lists can be found at www.ualberta.ca/~favero/code.html.

and the corresponding linear system

$$\mathcal{F}_{\mathbf{c}} = c_1 w^4 + c_2 w^3 z + c_3 xyzw + c_4 y^2 wz + c_5 x^3 z + c_6 w^2 z^2$$

of quartics in $\mathbf{P}^3$.

Then $\mathrm{Conv}(\Xi) = \Delta_{88}$ is the reflexive polytope of index 88 in Sage. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_8 \oplus H)$ since this appears as Reid families 46, 65, and 80 in the weighted-projective spaces $\mathbf{P}^3(5,6,22,33)$, $\mathbf{P}^3(3,5,11,14)$, and $\mathbf{P}^3(4,5,13,22)$, respectively.

Example 3.13. In this example we produce a family of $(E_8 \oplus E_7 \oplus H)$-polarized K3 surfaces in a blowup of $\mathbf{P}^3$. This was achieved independently by Clingher and Doran [CD12] and Vinberg [Vin13] using non-toric methods.

Consider the set

$$\Xi = \{(-1,-1,-1), (-1,-1,0), (0,0,0), (-1,1,0), (2,-1,0), (-1,-1,1), (0,-1,1)\}$$

and the corresponding linear system

$$\mathcal{F}_{\mathbf{c}} = c_1 w^4 + c_2 w^3 z + c_3 xyzw + c_4 y^2 wz + c_5 x^3 z + c_6 w^2 z^2 + c_7 xz^2 w$$

of quartics in $\mathbf{P}^3$.

Then $\mathrm{Conv}(\Xi) = \Delta_{221}$ is the reflexive polytope of index 221 in Sage. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_7 \oplus H)$ since this appears as Reid families 68, 83, and 92 in the weighted-projective spaces $\mathbf{P}^3(3,4,10,13)$, $\mathbf{P}^3(4,5,18,27)$, and $\mathbf{P}^3(3,5,11,19)$, respectively.

Remark 3.14. Bruzzo and Grassi (Theorem 3.8 of [BG12]) prove that, given the family $\mathcal{F}$ of anti-canonical hypersurfaces in a complete, simplicial toric variety X_Σ, the Picard rank of a very general hypersurface $X \in \mathcal{F}$ is the Picard rank of X_Σ. It is an interesting problem to find subloci of hypersurfaces of higher Picard rank than X_Σ. While our resolutions are of higher Picard rank, the hypersurfaces themselves are not, a priori, in the Noether-Lefschetz locus as they are sometimes not smooth.

3.4. Equivalences of exotic mirror constructions.

3.4.1. *Artebani-Comparin-Guilbot mirror symmetry.* We now prove Theorem 1.2. The following definition comes form [ACG16].

Definition 3.15. Let $\Delta_1 \subseteq \Delta_2$ be polytopes in $M_{\mathbb{Q}}$. We say that (Δ_1, Δ_2) is a **good pair** if Δ_1 and Δ_2^* are lattice polytopes with a unique interior lattice point.

If (Δ_1, Δ_2) is a good pair, then we can define a family of Calabi-Yau hypersurfaces by looking at the special linear system associated to $\Xi = \Delta_1 \cap M$ corresponding to anti-canonical sections in the toric variety $X_{\Sigma_{\Delta_2}}$. Note that $\nabla_{\Sigma_{\Delta_2}} = \Delta_2^*$ by the construction of the anti-canonical polytope Δ_2. Notice that if (Δ_1, Δ_2) is a good pair, then so is (Δ_2^*, Δ_1^*). In this situation, we may regard the lattice points of Δ_2^* as a linear system Ξ^* on $X_{\Sigma_{\Delta_1^*}}$ where $\nabla_{\Sigma_{\Delta_1^*}} = \Delta_1$. Artebani, Comparin, and Guilbot use this to propose the following mirror duality:

$$Z_{w, \Sigma_{\Delta_2}} \longleftrightarrow Z_{\hat{w}, \Sigma_{\Delta_1^*}},$$

where w is a generic member of the linear system $\mathrm{Conv}(\Delta_2^*) \cap N$ and $\hat{w}$ is a generic member of the linear system $\mathrm{Conv}(\Delta_1) \cap M$.

Fixing Δ_2 and choosing two good pairs (Δ_1, Δ_2) and (Δ_1', Δ_2), amounts to being able to choose different hypersurfaces in $X_{\Sigma_{\Delta_2}}$ which are symplectomorphic. Hence,

mirror symmetry predicts that the B-model of the mirror remains the same. This phenomenon is realized via derived categories.

