DEGENERATIONS, FIBRATIONS AND HIGHER RANK LANDAU-GINZBURG MODELS

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ABSTRACT. We study semi-stable degenerations of quasi-Fano varieties to unions of two pieces. We conjecture that the higher rank Landau-Ginzburg models mirror to these two pieces can be glued together to lower rank Landau-Ginzburg models which are mirror to the original quasi-Fano varieties. We prove this conjecture by relating their Euler characteristics, generalized functional invariants as well as periods. We also use it to conjecture a relation between the degenerations to the normal cones and the fibrewise compactifications of higher rank Landau-Ginzburg models. Furthermore, we use it to iterate the Doran-Harder-Thompson conjecture and obtain higher codimension Calabi-Yau fibrations.

CONTENTS

1. Introduction

We study a generalization of the conjecture of [DHT17] by considering a semi-stable degeneration of a quasi-Fano variety X (instead of a degeneration of a Calabi-Yau in [DHT17]) such that the central fibre is a union of two quasi-Fano varieties $X_1 \cup_{D_0} X_2$. We would like to study the relation of their mirrors.

Recall that, an (ordinary) Landau-Ginzburg (LG) model $(X^\vee, W)$ is a mirror partner of a quasi-Fano variety X with its smooth anticanonical divisor D such that the generic fibre of W is mirror to D. If D is not anticanonical (e.g. $D_0 \subset X_i$, for $i = 1, 2$, in the degeneration of a quasi-Fano variety described above) and we still want to define a mirror "LG model" for the pair (X, D) such that the generic fibre is mirror to D (with a smooth anticanonical divisor in D), then the generic fibre of the "LG model" for the pair (X, D) should also be an LG model. Therefore, one should consider an "LG model" whose generic fibre is another LG model. In other words, one should consider a higher rank LG model.

Therefore, in our set-up, we consider a degeneration of a quasi-Fano variety X which leads to a Tyurin degeneration of a smooth anticanonical divisor $D \subset X$ into $D_1 \cup_{D_0 \cap D_1 = D_0 \cap D_2} D_2$ where $D_1 \subset X_1$ and $D_2 \subset X_2$. We consider the hybrid LG models of $(X_1, D_0 + D_1)$ and $(X_2, D_0 + D_2)$ studied by S. Lee [Lee21]. In this paper, we will simply refer to hybrid LG models as higher rank LG models. In particular, a rank 1 LG model is an ordinary LG model, i.e., an LG model in the usual sense. In general, a higher rank LG model is a mirror partner of a quasi-Fano variety with a simple normal crossing anticanonical divisor. When the anticanonical divisor consists of two irreducible components, a higher rank LG model (of rank 2) consists of a pair $(X^\vee, h)$, where $X^\vee$ is a Kähler manifold and h is the multi-potential $h : X^\vee \to \mathbb{C}^2$ satisfying certain conditions (see Definition 2.5 for detail). Then we propose the following gluing conjecture.

Conjecture 1.1. *Given a semi-stable degeneration of (X, D) into the union of $(X_1, D_0 + D_1)$ and $(X_2, D_0 + D_2)$, the rank 1 LG model of (X, D) can be constructed topologically by gluing two rank 2 LG models of $(X_1, D_0 + D_1)$ and $(X_2, D_0 + D_2)$.*

Conjecture 1.1 can be considered as a generalization of the conjecture in [DHT17] to the log Calabi-Yau setting. We provide several pieces of evidence for this conjecture. In Section 2, we prove the topological version of this conjecture by computing Euler numbers. In other words, if a quasi-Fano variety X is obtained from $X_1 \cup_{D_0} X_2$ by smoothing and $(X^\vee, W)$ is obtained by gluing rank 2 LG models of $(X_1, D_0 + D_1)$ and $(X_2, D_0 + D_2)$, then

Theorem 1.2 (see Theorem 2.8)**.**

$$\chi(X) = (-1)^{\mathrm{d}} \chi(X^\vee, W^{-1}(t)),$$

where $\mathrm{d} = \dim_{\mathbb{C}} X$ *and* t *is a regular value of* W.

We refer to Theorem 2.8 for the precise statement.

In [DKY19], the Doran-Harder-Thompson conjecture is proved at the level of generalized functional invariants as well as periods for Calabi-Yau complete intersections in toric varieties. In this paper, we generalize it to the quasi-Fano complete intersections in toric varieties. In Section 3 and Section 4, we study two explicit examples in detail. In Section 3, we study a complete intersection of bidegrees $(3,1)$ and $(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. In Section 4, we study the quartic threefold in $\mathbb{P}^4$. In Section 5, we study toric complete intersections in general. We show that the generalized functional invariants of $(X^\vee, W)$, $(D^\vee, W_0)$, $(X_1^\vee, h_1)$ and $(X_2^\vee, h_2)$ satisfy a product relation (see Proposition 5.3). Using this product relation among generalized functional invariants, we obtain the relation among periods.

Theorem 1.3 (=Theorem 5.4). *The following Hadamard product relation holds for relative periods of* $(X^\vee, W)$, $(D^\vee, W_0)$, $(X_1^\vee, h_1)$ *and* $(X_2^\vee, h_2)$:

$$f_0^X(q) \star_q f_0^{D_0}(q) = \frac{1}{2\pi i} \oint f_0^{X_1}(q,y) \star_q f_0^{X_2}(q,y) \frac{dy}{y}.$$

The relation among bases of solutions to Picard-Fuchs equations holds in a similar form.

Theorem 1.4 (Theorem 5.5). *The Hadamard product relation among the bases of solutions to Picard-Fuchs equations is*

$$I^X(q) \star_q I^{D_0}(q) = \frac{1}{2\pi i} \oint I^{X_1}(q,y) \star_q I^{X_2}(q,y) \frac{dy}{y}.$$

The main difference between Theorem 1.3, Theorem 1.4 and [DKY19, Theorem 1.1 and Theorem 1.2] is the following. In [DKY19], absolute periods are obtained as the gluing (i.e., the residue integral of the Hadamard product) of two relative periods of rank 1 LG models. In Theorem 1.3 and Theorem 1.4, we glue relative periods of two rank 2 LG models to obtain the relative periods for rank 1 LG models. By the relative mirror theorem of [TY20a], relative periods of higher rank LG models are mirror to the genus zero Gromov-Witten invariants of simple normal crossings pairs defined in [TY20b].

Besides considering an LG model as a gluing of rank 2 LG models, one may also consider an LG model as a fibrewise compactification of a rank 2 LG model. More precisely, one can consider a quasi-Fano variety X with a smooth anticanonical divisor D or a simple normal crossing anticanonical divisor $D_1 + D_2$. Then the LG model mirrors to (X, D) should be considered as fibrewise compactification of the rank 2 LG model mirrors to $(X, D_1 + D_2)$. In Section 6, we consider a degeneration to the normal cone of the quasi-Fano variety X with respect to the divisor D_1. In this case, gluing two rank 2 LG models can be considered as providing a fibrewise compactification of a non-proper LG model of $(X, D_1 + D_2)$ to obtain a proper LG model of (X, D). In other words,

Conjecture 1.5 (=Conjecture 6.1). *The ordinary LG model of* (X, D) *is a (partial) compactification of the rank 2 LG model of* $(X, D_1 + D_2)$ *and the compactification can be obtained by gluing the rank 2 LG mirror of* $(\mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O}), D_1 + \tilde{D}_1)$ *to the rank 2 LG mirror of* $(X, D_1 + D_2)$.

Furthermore, we use this conjecture to study the iterative Tyurin degenerations of Calabi-Yau varieties. For Calabi-Yau varieties admitting such degenerations, we predict that the mirror has a fibration structure over higher dimensional bases. For Calabi-Yau complete intersections in toric varieties that admit two-step Tyurin degenerations, the mirror will have a Calabi-Yau fibration over $\mathbb{P}^1 \times \mathbb{P}^1$.

We illustrate this with an explicit example in Section 7, where we study an elliptic fibration structure of the mirror quintic threefold over a $\mathbb{P}^1 \times \mathbb{P}^1$ base. The fibration is by mirror cubic elliptic curves, mirror to the cubic curves arising in the pair of two-step Tyurin degenerations: $(5) \to (1) + (4) \to (1) + (1) + (3)$ and $(5) \to (2) + (3) \to (1) + (1) + (3)$. Moreover, the first of these degenerations corresponds to a mirror quartic K3-surface fibration and the second to a mirror sextic K3-surface fibration. There is a perfect correspondence between these K3-fibrations on the mirror quintic threefold compatible with the mirror cubic fibration and the two-step Tyurin degenerations of the quintic itself via the refinement of the nef partition. More generally, we expect the following.

Conjecture 1.6 (=Conjecture 7.3)**.** *If a Calabi-Yau variety X (or a log Calabi-Yau variety $X \smallsetminus D$) admits a semi-stable degeneration, connected to a point of maximal unipotent monodromy, such that the cone over the dual intersection complex of the central fibre is of dimension k, then the mirror of X (or $X \smallsetminus D$) admits a Calabi-Yau fibration structure with a k-dimensional base.*

To understand more general degenerations of Calabi-Yau varieties or more general iterative Tyurin degenerations of Calabi-Yau varieties, it is also necessary to consider degenerations of simple normal-crossings log Calabi-Yau pairs. Recall that Conjecture 1.1 considers a degeneration of a smooth pair. The conjecture can be generalized to the case when we consider a degeneration of a quasi-Fano variety X with a simple normal crossing anticanonical divisor D. In this case, the mirror of (X, D) itself is a higher rank LG model. Roughly speaking, the conjecture can be stated as follows.

Conjecture 1.7. *Given a quasi-Fano variety X with a simple normal crossings anticanonical divisor D, we consider a semi-stable degeneration of a quasi-Fano variety X into $X_1 \cup_{D_0} X_2$. The mirror of (X, D), which is a rank n LG model, can be obtained by topologically gluing two rank $(n + 1)$ LG models which are mirror to X_1 and X_2 with their corresponding simple normal crossing divisors.*

Remark 1.8. Recall that the generic fibre of a rank n LG model is a codimension n Calabi-Yau and the mirrors of Calabi-Yau varieties are Calabi-Yau varieties of the same dimension. Therefore, we may consider the mirrors of Calabi-Yau varieties as rank 0 LG models, that is, LG models with 0-dimensional bases which are just Calabi-Yau varieties without superpotentials. From this point of view, we may consider the Doran-Harder-Thompson conjecture as the rank 0 case of Conjecture 1.7. Conjecture 1.1 is a rank 1 case of Conjecture 1.7.

To prove a topological gluing formula, we need to have a proper definition of Hodge numbers of higher rank LG models. For simplicity, we consider the case when $n = 2$. Its generalization to $n > 2$ is straightforward. Hodge numbers of a rank 2 LG model $(X^\vee, h = (h_1, h_2))$ are defined in Definition 8.1:

$$h^{p,q}(X^\vee, (h_1, h_2)) := h^{p,q}(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2)),$$

where $t = (t_1, t_2)$ is a regular value of (h_1, h_2). One may consider this as a generalization of the Hodge numbers of LG models of [KKP17]. With this definition of Hodge numbers, we prove a topological gluing formula relating Euler numbers (see Theorem 8.4):

$$\chi(X) = (-1)^{\mathrm{d}}(\chi(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2))).$$

The relations among their generalized functional invariants and relative periods for toric complete intersections hold just like the rank 1 case.

2. Gluing rank 2 LG models

In [DHT17], Doran-Harder-Thompson conjectured a relation between mirrors of Calabi-Yau varieties and mirrors of quasi-Fano varieties under Tyurin degenerations of Calabi-Yau varieties. In this section, we consider a generalization of [DHT17] to degenerations of quasi-Fano varieties.

2.1. **Set-up.**

Definition 2.1. A smooth variety X is called quasi-Fano if its anticanonical linear system contains a smooth Calabi-Yau member and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

Definition 2.2 ([DHT17], Definition 2.1)**.** A Landau-Ginzburg (LG) model of a quasi-Fano variety X is a pair $(X^\vee, W)$ consisting of a Kähler manifold $X^\vee$ satisfying $h^1(X^\vee) = 0$ and a proper map $W : X^\vee \to \mathbb{C}$, where W is called the superpotential.

The generic fibre of W is expected to be mirror to a smooth anticanonical divisor $D \subset X$. The pair $(X^\vee, W)$ is considered as a mirror to the pair (X, D). A natural question to ask is how to extend this mirror symmetry to the case when D has more than one irreducible component. For this, we need the following generalization of LG models.

Definition 2.3. An LG model of rank 1 is the ordinary LG model in Definition 2.2. For $n \geq 2$, an LG model of rank n is a pair $(X^\vee, h)$, where

$$h := (h_1, \ldots, h_n) : X^\vee \to \mathbb{C}^n,$$

such that

- the generic fibre of h_i together with the restriction of $\hat{h}_i := (h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n)$ to this fibre is an LG model of rank $(n-1)$.
- by composing with the summation map $\Sigma : \mathbb{C}^n \to \mathbb{C}$, we get a non-proper ordinary LG model

$$W := \Sigma \circ h : X^\vee \to \mathbb{C}.$$

Remark 2.4. In this paper, we consider the case when the base of a rank n LG model is $\mathbb{C}^n$. In general, we can allow the base being more general n-dimensional varieties. This will be studied elsewhere.

Now we assume that $D = D_1 + D_2$ is a simple normal crossings anticanonical divisor of X and $D_{12} := D_1 \cap D_2$ is smooth. In Sukjoo Lee's thesis [Lee21], he defined the hybrid LG model which is expected to be mirror to the pair $(X, D_1 + D_2)$ where X is a Fano variety. Here, we consider the hybrid LG models for quasi-Fano varieties.

Definition 2.5 ([Lee21], Definition 1.4.1). A hybrid LG model of a quasi-Fano variety X with a simple normal crossings anticanonical divisor $D = D_1 + D_2$ is a pair

$$(X^\vee, h = (h_1, h_2) : X^\vee \to \mathbb{C}^2),$$

where $X^\vee$ is a Kähler manifold such that

- a generic fibre of h_1, denoted $D_1^\vee$, with

$$h|_{D_1^\vee} = h_2 : D_1^\vee \to \mathbb{C}$$

  is mirror to (D_1, D_{12});
- a generic fibre of h_2, denoted $D_2^\vee$, with

$$h|_{D_2^\vee} = h_1 : D_2^\vee \to \mathbb{C}$$

  is mirror to (D_2, D_{12});
- by composing with the summation map $\Sigma : \mathbb{C}^2 \to \mathbb{C}$, we get a non-proper ordinary LG model

$$W := \Sigma \circ h : X^\vee \to \mathbb{C}.$$

We will refer to hybrid LG models in Definition 2.5 as rank 2 LG models.