Corollary 3.16. *Let (Δ_1', Δ_2) and (Δ_1, Δ_2) be two good pairs. Then for a generic choice of*

$$\widehat{w} := \sum_{m \in \mathrm{Conv}(\Delta_2^*) \cap M} c_m x^m,$$

the two corresponding mirror coarse moduli spaces $Z_{\widehat{w}, \Sigma_{\Delta_1}}, Z_{\widehat{w}, \Sigma_{\Delta_1'}}$ are birational and the partial resolutions $\widetilde{\mathcal{Z}_{\widehat{w}, \Sigma_{\Delta_1}}}$ and $\widetilde{\mathcal{Z}_{\widehat{w}, \Sigma_{\Delta_1'}}}$ are derived equivalent. Moreover, if both $\mathcal{Z}_{\widehat{w}, \Sigma_{\Delta_1}}$ and $\mathcal{Z}_{\widehat{w}, \Sigma_{\Delta_1'}}$ are smooth, then they are derived equivalent.

Proof. Set $\Xi = \mathrm{Conv}(\Delta_2^*) \cap M$. Since Δ_2^* has a unique interior lattice point, Ξ is good. The polytopes Δ_1 and Δ_1' have a unique interior point and $\nabla_{\Sigma_{\Delta_1}} = \Delta_1$ and $\nabla_{\Sigma_{\Delta_1'}} = \Delta_1'$, hence $X_{\Sigma_{\Delta_1}}$ and $X_{\Sigma_{\Delta_1'}}$ have nef anti-canonical bundles. The result now follows from Theorem 2.7 and Corollary 2.10, depending on the smoothness of the stacks. $\square$

Remark 3.17. The birational aspect of this result was first proven as Proposition 3.5 in [ACG16].

3.4.2. *Clarke mirror symmetry.* While Clarke's construction was originally created in the Landau-Ginzburg setting, we will specialize to the Calabi-Yau case and give an interpretation along the lines of [ACG16]. In [Cla17b], Clarke proves his construction is a generalization of the Batyrev and Berglund-Hübsch-Krawitz constructions. As always, M and N are dual lattices and $\Sigma \subseteq N_\mathbb{R}$ a fan. We work over the field $\kappa = \mathbb{C}$. Recall from equation (2.1) that there is a right exact sequence:

$$(3.1) \qquad M \xrightarrow{\mathrm{div}_\Sigma} \mathbb{Z}^n \xrightarrow{\pi} \mathrm{Cl}(X_\Sigma) \to 0$$
$$m \mapsto \sum_{\rho \in \Sigma(1)} \langle u_\rho, m \rangle e_\rho.$$

Now consider a finite set $\Xi \subseteq \nabla_\Sigma^* \cap M$ with the origin in the interior of its convex hull. This gives another right exact sequence,

$$(3.2) \qquad N \xrightarrow{\mathrm{mon}_\Xi} \mathbb{Z}^\Xi \xrightarrow{q} \mathrm{coker}(\mathrm{mon}_\Xi) \to 0$$
$$n \mapsto \sum_{m \in \Xi} \langle m, n \rangle e_m.$$

The realization of X_Σ as a GIT quotient amounts to a choice of an S_Σ-linearization of $\mathcal{O}_{\mathbb{A}^n}$ or, equivalently, an element of $D \in \mathrm{Cl}(X_\Sigma)$. Indeed, the fan Σ can be recovered from (3.1) and D as the normal fan to the polytope $(\pi \otimes_\mathbb{Z} \mathbb{R})^{-1}(D \otimes_\mathbb{Z} \mathbb{R}) \cap \mathbb{R}_{\geq 0}^n$.

Similarly, a specific hypersurface with non-zero monomial coefficients in $\mathcal{F}_\Xi$ is a choice of coefficients $\mathbf{c} \in (\mathbb{C}^*)^\Xi$ which, up to reparametrization by the torus $N \otimes_\mathbb{Z} (\mathbb{C}^*)^\Xi$, amounts to a choice $q(\mathbf{c}) \in \mathrm{coker}(\mathrm{mon}_\Xi) \otimes_\mathbb{Z} \mathbb{C}$. Here, the $\mathbb{Z}$-module structure on $\mathbb{C}$ is the multiplicative structure.