2.2. **Topological gluing.** Given a quasi-Fano variety X with its smooth anticanonical divisor D, we consider a semi-stable degeneration of X into $X_1 \cup_{D_0} X_2$ such that it also gives a Tyurin degeneration of D into $D_1 \cup_{D_{12}} D_2$ where $D_1 \subset X_1$, $D_2 \subset X_2$ and $D_{12} = D_1 \cap D_0 = D_2 \cap D_0 = D_1 \cap D_2$.

Let $(X_1^\vee, h_1 = (h_{11}, h_{12}))$ and $(X_2^\vee, h_2 = (h_{21}, h_{22}))$ be rank 2 LG models mirror to $(X_1, D_0 + D_1 \in |-K_{X_1}|)$ and $(X_2, D_0 + D_2 \in |-K_{X_2}|)$ respectively. We will glue the rank 2 LG models along their first factors to form an LG model $(X^\vee, W)$ with a fibration over $\mathbb{P}^1$:

$$
\begin{array}{ccc}
X^\vee & \xrightarrow{\ W\ } & \mathbb{C} \\
{\scriptstyle \pi}\big\downarrow & & \\
\mathbb{P}^1 & &
\end{array}
$$

We choose r_1 and r_2 such that $|\lambda_1| \le r_1$ for every λ_1 in the critical locus of h_{11} and $|\lambda_2| \le r_2$ for every λ_2 in the critical locus of h_{21}. We assume that the fibres of h_{11} and h_{21} are topologically the same LG model which mirrors to (D_0, D_{12}).

Let $t = (t_1, t_2)$ be a regular value of h_i. The monodromy symplectmorphism ϕ_{11} associated to a clockwise loop around infinity of t_1 on $h_1^{-1}(t)$ should be the same as the monodromy symplectmorphism ϕ_{21}^{-1} associated to a counter-clockwise loop around infinity of t_1 on $h_2^{-1}(t)$. For each t_2, we choose local trivialization of $h_{i2}^{-1}(t_2)$ over $U_i = \{z \in \mathbb{C} : |z| > r_i\}$ and let $Q_{i,t_2} = h_{i1}^{-1}(U_i, t_2)$. Following [DHT17], one can glue $h_{12}^{-1}(t_2)$ to $h_{22}^{-1}(t_2)$ along Q_{1,t_2} and Q_{2,t_2} to produce the fibre of $X^\vee$ over t_2. Let $t_2 \in \mathbb{C}$ varies, then we obtain $X^\vee$ with the superpotential W such that $W|_{X_1^\vee} = h_{12}$ and $W|_{X_2^\vee} = h_{22}$. The gluing respects the fibrations h_{11} and h_{21}, so $X^\vee$ is equipped with a fibration over $\mathbb{P}^1$.

Remark 2.6. We can consider a rank 2 LG model as a family of ordinary LG models. Then gluing two rank 2 LG models can be considered as gluing two families of ordinary LG models to obtain a family of Calabi-Yau varieties and this family of Calabi-Yau varieties is the LG mirror of (X, D). Therefore, the gluing picture here is a family version of the original gluing picture of [DHT17].

Remark 2.7. There is an alternative view of the gluing picture. Instead of considering the gluing of two rank 2 LG models to the mirror of the quasi-Fano variety X, we can also consider mirror symmetry in the complement of a smooth anticanonical divisor D in X by forgetting the superpotential W. Then the mirror of the non-compact Calabi-Yau $X \smallsetminus D$ can be obtained by gluing two (ordinary) LG models of two non-compact varieties $X_1 \smallsetminus D_1$ and $X_2 \smallsetminus D_2$.

We have the following mirror relation between the Euler numbers of quasi-Fano varieties and the LG model constructed by the gluing picture described above.

Theorem 2.8. *Let X_1 and X_2 be d-dimensional quasi-Fano varieties which contain the same quasi-Fano hypersurface D_0, such that*

$$K_{X_1}|_{D_0} = -K_{X_2}|_{D_0} \quad and \ K_{D_1}|_{D_{12}} = -K_{D_2}|_{D_{12}},$$

where $D_{12} = D_0 \cap D_1 = D_0 \cap D_2$. Let $(X_1^\vee, h_1 = (h_{11}, h_{12}))$ and $(X_2^\vee, h_2 = (h_{21}, h_{22}))$ be rank 2 LG models mirror to $(X_1, D_0 + D_1 \in |-K_{X_1}|)$ and $(X_2, D_0 + D_2 \in |-K_{X_2}|)$ respectively. Suppose that the fibres of h_1 and h_2 are topologically the same Calabi-Yau manifold, which is topologically mirror to D_{12}. Let X be a quasi-Fano variety obtained from $X_1 \cup_{D_0} X_2$ by smoothing and let $D \subset X$ be a Calabi-Yau variety obtained from $D_1 \cup_{D_{12}} D_2$ by smoothing. Let $(X^\vee, W)$ be the LG model obtained by gluing rank 2 LG models $(X_1^\vee, h_1)$ and $(X_2^\vee, h_2)$. Then

$$\chi(X) = (-1)^d \chi(X^\vee, W^{-1}(t)), \quad and \ \chi(D) = (-1)^{d-1} \chi(W^{-1}(t))$$

for t a regular value of W, where χ is the Euler number.

Proof. The equality $\chi(D) = (-1)^{d-1}\chi(W^{-1}(t))$ was proved in [DHT17, Theorem 2.3]. It remains to prove $\chi(X) = (-1)^d \chi(X^\vee, W^{-1}(t))$.

The gluing picture gives the following relative Mayer-Vietoris sequence

$$(1) \qquad \cdots \to H^j(X^\vee, W^{-1}(t); \mathbb{C}) \to H^j(X_1^\vee, h_{12}^{-1}(t_1); \mathbb{C}) \oplus H^j(X_2^\vee, h_{22}^{-1}(t_2); \mathbb{C})$$
$$\to H^j(X_1^\vee \cap X_2^\vee, h_{12}^{-1}(t_1) \cap h_{22}^{-1}(t_2); \mathbb{C}) \to \cdots.$$

Recall that $X_1^\vee \cap X_2^\vee$ and $h_{12}^{-1}(t_1) \cap h_{22}^{-1}(t_2)$ are fibrations over annuli. Their cohomology can be computed using the Wang sequence [PS08, Theorem 11.33] and we conclude that $\chi(X_1^\vee \cap X_2^\vee, h_{12}^{-1}(t_1) \cap h_{22}^{-1}(t_2)) = 0$. Therefore, we have

$$(2) \qquad \chi(X^\vee, W^{-1}(t)) = \chi(X_1^\vee, h_{12}^{-1}(t_1)) + \chi(X_2^\vee, h_{22}^{-1}(t_2)).$$

Note that $(X_i^\vee, h_{i2})$ should be considered as the mirror of the complement of the divisor D_0 of X_i, because we forget about h_{i1} which corresponds to the divisor D_0. Therefore, we have

$$(3) \qquad \chi(X_i^\vee, h_{i2}^{-1}(t_i)) = (-1)^d \chi(X_i \smallsetminus D_0) = (-1)^d \chi(X_i) - (-1)^d \chi(D_0).$$

Combining (2) and (3), we have

$$(4) \qquad \chi(X^\vee, W^{-1}(t)) = (-1)^d (\chi(X_1) + \chi(X_2) - 2\chi(D_0)).$$

On the other hand, by [Lee06, Proposition IV. 6],

(5) $$\chi(X) = \chi(X_1) + \chi(X_2) - 2\chi(D_0).$$

Therefore, (4) and (5) together imply the identity:

$$\chi(X) = (-1)^{\mathrm{d}}\chi(X^\vee, W^{-1}(t))$$

$\square$

2.3. **Periods.** The definition of relative periods for LG models is given in [DKY19]. One can define periods for higher rank LG models similarly.

Definition 2.9 ([DKY19], Definition 2.2). Let $f: \mathcal{X} \to B$ be a family of Calabi-Yau manifolds. The morphism $g: B \to \mathcal{M}$, taking a point in the base to the corresponding point in the moduli space $\mathcal{M}$ of the fibre is called the *generalized functional invariant* of the family.

Definition 2.10. Given a (rank k) LG model and a choice of holomorphic n-form ω, we define the *periods* of the (rank k) LG model *relative to the (rank k) superpotential and ω* to be the period functions associated to the varying fibres of the (rank k) LG model obtained by integrating transcendental cycles across the n-form ω.

We refer to [DKY19] for more details about the definition of relative periods and how to compute them via generalized functional invariants. In mirror symmetry, under suitable assumptions, periods are mirror to generating functions of genus zero Gromov-Witten invariants (see, for example, [Giv98] and [LLY97]). For LG models which are mirror to smooth pairs (X, D), the relative periods considered in [DKY19] are mirror to generating functions of genus zero Gromov-Witten invariants of (X, D) by the relative mirror theorem in [FTY19]. Now, if we have higher rank LG models which are mirror to a simple normal crossings pairs (X, D), then the relative periods that we consider in this paper are mirror to genus zero Gromov-Witten invariants of (X, D) ([TY20a], see also Section 9).

3. EXAMPLE 1: A COMPLETE INTERSECTION IN $\mathbb{P}^4 \times \mathbb{P}^1$

In this section, we consider an example which is analogous to the example considered in [DKY19, Section 3]. Let $\tilde{Q}_4$ be a complete intersection of bidegrees (3,1) and (1,1) in $\mathbb{P}^4 \times \mathbb{P}^1$. The threefold $\tilde{Q}_4$ admits a degeneration

$$\tilde{Q}_4 \rightsquigarrow \tilde{X}_1 \cup_C \tilde{X}_2,$$

where

- $\tilde{X}_1$ is a hypersurface of bidegree $(3, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$;
- $\tilde{X}_2$ is a complete intersection of bidegrees $(3, 0), (1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$;
- C is a smooth cubic surface in $\mathbb{P}^3$.

Indeed, $\tilde{X}_1$ is the blow-up of $\mathbb{P}^3$ along the complete intersection of two cubic surfaces and $\tilde{X}_2$ is the blow-up of a cubic threefold C_3 along the complete intersection of two hyperplanes (degree one hypersurfaces in C_3).

The smooth anticanonical $K3$ surface in $\tilde{Q}_4$ is a complete intersection of bidegrees $(3, 1)$ and $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$. The degeneration of $\tilde{Q}_4$ leads to a Tyurin degeneration of this $K3$ surface into a hypersurface D_1 of bidegree $(3, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$ and a complete intersection D_2

of bidegrees $(3,0), (1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$. This Tyurin degeneration of the $K3$ surface is the degeneration considered in [DKY19, Section 3] in one dimensional lower (i.e. in dimension two).

3.1. Functional invariants.
The mirrors of $\tilde{Q}_4$, $\tilde{X}_1$ and $\tilde{X}_2$ (relative to their smooth anti-canonical divisors) can be written down explicitly following [Giv98]. The mirror of $\tilde{Q}_4$ is an LG model $(\tilde{Q}_4^\vee, W)$ where $\tilde{Q}_4^\vee$ is the compactification of

$$\left\{ (x_0, x_1, x_2, x_3, y) \in (\mathbb{C}^*)^5 \,\middle|\, x_1 + x_2 + \frac{q_1}{x_0 x_1 x_2 x_3} + \frac{q_0}{y} = 1; x_3 + y = 1 \right\}$$

and the superpotential

$$W_1 : \tilde{Q}_4^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, x_3, y) \mapsto x_0.$$

The LG model for $\tilde{X}_1$ is

$$W_1 : \tilde{X}_1^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, y_1) \mapsto y_1 + x_0,$$

where $\tilde{X}_1^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y_1) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + \frac{q_{1,1}}{x_0 x_1 x_2} + \frac{q_{0,1}}{y_1} = 1 \right\}.$$

The LG model for $\tilde{X}_2$ is

$$W_2 : \tilde{X}_2^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, y_2) \mapsto y_2 + x_0,$$

where $\tilde{X}_2^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y_2) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + \frac{q_{1,2}}{x_0 x_1 x_2 (1 - q_{0,2}/y_2)} = 1 \right\}.$$

The rank 2 LG models for $(\tilde{X}_1, D_1 + C)$ and $(\tilde{X}_2, D_2 + C)$ are the following:

$$h_1 : \tilde{X}_1^\vee \to \mathbb{C}^2$$
$$(x_0, x_1, x_2, y_1) \mapsto (x_0, y_1)$$

and

$$h_2 : \tilde{X}_2^\vee \to \mathbb{C}^2$$
$$(x_0, x_1, x_2, y_2) \mapsto (x_0, y_2),$$

respectively.

The LG mirror of the cubic surface C (relative to its smooth anticanonical divisor $D_{12} = C \cap D_1 = C \cap D_2$) is:

$$W_0 : C^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2) \mapsto x_0,$$

where $C^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2) \in (\mathbb{C}^*)^3 \,\middle|\, x_1 + x_2 + \frac{q_{1,0}}{x_0 x_1 x_2} = 1 \right\}.$$

By performing an appropriate change of variables, we see that all four of these families, which are mirrors to $(\tilde{Q}_4, K3)$, $(\tilde{X}_1, D_1 + C)$, $(\tilde{X}_2, D_2 + C)$ and (C, D_{12}) respectively, are fibred by the mirror cubic curve whose defining equation is the following:

$$\left\{ (\tilde{x}_1, \tilde{x}_2) \in (\mathbb{C}^*)^2 \,\middle|\, \tilde{x}_1 + \tilde{x}_2 + \frac{\tilde{\lambda}}{\tilde{x}_1 \tilde{x}_2} = 1 \right\}.$$

This allows us to conclude that they are families of mirror cubic curves and read off their functional invariants. For example, for the family $\tilde{X}_1^\vee$, we make the following change of variables:

$$\tilde{x}_1 = \frac{x_1}{1 - q_{0,1}/y_1}, \tilde{x}_2 = \frac{x_2}{1 - q_{0,1}/y_1}, \tilde{\lambda} = \frac{q_{1,1}}{x_0(1 - q_{0,1}/y_1)^3},$$

to obtain the mirror cubic family. We then read off the functional invariant for $\tilde{X}_1^\vee$ as $\lambda_1 = \frac{q_{1,1}}{x_0(1-q_{0,1}/y_1)^3}$. Similarly, we compute the functional invariants for $C^\vee$, $\tilde{Q}_4^\vee$ and $\tilde{X}_2^\vee$ by making appropriate changes of variables to match with the cubic mirror family. We obtain the following functional invariants for $\tilde{Q}_4^\vee$, $C^\vee$, $\tilde{X}_1^\vee$ and $\tilde{X}_2^\vee$ respectively:

$$\lambda = \frac{q_1}{x_0(1-y)(1-q_0/y)^3}, \quad \lambda_0 = \frac{q_{1,0}}{x_0}, \quad \lambda_1 = \frac{q_{1,1}}{x_0(1-q_{0,1}/y_1)^3}, \quad \lambda_2 = \frac{q_{1,2}}{x_0(1-q_{0,2}/y_2)}.$$

We glue the second factors of the bases of rank 2 LG models through the following change of variables:

$$(6) \qquad q_1 = q_{1,0} = q_{1,1} = q_{1,2}, \quad q_0 = q_{0,1}, \quad y = y_1 = q_{0,2}/y_2.$$

Proposition 3.1. *Under the identification (6), the following product relation holds among the functional invariants:*

$$(7) \qquad \lambda \cdot \lambda_0 = \lambda_1 \cdot \lambda_2.$$

The discriminant loci of these fibrations are determined by the functional invariants. Note that the mirror cubic family of elliptic curves has singular fibres of types I_3, I_1, IV^* over the points $0, \frac{1}{3^3}, \infty$ respectively.