Clarke's mirror construction simply exchanges the roles of the two right exact sequences above and the choices $D \otimes_\mathbb{Z} \mathbb{C}^*, q(\mathbf{c})$. More precisely, given X_Σ and an equivalence class $q(\mathbf{c})$ in a linear system Ξ, we get two stacks,

$$\mathcal{Z}_{q(\mathbf{c}), \Sigma}, \mathcal{Z}_{D \otimes_\mathbb{Z} \mathbb{C}^*, \Sigma_{\Xi, q(\mathbf{c})}},$$

where $\Sigma_{\Xi,q(\mathbf{c})}$ is the fan associated to (3.2) and $q(\mathbf{c})$, i.e., the normal fan to the polytope $(q \otimes_{\mathbb{Z}} \mathbb{R})^{-1}(\operatorname{im} q(\mathbf{c})) \cap \mathbb{R}^{\Xi}_{\geq 0}$.

Definition 3.18 (Clarke [Cla17b])**.** The stacks $\mathcal{Z}_{q(\mathbf{c}),\Sigma}$ and $\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi,q(\mathbf{c})}}$ are called mirror to one another.

Remark 3.19. To complete the symmetry, one can more generally choose the right exact sequence (3.1) and an element $\bar{D} \in \operatorname{Cl}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{C}$. This amounts to a choice of B-field on X_Σ [Cla17b].

Remark 3.20. Since we only choose an equivalence class of the defining polynomial w, the stacks above are only defined as subsets of $\mathcal{X}_\Sigma, \mathcal{X}_{\Sigma_\Xi}$ up to reparametrization by $\mathbb{G}_m^{\Sigma(1)}, \mathbb{G}_m^\Xi$.

If we choose different anti-canonical linear systems Ξ_1, Ξ_2 and non-zero coefficients

$$\mathbf{c}_i : \Xi_i \to \mathbb{C}^*,$$

then we get two smooth stacky Calabi-Yau hypersurfaces $\mathcal{Z}_{q(\mathbf{c}_1),\Sigma}$ and $\mathcal{Z}_{q(\mathbf{c}_2),\Sigma}$ in $\mathcal{X}_\Sigma$. While this varies only the complex structure of the Calabi-Yau hypersurface, the mirrors will lie in different toric varieties, which gives us the same phenomenon as before. We show this with a few extra assumptions on the toric variety X_Σ and linear systems Ξ_i:

Corollary 3.21. *Let X_Σ be a projective variety, let $\nabla_\Sigma \cap N$ be a good set, and let $\Xi_1, \Xi_2 \subseteq \nabla^*_\Sigma$ be finite sets such that $\operatorname{Conv}(\Xi_i)$ is full-dimensional with the origin in its interior. Choose coefficients $\mathbf{c}_i : \Xi_i \to \mathbb{C}^*$ and consider the stacky hypersurfaces $\mathcal{Z}_{q(\mathbf{c}_1),\Sigma}, \mathcal{Z}_{q(\mathbf{c}_2),\Sigma} \subseteq \mathcal{X}_\Sigma$. Then the two corresponding mirror coarse moduli spaces $Z_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1,q(\mathbf{c}_1)}}, Z_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2,q(\mathbf{c}_2)}}$ are birational. Moreover, the partial resolutions $\widetilde{\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1,q(\mathbf{c}_1)}}}$ and $\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2,q(\mathbf{c}_2)}}$ are derived equivalent.*

Proof. The mirror stack $\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_i,q(\mathbf{c}_i)}}$ is an anti-canonical member of $F_{\nabla_\Sigma \cap N}$ in $X_{\Sigma_{\Xi_i,q(\mathbf{c}_i)}}$. By looking at the exact sequence (3.2), we see that the rays of $\nabla_{\Sigma_{\Xi_i,q(\mathbf{c}_i)}}$ are the elements of Ξ_i. Therefore, $\nabla_{\Sigma_{\Xi_i,q(\mathbf{c}_i)}} = \operatorname{Conv}(\Xi_i) \subseteq M_{\mathbb{R}}$ Hence,

$$\nabla_\Sigma \subseteq \nabla^*_{\Sigma_{\Xi_1,q(\mathbf{c}_1)}} \cap \nabla^*_{\Sigma_{\Xi_2,q(\mathbf{c}_2)}} \subseteq N_{\mathbb{R}}.$$