Functional Invariant λ_1:

- $\lambda_1^{-1}(0) = \{y^3 = 0\} \cup \{(q_0 - y)^3 = \infty\} \cup \{x_0 = \infty\}$. The first two components then support I_9 fibres while the third component supports I_3-fibres.
- $\lambda_1^{-1}(\infty) = \{y^3 = \infty\} \cup \{x_0 = 0\} \cup \{(q_0 - y)^3 = 0\}$. Since $\lambda_1 = \infty$ is a type IV^* fibre, it follows that only the locus $x_0 = 0$ supports fibres with non-trivial monodromy.
- $\lambda_1^{-1}(\frac{1}{3^3})$ is described by the locus $x_0(y - q_0)^3 = 3^3 q_1 y^3$ and generically supports I_1-fibres.

Functional Invariant λ_2:

- $\lambda_2^{-1}(0) = \{x_0 = \infty\} \cup \{1 - y = \infty\}$; both components then support I_3-fibres.
- $\lambda_2^{-1}(\infty) = \{x_0 = 0\} \cup \{y = 1\}$, both of which support IV^*-fibres.
- $\lambda_2^{-1}(\frac{1}{3^3})$ is described by the locus $x_0(1 - y) = 3^3 q_1$ and generically supports I_1-fibres.

Functional Invariant λ_0:

- $\lambda_0^{-1}(0) = \{x_0 = \infty\}$ is a locus of I_3-fibres.
- $\lambda_0^{-1}(\infty) = \{x_0 = 0\}$ is a locus of IV^*-fibres.
- $\lambda_0^{-1}(\frac{1}{3^3})$ is described by $x_0 = 3^3 q_1$ and generically supports I_1-fibres.

Functional Invariant λ:

- $\lambda^{-1}(0) = \{y^3 = 0\} \cup \{(y - q_0)^3 = \infty\} \cup \{x_0 = \infty\} \cup \{(1 - y) = \infty\}$ with the first two components supporting I_9-fibres and the last two components supporting I_3-fibres.
- The locus $\lambda^{-1}(\infty) = \{x_0 = \infty\} \cup \{y = 1\} \cup \{(y - q_0)^3\}$ and the first two components support IV^*-fibres and the last components supports smooth fibres.
- The locus $\lambda^{-1}(\frac{1}{3^3})$ is described by $x_0(1 - y)(y - q_0)^3 = 3^3 q_1 y^3$ and supports I_1-fibres generically.

Note that the singular loci for the invariant λ is essentially the union of those corresponding to λ_1, λ_2 with the λ_0 factor eliminating excess branching at both $x_0 = 0$ and $x_0 = \infty$. Therefore one may also write the identity (7) as

$$\lambda = \frac{\lambda_1 \lambda_2}{\lambda_0}.$$

3.2. **Periods.** Recall that, the holomorphic period for the mirror cubic curve family is

$$f_0^{E^\vee}(\tilde{\lambda}) = \sum_{d \geq 0} \frac{(3d)!}{(d!)^3} \tilde{\lambda}^d.$$

Following [DKY19], the relative period for the LG model $(C^\vee, W_0)$ is

$$f_0^{C^\vee}(q_{1,0}, x_0) = f_0^{E^\vee}(\lambda_0) = \sum_{d \geq 0} \frac{(3d)!}{(d!)^3} (q_{1,0}/x_0)^d.$$

The relative period for the LG model $(\tilde{Q}_4^\vee, W)$ can be computed as a residue integral of the pullback of

$$\frac{1}{(1 - y)(1 - q_0/y)} f_0^{E^\vee}(\tilde{\lambda})$$

by the functional invariant

$$\lambda = \frac{q_1}{x_0(1 - y)(1 - q_0/y)^3}.$$

We have

$$\frac{1}{(1 - y)(1 - q_0/y)} f_0^{E^\vee}(\lambda) = \frac{1}{(1 - y)(1 - q_0/y)} \sum_{d_1 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_1}{x_0(1 - y)(1 - q_0/y)^3} \right)^{d_1}$$

$$= \frac{1}{(1 - q_0/y)} \sum_{d_1, d_{0,1} \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_1}{x_0(1 - q_0/y)^3} \right)^{d_1} \frac{(d_1 + d_{0,1})!}{d_1! d_{0,1}!} (y)^{d_{0,1}}$$

$$= \sum_{d_1, d_{0,1}, d_{0,2} \geq 0} \frac{(3d_1)!}{(d_1!)^3} (q_1/x_0)^{d_1} \frac{(d_1 + d_{0,1})!}{d_1! d_{0,1}!} \frac{(3d_1 + d_{0,2})!}{(3d_1)! d_{0,2}!} (y)^{d_{0,1}} (q_0/y)^{d_{0,2}},$$

where we used the identity

$$\frac{1}{(1 - x)^{k+1}} = \sum_{d=0}^{\infty} \frac{(d + k)!}{d! k!} x^d.$$

After taking a residue, we obtain the holomorphic relative period of $(\tilde{Q}_4^\vee, W)$:

$$
\begin{aligned}
f_0^{\tilde{Q}_4^\vee}(q_1, q_0, x_0) &= \frac{1}{2\pi i} \oint \frac{1}{(1-y)(1-q_0/y)} f_0^{E^\vee}(\lambda) \frac{dy}{y} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!}{(d_1!)^3} (q_1/x_0)^{d_1} \frac{(d_1+d_0)!}{d_1! d_0!} \frac{(3d_1+d_0)!}{(3d_1)! d_0!} (q_0)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1+d_0)!(d_1+d_0)!}{(d_1!)^4 (d_0!)^2} (q_1/x_0)^{d_1} q_0^{d_0}.
\end{aligned}
$$
(8)

Remark 3.2. The relative period (8) matches with the computation for the relative period of the LG model $(\tilde{Q}_4^\vee, W)$ when the relative period is pullback from the period of mirror quartic $K3$ surface family. In that case, the computation will be as follows. The smooth anticanonical divisor of $\tilde{Q}_4$ is a $K3$ surface. The period of the mirror $K3$ surface family is the following:

$$
\sum_{d_1, d_0 \geq 0} \frac{(3d_1+d_0)!(d_1+d_0)!}{(d_1!)^4 (d_0!)^2} (q_1)^{d_1} q_0^{d_0}.
$$
(9)

The generalized functional invariant for $(X^\vee, W)$ with respect to the mirror $K3$ surface family is

$$
\lambda = \frac{q_1}{x_0}.
$$

The relative period can be computed by pulling back the period of the mirror $K3$ surface family via the generalized functional invariants. The resulting relative period is precisely (8). The reason why these two computations match follows from the iterative structure of periods that we explained in [DKY19, Section 2.2.1]. Since the mirror $K3$ surface family is fibred by the mirror cubic curve family, the period (9) of the mirror $K3$ can be computed as the residue of the period of the mirror cubic via the functional invariant.

The relative periods for the rank 2 LG models can also be computed as pullback of the period of the mirror cubic via functional invariants:

$$
\begin{aligned}
f_0^{\tilde{X}_1}(q_{1,1}, q_{0,1}, y_1, x_0) &= \frac{1}{(1-q_{0,1}/y_1)} f_0^{E^\vee}(\lambda_1) \\
&= \frac{1}{(1-q_{0,1}/y_1)} \sum_{d_1 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_{1,1}}{x_0(1-q_{0,1}/y_1)^3} \right)^{d_1} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!(3d_1+d_0)!}{(d_1!)^3 (3d_1)! d_0!} (q_{1,1}/x_0)^{d_1} (q_{0,1}/y_1)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1+d_0)!}{(d_1!)^3 d_0!} (q_{1,1}/x_0)^{d_1} (q_{0,1}/y_1)^{d_0};
\end{aligned}
$$

$$f_0^{\tilde{X}_2}(q_{1,2}, q_{0,2}, y_2, x_0) = \frac{1}{(1 - q_{0,2}/y_2)} f_0^{E^\vee}(\lambda_2)$$

$$= \frac{1}{(1 - q_{0,2}/y_2)} \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_{1,2}}{x_0(1 - q_{0,2}/y_2)} \right)^{d_1}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!(d_1 + d_0)!}{(d_1!)^3 (d_1!) d_0!} (q_{1,2}/x_0)^{d_1} (q_{0,2}/y_2)^{d_0}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!(d_1 + d_0)!}{(d_1!)^4 d_0!} (q_{1,2}/x_0)^{d_1} (q_{0,2}/y_2)^{d_0}.$$

Under the identification (6), we can write relative periods as follows:

$$f_0^{C^\vee}(q_1, x_0) = f_0^{E^\vee}(\lambda_0) = \sum_{d \geq 0} \frac{(3d)!}{(d!)^3} (q_1/x_0)^d;$$

$$f_0^{\tilde{Q}_4^\vee}(q_1, q_0, x_0) = \sum_{d_1, d_0 \geq 0} \frac{(3d_1 + d_0)!(d_1 + d_0)!}{(d_1!)^4 (d_0!)^2} (q_1/x_0)^{d_1} q_0^{d_0};$$

$$f_0^{\tilde{X}_1}(q_1, q_0, y, x_0) = \sum_{d_1, d_0 \geq 0} \frac{(3d_1 + d_0)!}{(d_1!)^3 d_0!} (q_1/x_0)^{d_1} (q_0/y)^{d_0};$$

$$f_0^{\tilde{X}_2}(q_1, y, x_0) = \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!(d_1 + d_0)!}{(d_1!)^4 d_0!} (q_1/x_0)^{d_1} (y)^{d_0}.$$

Relative periods satisfy the Hadamard product formula similar to the Calabi-Yau case.

Theorem 3.3. *The holomorphic relative periods of rank 2 LG models can be glued together to form the holomorphic relative period of the LG model $(\tilde{Q}_4^\vee, W)$ with the correction given by the holomorphic relative period of the LG model $(C^\vee, W_0)$. More precisely, the relation is given by the Hadamard product*

$$f_0^{\tilde{Q}_4}(q_1, q_0, x_0) \star_{q_1} f_0^{C^\vee}(q_1, x_0) = \frac{1}{2\pi i} \oint f_0^{\tilde{X}_1}(q_1, q_0, y, x_0) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y, x_0) \frac{dy}{y},$$

where $\star_{q_1}$ means the Hadamard product with respect to the variable q_1 and we used (6) to identify variables.

Proof. The computation is straightforward:

$$\frac{1}{2\pi i} \oint f_0^{\tilde{X}_1}(q_1, q_0, y, x_0) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y, x_0) \frac{dy}{y}$$

$$= \frac{1}{2\pi i} \oint \sum_{d_1, d_{0,1}, d_{0,2} \geq 0} \frac{(3d_1 + d_{0,1})!}{(d_1!)^3 d_{0,1}!} \frac{(3d_1)!(d_1 + d_{0,2})!}{(d_1!)^4 d_{0,2}!} (q_1/x_0^2)^{d_1} (q_0/y)^{d_{0,1}} (y)^{d_{0,2}} \frac{dy}{y}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(3d_1 + d_0)!}{(d_1!)^3 d_0!} \frac{(3d_1)!(d_1 + d_0)!}{(d_1!)^4 d_0!} (q_1/x_0^2)^{d_1} q_0^{d_0}$$

$$= f_0^{\tilde{Q}_4}(q_1, q_0, x_0) \star_{q_1} f_0^{C^\vee}(q_1, x_0).$$

$\square$

Remark 3.4. Note that, in $f_0^{\tilde{Q}_4}$, $f_0^{C^\vee}$, $f_0^{\tilde{X}_1}$ and $f_0^{\tilde{X}_2}$, variables q_1 and x_0 always appear together as the factor q_1/x_0. We can also simply absorb the factor $1/x_0$ into q_1 by rescaling q_1. Then Theorem 3.3 stays the same.

Furthermore, the periods $f_0^{\tilde{Q}_4}$, $f_0^{C^\vee}$, $f_0^{\tilde{X}_1}$ and $f_0^{\tilde{X}_2}$ are solutions to systems of PDEs. We can write down the bases of solutions to these four systems of PDEs (Note that we absorb the factor $1/x_0$ into q_1 as mentioned in Remark 3.4):

$$I^{\tilde{Q}_4}(q_1, q_0) = q_1^H q_0^P \sum_{d_1, d_0 \geq 0}^{\infty} q_1^{d_1} q_0^{d_0} \frac{\prod_{k=1}^{3d_1+d_0}(3H + P + k) \prod_{k=1}^{d_1+d_0}(H + P + k)}{\prod_{k=1}^{d_1}(H + k)^4 \prod_{k=1}^{d_0}(P + k)^2} \qquad \mathrm{mod}\ H^2, P^2,$$

$$I^{C^\vee}(q_1) = q_1^H \sum_{d_1 \geq 0}^{\infty} q_1^{d_1} \frac{\prod_{k=1}^{3d_1}(3H + k)}{\prod_{k=1}^{d_1}(H + k)^3} \qquad \mathrm{mod}\ H^2,$$

$$I^{\tilde{X}_1}(q_{1,1}, x) = q_{1,1}^H x^P \sum_{d_1, d_0 \geq 0}^{\infty} q_{1,1}^{d_1} x^{d_0} \frac{\prod_{k=1}^{3d_1+d_0}(3H + P + k)}{\prod_{k=1}^{d_1}(H + k)^3 \prod_{k=1}^{d_0}(P + k)} \qquad \mathrm{mod}\ H^2, P^2,$$

$$I^{\tilde{X}_2}(q_{1,2}, y) = q_{1,2}^H y^P \sum_{d_1, d_0 \geq 0}^{\infty} q_{1,2}^{d_1} y^{d_0} \frac{\prod_{k=1}^{3d_1}(3H + k) \prod_{k=1}^{d_1+d_0}(H + P + k)}{\prod_{k=1}^{d_1}(H + k)^4 \prod_{k=1}^{d_0}(P + k)} \qquad \mathrm{mod}\ H^2, P^2,$$

where $q_1^H = e^{H \log q_1}$ and similarly for other prefactors of this form. They again satisfy the Hadamard product formula. The following theorem follows from a straightforward computation similar to the proof of Theorem 3.3.

Theorem 3.5. *We have the following Hadamard product relation*

$$(10) \qquad I^{\tilde{Q}_4}(q_1, q_0) \star_{q_1} I^{C^\vee}(q_1) = \frac{1}{2\pi i} \oint I^{\tilde{X}_1}(q_1, q_0/y) \star_{q_1} I^{\tilde{X}_2}(q_1, y) \frac{dy}{y} \qquad \mathrm{mod}\ H^2, P^2.$$

In (10), we treat $\log q_1$ as a variable that is independent from q_1. Alternatively, one may consider $\bar{I}(q_1, q_0) := I(q_1, q_0)/q_1^H q_0^P$ and write (10) in terms of $\bar{I}$.

4. EXAMPLE 2: QUARTIC THREEFOLD

In this section, we consider the semi-stable degeneration of a smooth quartic threefold Q_4 into a cubic threefold C_3 and the blow-up $\mathrm{Bl}_C \mathbb{P}^3$ of $\mathbb{P}^3$ along complete intersection center of degrees 3 and 4 hypersurfaces, that is,

$$Q_4 \rightsquigarrow C_3 \cup_{C_2} \mathrm{Bl}_C \mathbb{P}^3,$$

where C is the complete intersection of degrees 3 and 4 hypersurfaces in $\mathbb{P}^3$, and C_2 is the common intersection which is a cubic surface. Note that the smooth anticanonical $K3$ surface in Q_4 also degenerates. It degenerates into a cubic surface C_2' which is in C_3 and a blow-up $\mathrm{Bl}\,\mathbb{P}^2$ of $\mathbb{P}^2$ along a complete intersection of degrees 3 and 4 curves. The blown-up variety $\mathrm{Bl}\,\mathbb{P}^2$ is a hypersurface in $\mathrm{Bl}_C \mathbb{P}^3$. Therefore, we degenerate $(Q_4, K3)$ into (C_3, C_2') and $(\mathrm{Bl}_C \mathbb{P}^3, \mathrm{Bl}\,\mathbb{P}^2)$ intersecting along C_2.

We would like to write down the rank 2 LG models for $(C_3, C_2 + C_2)$ and $(\mathrm{Bl}_C \mathbb{P}^3, \mathrm{Bl}\,\mathbb{P}^2 + C_2)$ and glue them to the mirror LG model of $(Q_4, K3)$. Since $\mathrm{Bl}_C \mathbb{P}^3$ can be written as a

hypersurface in a toric variety, we can write down its LG model following Givental [Giv98]. For the rest of this section, we write

$$X := Q_4, \quad D_0 := C_2, \quad X_1 := C_3, \quad \text{and } \tilde{X}_2 := \mathrm{Bl}_C \, \mathbb{P}^3.$$

We also write D for the smooth anticanonical $K3$ surface in X. We write

$$D_1 := C_2' \subset X_1, \quad D_2 := \mathrm{Bl} \, \mathbb{P}^2 \subset X_2, \quad \text{and } D_{12} := D_1 \cap D_2.$$

4.1. **Functional invariants.** The mirror LG model of X is defined by

$$W_1 : X^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, y) \mapsto x_0,$$

where $X^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y) \in (\mathbb{C}^*)^4 \, \middle| \, x_1 + x_2 + y + \frac{q_{1,1}}{x_0 x_1 x_2 y} = 1 \right\}.$$

The mirror LG model of the cubic surface D_0 is defined by

$$W_1 : D_0^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2) \mapsto x_0,$$

where $D_0^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2) \in (\mathbb{C}^*)^3 \, \middle| \, x_1 + x_2 + \frac{q_{1,0}}{x_0 x_1 x_2} = 1 \right\}.$$

The mirror LG model of X_1 is

$$W_1 : X_1^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, y_1) \mapsto x_0 + y_1,$$

where $X_1^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y_1) \in (\mathbb{C}^*)^4 \, \middle| \, x_1 + x_2 + \frac{q_{1,1}}{x_0 x_1 x_2 y_1} = 1 \right\}.$$

Following [CCGK16, Section E], the quasi-Fano variety $\tilde{X}_2 = \mathrm{Bl}_C(\mathbb{P}^3)$ can be constructed as a hypersurface of degree $(3,1)$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3})$. The LG model of $\tilde{X}_2$ is defined as follows

$$W_2 : \tilde{X}_2^\vee \to \mathbb{C}$$
$$(x_0, x_1, x_2, y_2) \to x_0 + y_2,$$

where $\tilde{X}_2^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y_2) \in (\mathbb{C}^*)^4 \, \middle| \, x_1 + x_2 + \frac{q_{0,2}}{y_2} + \frac{q_{1,2} y_2}{x_0 x_1 x_2} = 1 \right\}.$$

The rank 2 LG models for $(X_1, D_1 + D_0)$ and $(\tilde{X}_2, D_2 + D_0)$ are

$$h_1 = (h_{11}, h_{12}) : X_1^\vee \to \mathbb{C}^2$$
$$(x_0, x_1, x_2, y_1) \mapsto (x_0, y_1),$$

and

$$h_2 = (h_{21}, h_{22}) : \tilde{X}_2^\vee \to \mathbb{C}^2$$
$$(x_0, x_1, x_2, y_2) \to (x_0, y_2)$$

respectively. Recall that the mirror cubic curve family is given by the compactification of

$$\left\{ (\tilde{x}_1, \tilde{x}_2) \in (\mathbb{C}^*)^2 \;\middle|\; \tilde{x}_1 + \tilde{x}_2 + \frac{\tilde{\lambda}}{\tilde{x}_1 \tilde{x}_2} = 1 \right\}.$$

Similar to the computation in Section 3.1, we obtain the functional invariants for $X^\vee$, $D_0^\vee$, $X_1^\vee$ and $\tilde{X}_2^\vee$ respectively :

$$(11) \qquad \lambda = \frac{q_1}{x_0 y (1 - y)^3}, \qquad \lambda_0 = \frac{q_{1,0}}{x_0}, \qquad \lambda_1 = \frac{q_{1,1}}{x_0 y_1}, \qquad \lambda_2 = \frac{q_{1,2} y_2}{x_0 (1 - q_{0,2}/y_2)^3}.$$

We set

$$(12) \qquad\qquad q_1 = q_{1,0} = q_{1,1} = q_{1,2} y_2, \quad y = y_1 = q_{0,2}/y_2.$$

The matching of singular fibres works similarly as in Section 3. We have

Proposition 4.1. *Under the identification (12), the following product relation holds among the functional invariants:*

$$(13) \qquad\qquad \lambda \cdot \lambda_0 = \lambda_1 \cdot \lambda_2.$$

4.2. **Periods.** Similar to Section 3, relative periods can be computed as pullbacks of the period for the mirror cubic curve family via the functional invariants. The relative period for $(X^\vee, W)$ is

$$f_0^{X^\vee}(q_1, x_0) = \sum_{d_1 \geq 0} (q_1/x_0)^{d_1} \frac{(4d_1)!}{(d_1!)^4}.$$

The relative period for $(D_0^\vee, W_0)$ is

$$f_0^{D_0^\vee}(q_{1,0}, x_0) = \sum_{d_1 \geq 0} (q_{1,0}/x_0)^{d_1} \frac{(3d_1)!}{(d_1!)^3}.$$

The relative period for $(X_1^\vee, h_1)$ is

$$f_0^{X_1^\vee}(q_{1,1}, y_1, x_0) = f_0^{E^\vee}(\lambda_1) = \sum_{d_1 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_{1,1}}{x_0 y_1} \right)^{d_1}.$$

The relative period for $(\tilde{X}_2^\vee, h_2)$ is

$$
\begin{aligned}
f_0^{\tilde{X}_2^\vee}(q_{1,2} y_2, q_{0,2}/y_2, x_0) &= \frac{1}{1 - q_{0,2}/y_2} f_0^{E^\vee}(\lambda_2) \\
&= \frac{1}{1 - q_{0,2}/y_2} \sum_{d_1 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_{1,2} y_2}{x_0 (1 - q_{0,2}/y_2)^3} \right)^{d_1} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1)!}{(d_1!)^3} (q_{1,2} y_2/x_0)^{d_1} \frac{(3d_1 + d_0)!}{(3d_1)! d_0!} (q_{0,2}/y_2)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(3d_1 + d_0)!}{(d_1)!^3 d_0!} (q_{1,2} y_2/x_0)^{d_1} (q_{0,2}/y_2)^{d_0}.
\end{aligned}
$$

Note that $q_1 = q_{1,1} = q_{1,2}y_2$ and $y = y_1 = q_{0,2}/y_2$ when we glue the rank 2 LG models. We may rewrite the relative period for $(\tilde{X}_2^\vee, h_2)$ as

$$f_0^{\tilde{X}_2^\vee}(q_1, y, x_0) = \sum_{d_1, d_0 \geq 0} \frac{(3d_1 + d_0)!}{(d_1)!^3 d_0!}(q_1/x_0)^{d_1}(y)^{d_0}.$$

Theorem 4.2. *We have the following Hadamard product relation:*

$$f_0^{X^\vee}(q_1, x_0) \star_{q_1} f_0^{D_0^\vee}(q_1, x_0) = \frac{1}{2\pi i} \oint f_0^{X_1^\vee}(q_1, y, x_0) \star_{q_1} f_0^{\tilde{X}_2^\vee}(q_1, y, x_0)\frac{dy}{y}.$$

As mentioned in Remark 3.4, we can absorb the factor $1/x_0$ into the factor q_1. These periods are solutions to systems of PDEs. The bases of solutions for the PDEs are as follows:

$$I^{Q_4}(q_1) = q_1^H \sum_{d \geq 0}\left(\frac{\prod_{k=1}^{4d}(4H + k)}{\prod_{k=1}^d(H + k)^4}\right)q_1^d;$$

$$I^{C^\vee}(q_1) = q_1^H \sum_{d_1 \geq 0}^\infty \frac{\prod_{k=1}^{3d_1}(3H + k)}{\prod_{k=1}^{d_1}(H + k)^3}q_1^{d_1};$$

$$I^{X_1}(q_1, y) = (q_1/y)^H \sum_{d_1 \geq 0}^\infty \frac{\prod_{k=1}^{3d_1}(3H + k)}{\prod_{k=1}^{d_1}(H + k)^3}(q_1/y)^{d_1};$$

$$I^{\tilde{X}_2}(q_1, y) = q_1^H y^P \sum_{d_1, d_0 \geq 0}^\infty \frac{\prod_{k=1}^{3d_1+d_0}(3H + P + k)}{\prod_{k=1}^{d_1}(H + k)^3 \prod_{k=1}^{d_0}(P + k)}q_1^{d_1}y^{d_0}.$$

Theorem 4.3. *The following Hadamard product relation holds:*

(14) $$I^{Q_4}(q_1) \star_{q_1} I^{C^\vee}(q_1) = \frac{1}{2\pi i} \oint I^{X_1}(q_1, y) \star_{q_1} I^{\tilde{X}_2}(q_1, y)\frac{dy}{y} \qquad \mathrm{mod}\ H^2,$$

where we set $P = H$. In (14), we treat $\log q_1$ as a variable that is independent from q_1. Alternatively, we can consider $\bar{I}(q_1) := I(q_1)/q_1^H$ and write (14) in terms of $\bar{I}$.

5. TORIC COMPLETE INTERSECTIONS

5.1. **Set-up.** Let X be a complete intersection in a toric variety Y defined by a generic section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s$, where each L_l is a nef line bundle for $0 \leq l \leq s$. Let $\rho_l = c_1(L_l)$, we assume that $-K_Y - \sum_{l=0}^s \rho_l$ is a sum of toric prime divisors (not necessarily nef). Without this assumption, the Hori-Vafa mirror may not be of the correct dimension.

Let $\mathbf{s}_l \in H^0(Y, L_l)$, for $0 \leq l \leq s$, be generic sections determining X. We consider a refinement of the partition with respect to L_0 such that $L_{0,1}, L_{0,2}$ are two nef line bundles and $L_0 = L_{0,1} \otimes L_{0,2}$. Let $\rho_{0,1} = c_1(L_{0,1})$ and $\rho_{0,2} = c_1(L_{0,2})$. Then we have two varieties X_1 and X_2 defined by generic sections of $E_1 = L_{0,1} \oplus L_1 \oplus \cdots \oplus L_s$ and $E_2 = L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s$ respectively.

Let $\mathbf{s}_{0,1} \in H^0(Y, L_{0,1})$ and $\mathbf{s}_{0,2} \in H^0(Y, L_{0,2})$ be the generic sections determining X_1 and X_2. We can construct a pencil of complete intersections as follows. Let

$$X' = \cap_{l=1}^s \{\mathbf{s}_l = 0\}.$$

We assume that X' is connected and quasi-smooth. Then we consider the pencil

$$\mathfrak{X} := \{t\mathbf{s}_0 - \mathbf{s}_{0,1}\mathbf{s}_{0,2} = 0\} \cap X'$$

in $\mathbb{A}^1 \times Y$ where t is a parameter on $\mathbb{A}^1$. $\mathfrak{X}$ has singularities along $t = \mathbf{s}_0 = \mathbf{s}_{0,1} = \mathbf{s}_{0,2} = 0$. We resolve the singularities of $\mathfrak{X}$ by blowing up $t = \mathbf{s}_{0,2} = 0$ inside $\mathbb{A}^1 \times Y$ and take the proper transform of $\mathfrak{X}$. This gives a degeneration of X such that the central fibre of the degeneration is the union of X_1 and $\tilde{X}_2$, where $\tilde{X}_2$ is given by blowing up X_2 along $X' \cap \{\mathbf{s}_0 = \mathbf{s}_{0,1} = \mathbf{s}_{0,2} = 0\}$. Two varieties X_1 and $\tilde{X}_2$ intersect along D_0 which is a complete intersection in the toric variety Y defined by a generic section of $L_{0,1} \oplus L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s$.

Let X be quasi-Fano, then its anticanonical linear system contains a smooth Calabi-Yau member. Let D be a smooth anticanonical divisor of X. Then the degeneration of X leads to a degeneration of D into $D_1 \cup_{D_{12}} \tilde{D}_2$ where $D_1 \subset X_1$, $\tilde{D}_2 \subset \tilde{X}_2$ and $D_{12} = D_1 \cap \tilde{D}_2 \subset D_0$. We do not assume $-K_Y - \sum_{l=0}^{s} \rho_l$ is nef on Y. But we assume that the restriction of $-K_Y - \sum_{l=0}^{s} \rho_l$ is nef on a toric subvariety $Y' \subset Y$ and D is defined by a generic section of $E' = L_0' \oplus L_1' \oplus \cdots \oplus L_s' \oplus L_{s+1}'$, where L_i', $0 \le i \le s+1$, are nef line bundles of Y' and L_i' are restrictions of L_i on Y' for $0 \le i \le s$. For example, let X be the blow-up of $\mathbb{P}^3$ along a complete intersection of degree 4 and 5 hypersurfaces. It can be realized as a degree $(4,1)$ hypersurface in the toric variety $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3})$, see [DKY19, Section 4]. Note that $-K_X$ is not nef. However, the smooth anticanonical divisor of X is a quartic $K3$ surface in $\mathbb{P}^3$. Therefore, this example is within our assumption. The Tyurin degeneration of D is obtained by a refinement of the nef partition which is induced by the refinement of the nef partition of E. In other words, we consider a refinement of the nef partition with respect to L_0' such that $L_0' = L_{0,1}' \otimes L_{0,2}'$, where $L_{0,1}', L_{0,2}'$ are two nef line bundles given by the restriction of $L_{0,1}$ and $L_{0,2}$ to Y' respectively.