Notice that $X_{\Sigma_{\Xi_i,q(\mathbf{c}_i)}}$ has a nef anti-canonical bundle since

$$\Xi_i \subseteq \operatorname{Conv}(\operatorname{Conv}(\nabla_\Sigma)^* \cap M)$$

has a unique interior lattice point. Therefore, $\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1,q(\mathbf{c}_1)}}, \mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2,q(\mathbf{c}_2)}}$ satisfy the hypotheses of Theorem 2.7. $\square$

Proof of Theorem 1.2. The mirror construction of Artebani-Comparin-Guilbot is a special case of Clarke's construction where $(\operatorname{Conv}(\Xi_1), \nabla^*_\Sigma) = (\Delta_1, \Delta_2)$ is a good pair, so the theorem follows from Corollary 3.21. Batyrev mirror symmetry is the special case of Corollary 3.16 where $\Delta_1 = \Delta_2$. Similarly, Berglund-Hübsch-Krawitz mirror symmetry is a special case by Theorem 2 of [ACG16] and does not require resolutions as BHK mirrors are smooth. $\square$

Remark 3.22. In the special case of Berglund-Hübsch-Krawitz mirror symmetry, the birational aspect was first proven by Shoemaker in [Sho14] and the derived aspect was proven in [FK16a]. The latter assumes that X_Σ is Gorenstein. The result above drops the Gorenstein assumption.

Remark 3.23. Aspinwall and Plesser [AP15] provide a general mirror construction for Landau-Ginzburg models associated to two Gorenstein cones which are contained in one another's duals. In their theory, they allow non-geometric phases as gauged linear sigma models. Comparing these phases relates to an algebraic condition which inspired our results; see Section 4 and especially Corollary 4.7 of [FK16a]. The method of the current paper is a toric description which focuses only on geometric phases related to Gorenstein cones of index 1.

Acknowledgments

References

[ACG16] Michela Artebani, Paola Comparin, and Robin Guilbot, *Families of Calabi-Yau hypersurfaces in $\mathbb{Q}$-Fano toric varieties* (English, with English and French summaries), J. Math. Pures Appl. (9) **106** (2016), no. 2, 319–341, DOI 10.1016/j.matpur.2016.02.012. MR3515305

[AP15] Paul S. Aspinwall and M. Ronen Plesser, *General mirror pairs for gauged linear sigma models*, J. High Energy Phys. **11** (2015), 029, front matter+32. MR3455065

[BFK12] M. Ballard, D. Favero, and L. Katzarkov, *Variation of Geometric Invariant Theory quotients and derived categories*, arXiv:1203.6643, to appear in Crelle.

[Bat94] Victor V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535. MR1269718

[BB96] Victor V. Batyrev and Lev A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. **126** (1996), no. 1, 183–203, DOI 10.1007/s002220050093. MR1408560

[Bel02] Sarah-Marie Belcastro, *Picard lattices of families of $K3$ surfaces*, Comm. Algebra **30** (2002), no. 1, 61–82, DOI 10.1081/AGB-120006479. MR1880661

[BH93] Per Berglund and Tristan Hübsch, *A generalized construction of mirror manifolds*, Nuclear Phys. B **393** (1993), no. 1-2, 377–391, DOI 10.1016/0550-3213(93)90250-S. MR1214325

[Bor13] Lev A. Borisov, *Berglund-Hübsch mirror symmetry via vertex algebras*, Comm. Math. Phys. **320** (2013), no. 1, 73–99, DOI 10.1007/s00220-013-1717-y. MR3046990

[BG12] Ugo Bruzzo and Antonella Grassi, *Picard group of hypersurfaces in toric 3-folds*, Internat. J. Math. **23** (2012), no. 2, 1250028, 14, DOI 10.1142/S0129167X12500280. MR2890472

[Cla14] Patrick Clarke, *A proof of the birationality of certain BHK-mirrors*, Complex Manifolds **1** (2014), 45–51, DOI 10.2478/coma-2014-0003. MR3333467

[Cla17a] Patrick Clarke, *Birationality and Landau-Ginzburg Models*, Comm. Math. Phys. **353** (2017), no. 3, 1241–1260, DOI 10.1007/s00220-017-2830-0. MR3652490

[Cla17b] Patrick Clarke, *Duality for toric Landau-Ginzburg models*, Adv. Theor. Math. Phys. **21** (2017), no. 1, 243–287, DOI 10.4310/ATMP.2017.v21.n1.a5. MR3636695

[CD12] Adrian Clingher and Charles F. Doran, *Lattice polarized K3 surfaces and Siegel modular forms*, Adv. Math. **231** (2012), no. 1, 172–212, DOI 10.1016/j.aim.2012.05.001. MR2935386

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322

[DGJ08] Charles Doran, Brian Greene, and Simon Judes, *Families of quintic Calabi-Yau 3-folds with discrete symmetries*, Comm. Math. Phys. **280** (2008), no. 3, 675–725, DOI 10.1007/s00220-008-0473-x. MR2399610

[FK16a] D. Favero and T. L. Kelly, *Derived Categories of BHK Mirrors*, arXiv:1602.05876.