We can realize $\tilde{X}_2$ as a complete intersection in a toric variety following [CCGK16, Section E]. Indeed, it is a hypersurface in the total space of $\pi : \mathbb{P}_{X_2}(\mathcal{O} \oplus i^*L_{0,2}^{-1}) \to X_2$ defined by a generic section of the line bundle $\pi^*i^*L_{0,1} \otimes \mathcal{O}(1)$, where $i : X_2 \hookrightarrow Y$ is the inclusion map. In other words, it is a complete intersection in the toric variety $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ given by a generic section of $\pi^*L_{0,2} \oplus \pi^*L_1 \oplus \cdots \oplus \pi^*L_s \oplus (\pi^*L_{0,1} \otimes \mathcal{O}(1))$ where we use the same π for the projection of $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ to the base Y.

Let $p_1, \ldots, , p_r \in H^2(Y, \mathbb{Z})$ be a nef integral basis. We write the toric divisors as

$$\mathbf{D}_j = \sum_{i=1}^{r} m_{ij} p_i, \quad 1 \le j \le m,$$

for some m_{ij}.

The partition of the toric divisors gives a partition of the variables $x_1, \ldots, x_m$ into $s + 2$ groups. Let $F_l(x)$ be the sum of x_i in each group for $l = 0, \ldots, s, s+1$. Let $F_{0,1}(x)$ and $F_{0,2}(x)$ correspond to the refinement of the nef partition with respect to L_0. We write $F_{0,1}(x) = x_{j_1} + \ldots, x_{j_a}$ and $F_{0,2}(x) = x_{j_{a+1}} + \ldots, x_{j_{a+b}}$. Note that we have $F_{0,1}(x) + F_{0,2}(x) = F_0(x)$.

Following [Giv98], the Hori-Vafa mirror of X is the LG model

$$W : X^\vee \to \mathbb{C}^*$$
$$(x_1, \ldots, x_m) \mapsto F_{s+1}(x),$$

where $X^\vee$ is the fibrewise Calabi-Yau compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^{m} x_j^{m_{ij}} = q_i, i = 1, \ldots, r; F_l(x) = 1, l = 0, \ldots, s \right. \right\}.$$

The mirror for D_0 is the LG model

$$W_0 : D_0^\vee \to \mathbb{C}^*$$
$$(x_1, \ldots, x_m) \mapsto F_{s+1}(x),$$

where $D_0^\vee$ is the fibrewise Calabi-Yau compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^{m} x_j^{m_{ij}} = q_i, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1, F_{0,2}(x) = 1 \right\}.$$

The mirror for $(X_1, D_0 + D_1)$ is the rank 2 LG model

$$h_1 : X_1^\vee \to (\mathbb{C}^*)^2$$
$$(x_1, \ldots, x_m) \mapsto (F_{s+1}(x), F_{0,2}(x)),$$

where $X_1^\vee$ is the fibrewise compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^{m} x_j^{m_{ij}} = q_{i,1}, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1 \right\}.$$

$\tilde{X}_2$ is a complete intersection in the toric variety $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ given by a generic section of $L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s \oplus (\pi^* L_{0,1} \otimes \mathcal{O}(1))$. The rank 2 LG model for $(\tilde{X}_2, D_0 + \tilde{D}_2)$ is

$$h_2 : \tilde{X}_2^\vee \to (\mathbb{C}^*)^2$$
$$(x_1, \ldots, x_m, y_2) \mapsto (F_{s+1}(x), y_2),$$

where $\tilde{X}_2^\vee$ is the fibrewise compactification of

$$\left\{ (x_1, \ldots, x_m, y_2) \in (\mathbb{C}^*)^{m+1} \left| \left(\prod_{j=1}^{m} x_j^{m_{ij}} \right) y_2^{-\sum_{k=a+1}^{a+b} m_{ij_k}} = q_{i,2}, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) + \frac{q_{0,2}}{y_2} = 1, F_{0,2}(x) = 1 \right\}.$$

5.2. **Generalized functional invariants.** Then, we can compute the generalized functional invariants. Recall that $X^\vee$ is the compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^{m} x_j^{m_{ij}} = q_i, i = 1, \ldots, r; F_l(x) = 1, l = 0, \ldots, s \right. \right\}.$$

Set $F_{0,2}(x) = y$, then $F_{0,1}(x) = 1 - y$. For $y \neq 1$, the following change of variables can give us $D_0^\vee$.

$$\bar{x}_{j_k} = \frac{x_{j_k}}{1-y}, k = 1, \ldots, a; \quad \bar{x}_{j_k} = \frac{x_{j_k}}{y}, k = a+1, \ldots, a+b;$$

and

$$\lambda_i = \frac{q_i}{\prod_{k=1}^{a}(1-y)^{m_{ij_k}} \prod_{k=a+1}^{a+b}(y)^{m_{ij_k}}}, i = 1 \ldots, r.$$

Therefore, the generalized functional invariant for $X^\vee$ is $\vec{\lambda} = (\lambda_1, \ldots, \lambda_r)$, where

$$(15) \qquad \lambda_i = \frac{q_i}{\prod_{k=1}^{a}(1-y)^{m_{ij_k}} \prod_{k=a+1}^{a+b}(y)^{m_{ij_k}}}, i = 1 \ldots, r.$$

For $X_1^\vee$, we also set $F_{0,2}(x) = y_1$. By similar computation, the generalized functional invariant is $\vec{\lambda}_1 = (\lambda_{1,1}, \ldots, \lambda_{r,1})$, where

$$(16) \qquad \lambda_{i,1} = \frac{q_{i,1}}{\prod_{k=a+1}^{a+b} y_1^{m_{ij_k}}}, i = 1 \ldots, r.$$

Similarly, the generalized functional invariant for $\tilde{X}_2$ is $\vec{\lambda}_2 = (\lambda_{1,2}, \ldots, \lambda_{r,2})$, where

$$(17) \qquad \lambda_{i,2} = \frac{q_{i,2}}{y_2^{-\sum_{k=a+1}^{a+b} m_{ij_k}} \prod_{k=1}^{a}(1 - q_{0,2}/y_2)^{m_{ij_k}}}, i = 1 \ldots, r.$$

We set

$$(18) \qquad q_i = q_{i,1} = q_{i,2} y_2^{\sum_{k=a+1}^{a+b} m_{ij_k}}, i = 1, \ldots, r, \quad y = y_1 = q_{0,2}/y_2.$$

Proposition 5.1. *Under the identification (18), the generalized functional invariants satisfy the product relation*

$$\lambda_i q_i = \lambda_{i,1} \lambda_{i,2}, \quad i = 1, \ldots, r.$$

Remark 5.2. Note that the generalized functional invariants in here are with respect to the mirror family of D_0. This is slightly different from the generalized functional invariants considered in Section 3 and Section 4 where the functional invariants are with respect to the mirror family of the smooth anticanonical divisor D_{12} of D_0. In Proposition 5.1 q_i is considered as the generalized functional invariants for D_0 with respect to itself.

We can also compute the generalized functional invariants when the base family is the mirror to the smooth anticanonical divisor D_{12} of D_0. Recall that we have a Tyurin degeneration of D into $D_1 \cup_{D_{12}} \tilde{D}_2$, where $D_1 \subset X_1$, $\tilde{D}_2 \subset \tilde{X}_2$, induced by a refinement of nef partition. If we assume that $-K_Y - \sum_{l=0}^{s} c_1(L_l)$ is nef, then $D_{12}^\vee$ is defined by the fibrewise Calabi-Yau compactification of

$$(19) \qquad \left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^{m} x_j^{m_{ij}} = q_i, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s+1; F_{0,1}(x) = 1, F_{0,2}(x) = 1 \right\}.$$

If we only assume that the restriction of $-K_Y - \sum_{l=0}^{s} c_1(L_l)$ is nef on a toric subvariety $Y' \subset Y$, then $D_{12}^\vee$ is a specialization of (19) to $q_i = 0$ for some $i \in \{1, \ldots, r\}$. We will simply use (19) to do the computation and we can specialize to $q_i = 0$, for some $i \in \{1, \ldots, r\}$, when it is necessary.

Then, we can compute generalized functional invariants with respect to $D_{12}^\vee$. Recall that the mirror for D_0 is the LG model

$$W_0 : D_0^\vee \to \mathbb{C}^*$$
$$(x_1, \ldots, x_m) \mapsto F_{s+1}(x),$$

where $D_0^\vee$ is the fibrewise Calabi-Yau compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \,\middle|\, \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; \right.$$
$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1, F_{0,2}(x) = 1 \right\}.$$

Let I_{s+1} be the index set for the summand of $F_{s+1}(x)$. We set $F_{s+1}(x) = y_0$ and consider the change of variables

$$\bar{x}_j = \frac{x_j}{y_0}, j \in I_{s+1}$$

and

$$\lambda_{i,0} = \frac{q_i}{\prod_{j \in I_{s+1}}^{a+b} y_0^{m_{ij}}}, i = 1 \ldots, r.$$

Generalized functional invariants of $X^\vee$, $X_1^\vee$ and $\tilde{X}_2^\vee$ with respect to $D_{12}^\vee$ can also be computed. The computation is similar to the computation for (15), (16) and (17) except that there is an extra change of variables

$$\bar{x}_j = \frac{x_j}{y_0}, j \in I_{s+1}.$$

Therefore, the generalized functional invariants of $X^\vee$, $X_1^\vee$ and $\tilde{X}_2^\vee$ with respect to $D_{12}^\vee$ are

$$\lambda_i' = \lambda_i \frac{1}{\prod_{j \in I_{s+1}}^{a+b} y_0^{m_{ij}}}, i = 1 \ldots, r;$$

$$\lambda_{i,1}' = \lambda_{i,1} \frac{1}{\prod_{j \in I_{s+1}}^{a+b} y_0^{m_{ij}}}, i = 1 \ldots, r;$$

$$\lambda_{i,2}' = \lambda_{i,2} \frac{1}{\prod_{j \in I_{s+1}}^{a+b} y_0^{m_{ij}}}, i = 1 \ldots, r.$$

Then, we again have the relation among generalized functional invariants

Proposition 5.3. *Under the identification (18), the generalized functional invariants satisfy the product relation*

$$\lambda_i' \lambda_{i,0} = \lambda_{i,1}' \lambda_{i,2}', i = 1 \ldots, r.$$

5.3. **Periods.** Recall that $p_1, \ldots, , p_r \in H^2(Y, \mathbb{Z})$ is a nef integral basis and the toric divisors are

$$\mathbf{D}_j = \sum_{i=1}^r m_{ij} p_i, \quad 1 \le j \le m,$$

for some m_{ij}.

If we assume that $\rho_{s+1} = -K_Y - \sum_{l=0}^s c_1(L_l)$ is nef on Y, then the relative period for $(D_0^\vee, W_0)$ is

$$(20) \qquad f_0^{D_0}(\lambda_0) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle \langle \rho_{s+1}, d \rangle! \lambda_0^d.$$

Suppose ρ_{s+1} is nef when restricted to a toric subvariety $Y' \subset Y$. Then the relative period for $(D_0^\vee, W_0)$ is a restriction of (20) to $q_i = 0$ for some of $i \in \{1, \ldots, r\}$.

We can compute the holomorphic period for $X^\vee$ via generalized functional invariants:

$$\frac{1}{(1-y)} f_0^{D_0}(\lambda')$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \frac{\prod_{l=1}^{s+1} \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \langle \rho_{s+1}, d \rangle! \frac{\lambda_0^d}{(1-y)^{1+\langle \rho_{0,1}, d \rangle} y^{\langle \rho_{0,2}, d \rangle}}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \sum_{d_{0,1} \geq 0} \frac{\prod_{l=1}^{s+1} \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \langle \rho_{s+1}, d \rangle! \frac{(d_{0,1} + \langle \rho_{0,1}, d \rangle)! \lambda_0^d}{d_{0,1}! \langle \rho_{0,1}, d \rangle! y^{\langle \rho_{0,2}, d \rangle}} y^{d_{0,1}}.$$

We obtain the holomorphic period for $X^\vee$:

$$f_0^X(\lambda_0) = \frac{1}{2\pi i} \oint \frac{1}{(1-y)} f_0^{D_0}(\lambda') \frac{dy}{y}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} (\langle \rho_{0,1}, d \rangle + \langle \rho_{0,2}, d \rangle)! \langle \rho_{s+1}, d \rangle! \lambda_0^d.$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \frac{\prod_{l=0}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \langle \rho_{s+1}, d \rangle! \lambda_0^d.$$

The holomorphic periods for $X_1^\vee$ and $\tilde{X}_2^\vee$ are computed in a similar way, we have

$$f_0^{X_1}(\lambda_0, y_1) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \langle \rho_{s+1}, d \rangle! \frac{\lambda_0^d}{(y_1)^{\langle \rho_{0,2}, d \rangle}};$$

$$f_0^{\tilde{X}_2}(\lambda_0, y) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle \mathbf{D}_j, d \rangle \geq 0}} \sum_{d_0 \geq 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle \mathbf{D}_j, d \rangle!} \frac{(d_0 + \langle \rho_{0,1}, d \rangle)! \langle \rho_{0,2}, d \rangle! \lambda_0^d}{d_0!} \langle \rho_{s+1}, d \rangle! (y)^{d_0}.$$

Recall that $y = y_1 = q_{0,2}/y_2$ and $\lambda_{i,0} = \frac{q_i}{\prod_{j \in I_{s+1}}^{a+b} y_0^{m_{ij}}}, i = 1 \ldots, r$. Since λ_0 and q are only differ by a scaling factor, we will simply replace λ_0 by q for the rest of the section. Then we have

Theorem 5.4. *The following Hadamard product relation holds*

$$f_0^X(q) \star_q f_0^{D_0}(q) = \frac{1}{2\pi i} \oint f_0^{X_1}(q, y) \star_q f_0^{\tilde{X}_2}(q, y) \frac{dy}{y}.$$

Similarly, we have the Hadamard product relation among Picard-Fuchs operators. Let

$$I^X(q) = e^{\sum_{i=0}^r p_i \log q_i} \sum_{d \in NE(Y)_\mathbb{Z}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (\mathbf{D}_j + k)}{\prod_{k=-\infty}^{\langle \mathbf{D}_j, d \rangle} (\mathbf{D}_j + k)} \right) \left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right) \prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k) q^d;$$

$$I^{D_0}(q) = e^{\sum_{i=1}^r p_i \log q_i} \sum_{d \in NE(Y)_\mathbb{Z}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (\mathbf{D}_j + k)}{\prod_{k=-\infty}^{\langle \mathbf{D}_j, d \rangle} (\mathbf{D}_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right) \prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k) q^d;$$

$$I^{X_1}(q,y) = e^{\sum_{i=1}^r p_i \log \lambda_{i,1}} \sum_{d \in NE(Y)_\mathbb{Z}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (\mathbf{D}_j + k)}{\prod_{k=-\infty}^{\langle \mathbf{D}_j, d \rangle} (\mathbf{D}_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right) \prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k) \frac{q^d}{y^{\langle \rho_{0,2}, d \rangle}},$$

where $\lambda_{i,1} = \frac{q_{i,1}}{\prod_{k=a+1}^{a+b} y_1^{m_{ij_k}}}, i = 1 \ldots, r$;

$$I^{\tilde{X}_2}(q,y) = e^{\sum_{i=1}^r p_i \log q_i + p_0 \log y} \sum_{d \in NE(Y)_\mathbb{Z}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (\mathbf{D}_j + k)}{\prod_{k=-\infty}^{\langle \mathbf{D}_j, d \rangle} (\mathbf{D}_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \frac{\left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle + d_0} (\rho_{0,1} + p_0 + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right)}{\prod_{k=1}^{d_0} (p_0 + k)} \prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k) y^{d_0} q^d.$$

Theorem 5.5. *The Hadamard product relation among the bases of solutions to Picard-Fuchs equations is*

$$I^X(q) \star_q I^{D_0}(q) = \frac{1}{2\pi i} \oint I^{X_1}(q,y) \star_q I^{\tilde{X}_2}(q,y) \frac{dy}{y},$$

where we set $p_0 = \rho_{0,2}$. In the Hadamard product $\star_q$, we treat $\log q_i$ as a variable that is independent from q_i. Alternatively, we can consider $\bar{I}(q) := I(q)/(\sum p_i \log q_i)$ and write Hadamard product relation for $\bar{I}$.

Remark 5.6. The Hadamard product relation between periods is also true if we remove the factor $\langle \rho_{s+1}, d \rangle!$ from each period. Similarly, we can remove the factor $\prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k)$ from I-functions appeared in Theorem 5.5. Then the hypergeometric series are related to different enumerative invariants. For example, $I^X(q)$ is related to a generating function of genus zero relative Gromov-Witten invariants of (X, D). Removing the factor $\prod_{k=1}^{\langle \rho_{s+1}, d \rangle} (\rho_{s+1} + k)$ from $I^X(q)$, it is related to a generating function of genus zero absolute Gromov-Witten invariants X instead of relative Gromov-Witten invariants of (X, D).

6. Degeneration to the normal cones

In this section, we consider a special kind of degeneration: degeneration to the normal cones. Let X be a quasi-Fano variety with its smooth anticanonical divisor D. We can consider the degeneration to the normal cone of X with respect to a divisor D_1. More

precisely, let $\mathcal{X}$ be the blow-up of $X \times \mathbb{A}^1$ along the subvariety $D_1 \times \{0\}$. There is a projection map $\pi : \mathcal{X} \to \mathbb{A}^1$ to the second factor. The central fibre $\pi^{-1}(0)$ is $X \cup_{D_1} \mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O})$.

Let $D_1 + D_2 \in |-K_X|$ be simple normal crossings and $D_{12} = D_1 \cap D_2$. Then, the degeneration of X also gives a Tyurin degeneration of D into $D_2 \subset X$ and a blow-up $\tilde{D}_1$ of D_1. The blown-up variety $\tilde{D}_1$ is in $\mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O})$. Therefore, the degeneration that we are considering is the following:

$$(X, D) \rightsquigarrow (X, D_2) \cup_{(D_1, D_{12})} (\mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O}), \tilde{D}_1).$$

Then Conjecture 1.1 states that the LG mirror of (X, D) can be obtained by gluing rank 2 LG mirrors of $(X, D_1 + D_2)$ and $(\mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O}), D_1 + \tilde{D}_1)$. Recall that, suppose $(X^\vee, h = (h_1, h_2))$ is the rank 2 LG mirror of $(X, D_1 + D_2)$, then $W = h_1 + h_2 : X^\vee \to \mathbb{C}$ is a non-proper LG model. By mirror symmetry, it is expected that the LG model $(\bar{X}^\vee, \bar{W})$ of (X, D) is a compactification of the non-proper LG model $(X^\vee, W)$. Our conjecture suggests the following.

Conjecture 6.1. *Given a quasi-Fano variety X with its smooth anticanonical divisor D. Let $D_1 + D_2$ be a simple normal crossings anticanonical divisor of X. The ordinary LG model of (X, D) is a (partial) compactification of the rank 2 LG model of $(X, D_1 + D_2)$ and the compactification can be obtained by gluing the rank 2 LG mirror of $(\mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O}), D_1 + \tilde{D}_1)$ to the rank 2 LG mirror of $(X, D_1 + D_2)$.*

When $D \subset X$ is a toric complete intersection of a toric variety determined by a nef partition. The refinement of the nef partition discussed in [DHT17, Section 3] gives a degeneration of (X, D) which is compatible with a Tyurin degeneration of D. One can obtain the relation between their generalized functional invariants, periods, and I-functions following the procedure of [DKY19, Section 5]. This is similar to the discussion in Section 5, but slightly different. Suppose X is a complete intersection in a toric variety Y defined by a generic section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s$ and $D \subset X$ is a Calabi-Yau intersection in Y defined by a generic section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s \oplus L_{s+1}$. For degeneration to the normal cones, we consider a refinement of the nef partition with respect to the part L_{s+1}. While, in Section 5, we consider a refinement of the nef partition with respect to L_i for $i \in \{0, 1, \ldots, s\}$. We will work out an example of the degeneration to the normal cone explicitly in the next section. The general case works similarly.

6.1. **An example.** In this section, we will consider an example of the degeneration to the normal cone. We consider a degeneration to the normal cone of $\mathbb{P}^3$ with respect to a smooth cubic surface S:

$$\mathbb{P}^3 \rightsquigarrow \mathbb{P}^3 \cup_S \mathbb{P}(N_S \oplus O_S).$$

Then a quartic $K3$ surface in $\mathbb{P}^3$ degenerates to $\mathbb{P}^2 \subset \mathbb{P}^3$ and a blow-up of S along a complete intersection of a degree one and a degree four hypersurfaces. This blown-up variety $\tilde{D}$ is a hypersurface in $Y := \mathbb{P}(N_S \oplus O_S)$ defined by the vanishing locus of a generic section of $\mathcal{O}_Y(1) \otimes \pi^* \mathcal{O}_S(1)$, where $\pi : Y \to S$ is the projection.

Therefore, what we consider is the degeneration of $(\mathbb{P}^3, K3)$ into $(\mathbb{P}^3, \mathbb{P}^2)$ and $(\mathbb{P}(N_S \oplus O_S), \tilde{D})$ intersecting along S. We study the relation among their mirrors. The mirror of

$(\mathbb{P}^3, K3)$ is an LG model given by the fibrewise compactification of

$$W : (\mathbb{C}^*)^3 \to \mathbb{C}$$

$$(x_0, x_1, x_2) \mapsto x_0 + x_1 + x_2 + \frac{q_1}{x_0 x_1 x_2}.$$

It can be rewritten as

$$W : X^\vee \to \mathbb{C}$$

$$(x_0, x_1, x_2, y) \mapsto y,$$

where $X^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y_1) \in (\mathbb{C}^*)^4 \,\middle|\, x_0 + x_1 + x_2 + \frac{q_1}{x_0 x_1 x_2} = y \right\}.$$

The rank 2 LG model mirrors to $(\mathbb{P}^3, \mathbb{P}^2 + S)$ is given by

$$h_1 := (h_{11}, h_{12}) : (\mathbb{C}^*)^3 \to (\mathbb{C})^2$$

$$(x_0, x_1, x_2) \mapsto (x_0, y_1 := \frac{q_{1,1}}{x_0 x_1 x_2} + x_1 + x_2).$$

The rank 2 LG model mirrors to $(\mathbb{P}(N_S \oplus O_S), \tilde{D} + S)$ is given by.

$$h_2 : X_2^\vee \to (\mathbb{C})^2$$

$$(x_0, x_1, x_2, y, y_2) \mapsto (y, y_2),$$

where $X_2^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2, y, y_2) \in (\mathbb{C}^*)^5 \,\middle|\, x_1 + x_2 + \frac{q_{1,2}}{x_0 x_1 x_2 (y - x_0)^3} = 1, (y - x_0) y_2 = q_{2,2} \right\}.$$

Recall that the mirror LG model of the cubic surface D_0 is defined by

$$W_1 : D_0^\vee \to \mathbb{C}$$

$$(x_0, x_1, x_2) \mapsto x_0,$$

where $D_0^\vee$ is the fibrewise compactification of

$$\left\{ (x_0, x_1, x_2) \in (\mathbb{C}^*)^3 \,\middle|\, x_1 + x_2 + \frac{q_{1,0}}{x_0 x_1 x_2} = 1 \right\}.$$

The functional invariants are the following:

$$(21) \qquad \lambda = \frac{q_1}{x_0 (y - x_0)^3}, \quad \lambda_0 = \frac{q_{1,0}}{x_0}, \quad \lambda_1 = \frac{q_{1,1}}{x_0 y_1^3}, \quad \lambda_2 = \frac{q_{1,2} y_2^3}{(y - q_{2,2}/y_2) q_{2,2}^3}.$$

We have the following identification of variables

$$(22) \qquad q_1 = q_{1,0} = q_{1,1} = \frac{q_{1,2} y_2^3}{q_{2,2}^3}, \quad y_1 = q_{2,2}/y_2 = y - x_0.$$

The relation among their functional invariants are the following

$$\lambda \cdot \lambda_0 = \lambda_1 \cdot \lambda_2.$$

The relative period for $(S^\vee, W_0)$ is

$$f_0^{S^\vee}(q_{1,0}, x_0) = \sum_{d_1 \geq 0} (q_{1,0}/x_0)^{d_1} \frac{(3d_1)!}{(d_1!)^3}.$$

The relative period of $(\mathbb{P}^3, K3)$ can be computed as a residue integral of the pullback of

$$\frac{1}{(1 - x_0/y)} f_0^{E^\vee}(\tilde{\lambda}).$$

by the functional invariant

$$\lambda = \frac{q_1}{x_0(y - x_0)^3} = \frac{q_1}{x_0 y^3} \frac{1}{(1 - x_0/y)^3}.$$

Therefore, we have

$$\frac{1}{(1 - x_0/y)} f_0^{E^\vee}(\lambda) = \frac{1}{(1 - x_0/y)} \sum_{d_1 \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_1}{x_0 y^3} \frac{1}{(1 - x_0/y)^3} \right)^{d_1}$$

$$= \sum_{d_1, d_{0,1} \geq 0} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_1}{x_0 y^3} \right)^{d_1} \frac{(3d_1 + d_{0,1})!}{(3d_1)! d_{0,1}!} (x_0/y)^{d_{0,1}}$$

$$= \sum_{d_1, d_{0,1} \geq 0} \frac{(3d_1 + d_{0,1})!}{(d_1!)^3 d_{0,1}!} \left(\frac{q_1}{x_0 y^3} \right)^{d_1} \left(\frac{x_0}{y} \right)^{d_{0,1}}$$

After taking a residue, we obtain the holomorphic period of $(\mathbb{P}^3, K3)$:

$$f_0^{X^\vee}(q_1, y) = \frac{1}{2\pi i} \oint \frac{1}{(1 - x_0/y)} f_0^{E^\vee}(\lambda) \frac{dx_0}{x_0}$$

$$= \sum_{d_1 \geq 0} \frac{(3d_1 + d_1)!}{(d_1!)^3 d_1!} \left(\frac{q_1}{y^3} \right)^{d_1} \left(\frac{1}{y} \right)^{d_1}$$

$$= \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_1}{y^4} \right)^{d_1}.$$

This again matches with the previous computation of relative periods in [DKY19].

Relative period for the rank 2 LG model of $(\mathbb{P}^3, \mathbb{P}^2 + S)$ is

$$f_0^{X_1^\vee}(q_{1,1}, y_1, x_0) = \sum_{d_1 \geq 0} \left(\frac{q_{1,1}}{x_0 y_1^3} \right)^{d_1} \frac{(3d_1)!}{(d_1!)^3}.$$

Now, we compute the relative period for the rank 2 LG model of $(\mathbb{P}(N_S \oplus O_S), \tilde{D} + S)$. Recall that the functional invariant is

$$\lambda_2 = \frac{q_{1,2} y_2^3}{(y - q_{2,2}/y_2) q_{2,2}^3} = \frac{q_{1,2} y_2^3}{y q_{2,2}^3} \frac{1}{1 - q_{2,2}/(y_2 y)}.$$

The relative period is

$$\frac{1}{1 - q_{2,2}/(y_2 y)} f_0^{E^\vee}(\lambda_2)$$

$$=\frac{1}{1 - q_{2,2}/(y_2 y)} \sum_{d_1 \geq 0} \left(\frac{q_{1,2} y_2^3}{y q_{2,2}^3}\right)^{d_1} \left(\frac{1}{1 - q_{2,2}/(y_2 y)}\right)^{d_1} \frac{(3d_1)!}{(d_1!)^3}$$

$$=\sum_{d_1, d_{0,2} \geq 0} \left(\frac{q_{1,2} y_2^3}{y q_{2,2}^3}\right)^{d_1} \frac{(d_1 + d_{0,2})!}{d_1! d_{0,2}!} \left(\frac{q_{2,2}}{y_2 y}\right)^{d_{0,2}} \frac{(3d_1)!}{(d_1!)^3}.$$

Under the identification (22), the relative period can be rewritten as

$$f_0^{X_2^\vee}(q_1, y_1, y) = \sum_{d_1, d_{0,2} \geq 0} \left(\frac{q_1}{y}\right)^{d_1} \frac{(d_1 + d_{0,2})!}{d_1! d_{0,2}!} \left(\frac{y_1}{y}\right)^{d_{0,2}} \frac{(3d_1)!}{(d_1!)^3}.$$

Then we can glue periods as follows

$$\frac{1}{2\pi i} \oint f_0^{X_1^\vee}(q_1, y_1, x_0) \star_{q_1} f_0^{X_2^\vee}(q_1, y_1, y) \frac{dy_1}{y_1}$$

$$=\frac{1}{2\pi i} \oint \left(\sum_{d_1 \geq 0} \left(\frac{q_1}{x_0 y_1^3}\right)^{d_1} \frac{(3d_1)!}{(d_1!)^3}\right) \star_{q_1} \left(\sum_{d_1, d_{0,2} \geq 0} \left(\frac{q_1}{y}\right)^{d_1} \frac{(d_1 + d_{0,2})!}{d_1! d_{0,2}!} \left(\frac{y_1}{y}\right)^{d_{0,2}} \frac{(3d_1)!}{(d_1!)^3}\right) \frac{dy_1}{y_1}$$

$$=\sum_{d_1 \geq 0} \left(\frac{q_1}{x_0}\right)^{d_1} \frac{(3d_1)!}{(d_1!)^3} \left(\frac{q_1}{y}\right)^{d_1} \frac{(d_1 + 3d_1)!}{d_1!(3d_1)!} \left(\frac{1}{y}\right)^{3d_1} \frac{(3d_1)!}{(d_1!)^3}$$

$$=\left(\sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_1}{y^4}\right)^{d_1}\right) \star_{q_1} \left(\sum_{d_1 \geq 0} \left(\frac{q_1}{x_0}\right)^{d_1} \frac{(3d_1)!}{(d_1!)^3}\right)$$

$$(23) \qquad =f_0^{X^\vee}(q_1, y) \star_{q_1} f_0^{S^\vee}(q_1, x_0).$$

Just like previous examples, these periods are solutions to systems of PDEs. We can write down the bases of solutions for the systems of PDEs and they also satisfy the Hadamard product relation.