[FK16b] D. Favero and T. L. Kelly, *Fractional Calabi-Yau Categories via Landau-Ginzburg Models*, arXiv:1610:04918, to appear in Algebraic Geometry.

[FK17] David Favero and Tyler L. Kelly, *Proof of a conjecture of Batyrev and Nill*, Amer. J. Math. **139** (2017), no. 6, 1493–1520, DOI 10.1353/ajm.2017.0038. MR3730928

[Fl00] A. R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173. MR1798982

[GS] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at http://www.math.uiuc.edu/Macaulay2/

[GP90] B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau moduli space*, Nuclear Phys. B **338** (1990), no. 1, 15–37, DOI 10.1016/0550-3213(90)90622-K. MR1059831

[H-L15] Daniel Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912, DOI 10.1090/S0894-0347-2014-00815-8. MR3327537

[HW12] Manfred Herbst and Johannes Walcher, *On the unipotence of autoequivalences of toric complete intersection Calabi-Yau categories*, Math. Ann. **353** (2012), no. 3, 783–802, DOI 10.1007/s00208-011-0704-x. MR2923950

[Kaw05] Yujiro Kawamata, *Log crepant birational maps and derived categories*, J. Math. Sci. Univ. Tokyo **12** (2005), no. 2, 211–231. MR2150737

[Kel13] Tyler L. Kelly, *Berglund-Hübsch-Krawitz mirrors via Shioda maps*, Adv. Theor. Math. Phys. **17** (2013), no. 6, 1425–1449. MR3262528

[KM12] Masanori Kobayashi and Makiko Mase, *Isomorphism among families of weighted K3 hypersurfaces*, Tokyo J. Math. **35** (2012), no. 2, 461–467, DOI 10.3836/tjm/1358951330. MR3058718

[Kra09] Marc Krawitz, *FJRW rings and Landau-Ginzburg mirror symmetry*, ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–University of Michigan. MR2801653

[KS92] Maximilian Kreuzer and Harald Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. **150** (1992), no. 1, 137–147. MR1188500

[Li16] Zhan Li, *On the birationality of complete intersections associated to nef-partitions*, Adv. Math. **299** (2016), 71–107, DOI 10.1016/j.aim.2016.05.006. MR3519464

[Mul09] Joshua P. Mullet, *Toric Calabi-Yau hypersurfaces fibered by weighted K3 hypersurfaces*, Comm. Anal. Geom. **17** (2009), no. 1, 107–138, DOI 10.4310/CAG.2009.v17.n1.a5. MR2495835

[Rei87] Miles Reid, *The moduli space of 3-folds with K = 0 may nevertheless be irreducible*, Math. Ann. **278** (1987), no. 1-4, 329–334, DOI 10.1007/BF01458074. MR909231

[Roh04] F. Rohsiepe, *Lattice polarized toric K3 surfaces*, arXiv:hep-th/0409290.

[Sage] SageMath, the Sage Mathematics Software System (Version 8.1), The Sage Developers, 2018, http://www.sagemath.org.

[Sho14] Mark Shoemaker, *Birationality of Berglund-Hübsch-Krawitz mirrors*, Comm. Math. Phys. **331** (2014), no. 2, 417–429, DOI 10.1007/s00220-014-2121-y. MR3238520

[Vin13] E. B. Vinberg, *On the algebra of Siegel modular forms of genus 2*, Trans. Moscow Math. Soc., posted on 2013, 1–13, DOI 10.1090/s0077-1554-2014-00217-x. MR3235787

[Wło97] Jarosław Włodarczyk, *Decomposition of birational toric maps in blow-ups & blow-downs*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 373–411, DOI 10.1090/S0002-9947-97-01701-7. MR1370654

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA CANADA
Email address: doran@math.ualberta.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA CANADA – AND – KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL, REPUBLIC OF KOREA
Email address: favero@ualberta.ca

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB UNITED KINGDOM
Email address: tlk20@dpmms.cam.ac.uk