7. Iterating the Doran-Harder-Thompson conjecture

In this section, we apply the conjecture of gluing rank 2 LG models in Section 2 to iterate the Doran-Harder-Thompson conjecture.

We consider a Tyurin degeneration of a Calabi-Yau variety X such that the central fibre of the degeneration is a simple normal crossing variety $X_1 \cup_D X_2$ where X_1 and X_2 are quasi-Fano varieties and D is their common smooth anti-canonical divisor. We further degenerate the pairs (X_1, D) and (X_2, D) into $(X_{11}, D_1) \cup_{D_{10}} (X_{12}, D_2)$ and $(X_{21}, D_1) \cup_{D_{20}} (X_{22}, D_2)$ respectively, such that the degeneration of D into $D_1 \cup_{D_1 \cap D_2} D_2$ is a Tyurin degeneration. Then by [DHT17] and Conjecture 1.1, we expect that the mirror of X can be obtained by gluing four rank 2 LG models.

Suppose X is a Calabi-Yau complete intersection in a toric variety given by a nef partition. A refinement of the nef partition gives a Tyurin degeneration of X into a union of quasi-Fano varieties X_1 and X_2 meeting normally along D. Following [DKY19, Section 5], X_1, X_2 and D are still complete intersections in certain toric varieties. We can further degenerate X_1

and X_2 following Section 5. Then X_{11}, X_{12}, X_{21} and X_{22} are all toric complete intersections. The relation among their generalized functional invariants follows from the relation that we obtained in [DKY19, Section 5] and Section 5. We explain it here in full detail for the degeneration of the quintic threefold.

We consider the degeneration of the quintic threefold $Q_5 \subset \mathbb{P}^4$ into $(\mathbb{P}^3, K3)$ and $(\mathrm{Bl}_{C_{1,5}} Q_4, K3)$, where $\mathrm{Bl}_{C_{1,5}} Q_4$ is the blow-up of a quartic fourfold Q_4 along a complete intersection of degree 1 and 5 hypersurfaces. Then we further degenerate $(\mathbb{P}^3, K3)$ into $(\mathbb{P}^3, \mathbb{P}^2)$ and $(\mathbb{P}(N_S \oplus \mathcal{O}), \tilde{D})$ intersecting along a cubic surface S_1 which is described in Section 6.1. We also degenerate $(\mathrm{Bl}_{C_{1,5}} Q_4, K3)$ into $(\mathrm{Bl}_{C_{1,5}} \mathbb{P}^3, \mathbb{P}^2)$ and $(\mathrm{Bl}_{C_{1,4}} \mathrm{Bl}_{C_{1,5}} Q_3, \tilde{D})$ intersecting along a common divisor S_2.

The blown-up varieties can be realized as complete intersections in toric varieties. The variety $\mathrm{Bl}_{C_{1,5}} Q_4$ has been studied in [DKY19, Section 4.2]. It is a complete intersection of degrees $(1,1)$ and $(4,0)$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4})$. The variety $\mathrm{Bl}_{C_{1,5}} \mathbb{P}^3$ can be constructed similarly. It is a hypersurface of degree $(1,1)$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3})$. Similarly, $\mathrm{Bl}_{C_{1,4}} \mathrm{Bl}_{C_{1,5}} Q_3$ can be considered as follows. Let $X' = \mathrm{Bl}_{C_{1,5}} Q_3$. Then X' is a complete intersection given by the zero locus of generic sections of lines bundles $\mathcal{O}_Y(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^4}(1)$ and $\pi^* \mathcal{O}_{\mathbb{P}^4}(3)$ in the toric variety

$$\pi : Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4}) \to \mathbb{P}^4.$$

Then $\mathrm{Bl}_{C_{1,4}} \mathrm{Bl}_{C_{1,5}} Q_3 = \mathrm{Bl}_{C_{1,4}} X'$ is a complete intersection given by the zero locus of generic sections of lines bundles $\mathcal{O}_{Y'}(1) \otimes (\pi')^* \pi^* \mathcal{O}_{\mathbb{P}^4}(1)$, $(\pi')^* \mathcal{O}_Y(1) \otimes (\pi')^* \pi^* \mathcal{O}_{\mathbb{P}^4}(1)$ and $(\pi')^* \pi^* \mathcal{O}_{\mathbb{P}^4}(3)$ in

$$\pi' : Y' = \mathbb{P}_Y(\pi^* \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_Y) \to Y.$$

The mirrors can be written down following Section 5. Recall that the functional invariant for the mirror of $(\mathbb{P}^3, K3)$ with respect to the mirror cubic curve family is

$$\lambda_1 = \frac{q_{1,1}}{x_0(y - x_0)^3}$$

as in (21). We can compute the functional invariants for mirrors of Q_5, $(\mathrm{Bl}_{C_{1,4}} Q_4, K3)$ and the quartic $K3$:

$$\lambda = \frac{q_1}{x_0(1-y)(y-x_0)^3}, \lambda_2 = \frac{q_{1,2} y_2^4}{x_0(1-x_0)^3(1-q_2/y_2)}, \lambda_0 = \frac{q_{1,0}}{x_0(1-x_0)^3}.$$

Under the identifications

$$y = q_2/y_2, \quad \text{and } q_1 = q_{1,1} = q_{1,2} y_2^4 = q_{1,0},$$

we have the product relation among functional invariants

$$\lambda \lambda_0 = \lambda_1 \lambda_2.$$

Remark 7.1. Note that in our earlier paper [DKY19], we explained how to glue two functional invariants $\lambda_1 = \frac{Q_1}{(y-1)^4}$ and $\lambda_2 = \frac{Q_2}{y}$ to get $\lambda = \frac{Q}{y(y-1)^4}$. By replacing Q_i with the expression λ_0 above, we obtain the expressions λ_1, λ_2. In this sense, the gluing formula above is really the same as starting with our old one and then replacing the parameters with λ_0, as was described above in Section 5.

Here is a brief analysis of the singular loci for the mirror cubic elliptic curve fibration of the mirror quintic threefold. Recall once more that the mirror cubic family of elliptic curves has singular fibres of types I_3, I_1, IV^* over $0, \frac{1}{3^3}, \infty$ respectively.

Functional invariant λ_1:

- $\lambda_1^{-1}(0) = \{x_0 = \infty\} \cup \{(y - x_0)^3 = \infty\}$ supports I_3 and I_9 singular fibres respectively.
- $\lambda_1^{-1}(\infty) = \{x_0 = 0\} \cup \{(y - x_0)^3 = 0\}$; the first component supports IV^*-fibres with the second component supporting smooth fibres.
- The locus $\lambda_1^{-1}(\frac{1}{3^3}) = \{x_0(y - x_0)^3 = 3^3 q_1\}$ supports I_1-fibres generically.

Functional invariant λ_2:

- $\lambda_2^{-1}(0) = \{x_0 = \infty\} \cup \{y = 1\} \cup \{(1 - x_0)^3 = \infty\}$ with the first two components supporting I_3-fibres and the last supporting I_9-fibres.
- $\lambda_2^{-1}(\infty) = \{x_0 = 0\} \cup \{(1 - y) = \infty\} \cup \{(1 - x_0)^3 = 0\}$. The first two components support IV^*-fibres with the third component supporting smooth fibres.
- The locus $\lambda_2^{-1}(\frac{1}{3^3}) = \{x_0(1 - x_0)^3)(1 - y) = 3^3 q_1\}$ supports I_1-fibres.

Functional invariant λ_0:

- $\lambda_0^{-1}(0) = \{x_0 = \infty\} \cup \{(1 - x_0)^3 = \infty\}$ supports I_3 and I_9 singular fibres respectively.
- $\lambda_0^{-1}(\infty) = \{x_0 = 0\} \cup \{(1 - x_0)^3 = 0\}$. The first component supports IV^*-fibres with the second component supporting smooth fibres.
- The locus $\lambda_0^{-1}(\frac{1}{3^3}) = \{x_0(1 - x_0)^3 = 3^3 q_1\}$ supports I_1-fibres.

Functional invariant λ:

- $\lambda^{-1}(0) = \{x_0 = \infty\} \cup \{(1 - y) = \infty\} \cup \{(y - x_0)^3 = \infty\}$, with the first two components supporting I_3 fibres and the last component supporting I_9-fibres.
- $\lambda^{-1}(\infty) = \{x_0 = 0\} \cup \{(1 - y) = 0\} \cup \{(y - x_0)^3 = 0\}$; the first two component supporting IV^*-fibres with the last component supporting smooth fibres.
- The locus $\lambda^{-1}(\frac{1}{3^3}) = \{x_0(1 - y)(y - x_0)^3 = 3^3 q_1\}$ supports I_1-fibres.

The singular loci of λ are essentially the union of the singular loci of λ_1, λ_2 with the exception of the removal of one factor of $x_0 = 0$ and $x_0 = \infty$ that is accounted for in λ_0.

Now we further degenerate $(\mathbb{P}^3, K3)$ and $(\mathrm{Bl}_{C_{1,4}} Q_4, K3)$. By Proposition 5.1 and (21), the functional invariants λ_1 and λ_2 satisfy the following relations:

$$\lambda_1 \lambda_{1,0} = \lambda_{1,1} \lambda_{1,2}$$

and

$$\lambda_2 \lambda_{2,0} = \lambda_{2,1} \lambda_{2,2},$$

where $\lambda_{i,j}$ are functional invariants of degenerated pieces (and the intersections of degenerated pieces) with respect to the mirror cubic family; the identities are obtained under appropriate identifications of variables. Therefore, we have the following relation among functional invariants

$$(24) \qquad \lambda \lambda_0 \lambda_{1,0} \lambda_{2,0} = \lambda_{1,1} \lambda_{1,2} \lambda_{2,1} \lambda_{2,2}.$$

Now, we would like to show that the period for the mirror quintic can be obtained by gluing relative periods of four rank 2 LG models. As computed in [DKY19, Section 2.2.1], the period of the mirror quintic can be obtained from the residue integral of the pullback

of the period of its internal fibration of mirror quartic $K3$ by the generalized functional invariant $\lambda = \frac{q}{y(1-y)^4}$. Similarly, we can compute the period of the mirror quintic by taking double residue integrals of the pullback of the period of its internal elliptic fibration by the functional invariant $\lambda = \frac{q_1}{x_0(1-y)(y-x_0)^3}$. The computation is as follows. Recall that, the holomorphic period for the mirror cubic curve family is

$$f_0^{E^\vee}(\tilde{\lambda}) = \sum \frac{(3d)!}{(d!)^3}\tilde{\lambda}^d.$$

Then

$$\frac{1}{(1-y)(1-x_0/y)}f_0^{E^\vee}(\lambda) = \frac{1}{(1-y)(1-x_0/y)}\sum_{d_1 \geq 0}\frac{(3d_1)!}{(d_1!)^3}\left(\frac{q_1}{x_0 y^3(1-y)(1-x_0/y)^3}\right)^{d_1}$$

$$= \frac{1}{(1-x_0/y)}\sum_{d_1, d_{0,1} \geq 0}\frac{(3d_1)!}{(d_1!)^3}\left(\frac{q_1}{x_0 y^3(1-x_0/y)^3}\right)^{d_1}\frac{(d_1+d_{0,1})!}{d_1!d_{0,1}!}(y)^{d_{0,1}}$$

$$= \sum_{d_1, d_{0,1}, d_{0,2} \geq 0}\frac{(3d_1)!}{(d_1!)^3}\left(\frac{q_1}{x_0 y^3}\right)^{d_1}\frac{(d_1+d_{0,1})!}{d_1!d_{0,1}!}\frac{(3d_1+d_{0,2})!}{(3d_1)!d_{0,2}!}(y)^{d_{0,1}}\left(\frac{x_0}{y}\right)^{d_{0,2}}.$$

We take the residue integrals to obtain the holomorphic period of the mirror of $X = Q_5$:

$$f_0^{X^\vee}(q_1) = \frac{1}{(2\pi i)^2}\oint\oint\frac{1}{(1-y)(1-x_0/y)}f_0^{E^\vee}(\lambda)\frac{dy}{y}\frac{dx_0}{x_0}$$

$$= \sum_{d_1, d_0 \geq 0}\frac{(3d_1)!}{(d_1!)^3}(q_1)^{d_1}\frac{(d_1+4d_1)!}{d_1!(4d_1)!}\frac{(3d_1+d_1)!}{(3d_1)!d_1!}$$

$$= \sum_{d_1 \geq 0}\frac{(5d_1)!}{(d_1!)^5}q_1^{d_1}.$$

Similarly, we can compute relative periods for the mirrors of $(X_1, D) := (\mathbb{P}^3, K3)$, $(X_2, D) := (\mathrm{Bl}_{C_{1,4}} Q_4, K3)$ and the quartic $D := K3$ as

$$f_0^{\tilde{X}_1^\vee}(q_1, y) = \frac{1}{2\pi i}\oint\frac{1}{(1-x_0/y)}f_0^{E^\vee}(\lambda_1)\frac{dx_0}{x_0},$$

$$f_0^{\tilde{X}_2^\vee}(q_1, y) = \frac{1}{2\pi i}\oint\frac{1}{(1-y)(1-x_0)}f_0^{E^\vee}(\lambda_2)\frac{dx_0}{x_0}$$

$$f_0^{\tilde{D}^\vee}(q_1) = \frac{1}{2\pi i}\oint\frac{1}{(1-x_0)}f_0^{E^\vee}(\lambda_0)\frac{dx_0}{x_0}.$$

We have

(25) $$f_0^{X^\vee}(q_1) \star_{q_1} f_0^{\tilde{D}^\vee}(q_1) = \frac{1}{2\pi i}\oint f_0^{\tilde{X}_1^\vee}(q_1, y) \star_{q_1} f_0^{\tilde{X}_2^\vee}(q_1, y)\frac{dy}{y}.$$

On the other hand, $f_0^{\tilde{X}_1^\vee}(q_1, y)$ and $f_0^{\tilde{X}_2^\vee}(q_1, y)$ satisfy the following product relations by Theorem 5.4 and Equation (23):

$$f_0^{X^\vee} \star_{q_1} f_0^{\tilde{D}^\vee} \star_{q_1} f_0^{D_{10}^\vee} \star_{q_1} f_0^{D_{20}^\vee}$$

$$= \frac{1}{(2\pi i)^2}\oint\left(\left(\oint f_0^{\tilde{X}_{11}^\vee} \star_{q_1} f_0^{\tilde{X}_{12}^\vee}\frac{dy_1}{y_1}\right) \star_{q_1}\left(\oint f_0^{\tilde{X}_{21}^\vee} \star_{q_1} f_0^{\tilde{X}_{22}^\vee}\frac{dy_2}{y_2}\right)\right)\frac{dy}{y}.$$

These holomorphic periods are solutions to systems of PDEs. This relation between periods is also true for the bases of solutions for the systems of PDEs. In general, such a relation is true when we consider iterating Tyurin degenerations of Calabi-Yau complete intersections in toric varieties.

In the above example of quintic, the degeneration is induced from a refinement of the nef partition $(5) \to (1) + (4)$. There is another refinement of the nef partition that one can consider: $(5) \to (2) + (3)$. In this case, the quintic threefold is degenerated into a blow-up $\mathrm{Bl}_{3,5} Q_2$ of quadric threefold Q_2 along complete intersection of degree 3 and 5 hypersurfaces and a cubic threefold Q_3. The intersection of $\mathrm{Bl}_{3,5} Q_2$ and Q_3 is a sextic $K3$ surface in $\mathbb{P}^4$ whose mirror is a family of M_3-polarized $K3$ surfaces. The family of M_3-polarized $K3$ surfaces also admits an internal fibration structure over $\mathbb{P}^1$ whose generic fiber is a mirror cubic curve. One can compute the functional invariants of the mirrors of Q_5, $(\mathrm{Bl}_{3,5} Q_2, K3)$, $(Q_3, K3)$ and the sextic $K3$ with respect to the mirror cubic family. The computation is similar to the previous example and we have a product relation among functional invariants

$$\lambda \lambda_0 = \lambda_1 \lambda_2.$$

We can further degenerate $(\mathrm{Bl}_{3,5} Q_2, K3)$ and $(Q_3, K3)$. We degenerate $(\mathrm{Bl}_{3,5} Q_2, K3)$ into $(\mathrm{Bl}_{3,5} \mathbb{P}^3, C)$ and $(\mathrm{Bl}_{1,2} \mathrm{Bl}_{3,5} \mathbb{P}^3, \tilde{C})$ where C is a cubic surface and $\tilde{C}$ is a blow-up of a cubic surface. We degenerate $(Q_3, K3)$ into (Q_3, C) and $(\mathbb{P}_C(N_C \oplus O_C), \tilde{C})$. We then also have the relation for functional invariants of the two-step degenerations which takes the same form as Identity (24).

Remark 7.2. It is natural to ask: viewing $Q_5^\vee$ as a family of elliptic covers over $\mathbb{P}^1 \times \mathbb{P}^1$, what are the families of rational curves on the base that support $K3$ surfaces? The x_0 and $(y-1)$ slices give rise to families of M_3-polarized K3 surfaces and we obtain M_2-polarized $K3$ surfaces by considering $(y-x_0)$-slices. Are there others?

The answer is no, and here is one way to see this. Firstly, any rational curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve of type $(1, d)$ or $(d, 1)$. That is, in an affine chart, it can be represented as the graph of a rational function given by the ratio of two degree d-polynomials. In turn, by replacing $(x_0, 1-y)$ with the corresponding expressions in λ above, we obtain the generalized functional invariant of the corresponding (family of) elliptic fibration(s). Unless $d = 0$ or $d = 1$, the degree of the resulting functional invariant will be at least 5. On the other hand, Riemann-Hurwitz and fibre-type considerations readily imply that the maximum degree generalized functional invariant for a genus 0-cover of the λ-line that gives rise to a $K3$ surface is 4.

The above degenerations of quintic threefold are mirror to the elliptic fibration of the mirror quintic over the $\mathbb{P}^1 \times \mathbb{P}^1$ base. In general, one can expect the following generalization of the Doran-Harder-Thompson conjecture, which is also mentioned in [BD21] and [DT21],

Conjecture 7.3. *If a Calabi-Yau variety X (or a log Calabi-Yau variety $X \smallsetminus D$) admits a semi-stable degeneration, connected to a point of maximal unipotent monodromy, such that the cone over the dual intersection complex of the central fibre is of dimension k, then the mirror of X (or $X \smallsetminus D$) admits a Calabi-Yau fibration structure with a k-dimensional base.*

There are already some evidence of this conjecture. Given a semi-stable degeneration, one can consider the Clemens-Schmid exact sequence relating the geometry of the central fibre of the degeneration and the geometry of a nearby smooth fibre. On the mirror side, if a

smooth variety admits a projective fibration over a projective base, Doran-Thompson [DT21] introduce a four-term long exact sequence relating the cohomology of this variety and the cohomology of the open set obtained removing the preimage of a general linear section. This sequence is considered to be the mirror of the Clemens-Schmid sequence. It requires neither a restriction to Calabi-Yau varieties nor a restriction to codimension one fibrations.

8. Degenerations for simple normal crossings pairs

In previous sections, we consider the degeneration of the pair (X, D) where X is a quasi-Fano variety and D is a smooth anticanonical divisor of X. The story can be generalized to the case when D is a simple normal crossings divisor. Let

$$D_1, \ldots, D_n \subset X$$

be smooth irreducible divisors and

$$D = D_1 + \ldots + D_n \in |-K_X|$$

be simple normal crossings. We consider a degeneration of (X, D) into

$$(X_1, D_{11} + D_{12} + \ldots + D_{1n}), \quad \text{and } (X_2, D_{21} + D_{22} + \ldots + D_{2n}),$$

and

$$D_{1i} \cup_{D_{1i} \cap D_0 = D_{2i} \cap D_0} D_{2i}$$

is a degeneration of D_i. We consider the higher rank LG models for (X, D), $(X_1, D_0 + D_{11} + D_{12} + \ldots + D_{1n})$ and $(X_2, D_0 + D_{21} + D_{22} + \ldots + D_{2n})$. The definition of higher rank LG models when D has more than two components is a direct generalization of rank 2 LG models defined in Definition 2.5. As mentioned in Definition 2.3, a rank n LG model can be defined inductively in terms of LG models of lower ranks. We also refer to [Lee21] for the definition.

The topological gluing for higher rank LG models is similar to the topological gluing described in Section 2.2. We glue the rank $(n+1)$ LG models of $(X_1, D_0 + D_{11} + D_{12} + \ldots + D_{1n})$ and $(X_2, D_0 + D_{21} + D_{22} + \ldots + D_{2n})$ along the first component to get the rank n LG models for (X, D).

A natural question to ask is whether their Hodge numbers and Euler characteristics are still related. To answer this question, we would like to ask what is a reasonable definition of Hodge numbers of higher rank LG models? We examine it for the case when D contains two irreducible components. The following discussions can be generalized to the cases when D contains more than two irreducible components.

Let $(X, D_1 + D_2)$ be a log Calabi-Yau simple normal crossings pair. Let $(X^\vee, h = (h_1, h_2))$ be the mirror rank 2 LG models.

Definition 8.1. We define the Hodge numbers of rank 2 LG models as

$$h^{p,q}(X^\vee, (h_1, h_2)) := h^{p,q}(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2)),$$

where $t = (t_1, t_2)$ is a regular value of (h_1, h_2).

Conjecture 8.2. *If the rank 2 LG model $(X^\vee, h)$ is mirror to $(X, D_1 + D_2)$, then*

$$h^{n-q,p}(X) = h^{p,q}(X^\vee, (h_1, h_2)).$$

Here are some reasons for the definition and the conjecture. Let $D_1^\vee = h_1^{-1}(t_1)$ and $D_2^\vee = h^{-1}(t_2)$. Note that $X \smallsetminus (D_1 \cup_{D_1 \cap D_2} D_2)$ is mirror to $X^\vee$. Therefore, we expect to have the following relation between Euler characteristics under mirror symmetry:

$$(26) \qquad \chi(X \smallsetminus (D_1 \cup_{D_1 \cap D_2} D_2)) = (-1)^{\mathrm{d}} \chi(X^\vee).$$

Furthermore, by assumption, $(D_1, D_1 \cap D_2)$ is mirror to $(D_1^\vee, h_2)$, where $D_1^\vee := h_1^{-1}(t_1)$. Similarly, $(D_2, D_1 \cap D_2)$ is mirror to $(D_2^\vee, h_1)$, where $D_2^\vee := h_2^{-1}(t_2)$. Then, we have the following relation

$$(27) \qquad \chi(D_1) = (-1)^{\mathrm{d}-1} \chi(D_1^\vee, h_2^{-1}(t_2)), \quad \text{and } \chi(D_2) = (-1)^{\mathrm{d}-1} \chi(D_2^\vee, h_1^{-1}(t_1)).$$

Since $h^{-1}(t) := h_1^{-1}(t_1) \cap h_2^{-1}(t_2)$ is mirror to $D_1 \cap D_2$, we have

$$(28) \qquad \chi(D_1 \cap D_2) = (-1)^{\mathrm{d}-2} \chi(h^{-1}(t)).$$

Combining (26), (27) and (28) together, we have

$$
\begin{aligned}
\chi(X) &= \chi(X \smallsetminus (D_1 \cup_{D_1 \cap D_2} D_2)) + \chi(D_1 \cup_{D_1 \cap D_2} D_2) \\
&= (-1)^{\mathrm{d}} \chi(X^\vee) + \chi(D_1) + \chi(D_2) - \chi(D_1 \cap D_2) \\
&= (-1)^{\mathrm{d}} \chi(X^\vee) + (-1)^{\mathrm{d}-1} \chi(D_1^\vee, h_2^{-1}(t_2)) + (-1)^{\mathrm{d}-1} \chi(D_2^\vee, h_1^{-1}(t_1)) - (-1)^{\mathrm{d}-2} \chi(h^{-1}(t)) \\
&= (-1)^{\mathrm{d}} \chi(X^\vee, h^{-1}(t)) + (-1)^{\mathrm{d}-1} \chi(D_1^\vee, h_2^{-1}(t_2)) + (-1)^{\mathrm{d}-1} \chi(D_2^\vee, h_1^{-1}(t_1)) \\
&= (-1)^{\mathrm{d}} \left(\chi(X^\vee, h^{-1}(t)) - \chi(D_1^\vee, h_2^{-1}(t_2)) - \chi(D_2^\vee, h_1^{-1}(t_1)) \right) \\
&= (-1)^{\mathrm{d}} \left(\chi(X^\vee, D_1^\vee \cup D_2^\vee) \right).
\end{aligned}
$$

This is the expected relation between Euler characteristics of X and $(X^\vee, h)$.

Recall that $W := h_1 + h_2 : X^\vee \to \mathbb{C}$ is a non-proper LG model. Naturally, one can ask what is the relation between Hodge numbers of rank 2 LG models and Hodge numbers of proper LG model $(\bar{X}^\vee, \bar{W})$ which is mirror to (X, D) when D is a smooth anticanonical divisor of X? We have the following relation between their Euler characteristics. The relation between their Hodge numbers may be studied elsewhere.

Theorem 8.3. *Given a quasi-Fano variety X with a smooth anticanonical divisor $D \in |-K_X|$. Let $(\bar{X}^\vee, W)$ be the LG mirror of (X, D) and $(X^\vee, h)$ be the rank 2 LG mirror of $(X, D_1 + D_2)$, where $D_1 + D_2 \in |-K_X|$ is simple normal crossing, then*

$$\chi(\bar{X}^\vee, \bar{W}^{-1}(t)) = \chi(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2)).$$

Proof. We can consider the degeneration to the normal cone of X with respect to D_2. Then (X, D) degenerates to

$$(X, D_1) \cup_{(D_2, D_1 \cap D_2)} (\mathbb{P}(N_{D_2} \oplus \mathcal{O}), \tilde{D}_2).$$

By Conjecture 1.1, the LG model $(\bar{X}^\vee, \bar{W})$ is obtained by gluing two rank 2 LG models (X, h) and $(P^\vee, h_p)$, where $(P^\vee, h_p)$ is the rank 2 LG model mirrors to $(\mathbb{P}(N_{D_2} \oplus \mathcal{O}), \tilde{D}_2 + D_2)$. By the relative Mayer-Vietoris sequence (1), we have

$$\chi(\bar{X}^\vee, \bar{W}^{-1}(t)) = \chi(X^\vee, h_1^{-1}(t_1)) + \chi(P^\vee, h_{p,1}^{-1}(t_2)).$$

Note that

$$\chi(X^\vee, h_1^{-1}(t_1)) = \chi(X^\vee, h^{-1}(t)) - \chi(h_1^{-1}(t_1), h^{-1}(t)).$$

On the other hand, since $X \cup_{D_2} \mathbb{P}(N_{D_2} \oplus \mathcal{O})$ is smoothable to X, by [Lee06, Proposition IV. 6], we have

$$\chi(X) = \chi(X) + \chi(\mathbb{P}(N_{D_2} \oplus \mathcal{O})) - 2\chi(D_2).$$

Hence $\chi(\mathbb{P}(N_{D_2} \oplus \mathcal{O}) = 2\chi(D_2)$. Recall that $(P^\vee, h_p)$ is the rank 2 LG model mirrors to $(\mathbb{P}(N_{D_2} \oplus \mathcal{O}), \tilde{D}_2 + D_2)$. Therefore, $(P^\vee, h_{p,1})$ is mirror to the complement of the divisor D_2 in $\mathbb{P}(N_{D_2} \oplus \mathcal{O})$. Hence,

$$\begin{aligned}
\chi(P^\vee, h_{p,1}^{-1}(t_2)) &= (-1)^{\mathrm{d}} \chi(\mathbb{P}(N_{D_2} \oplus \mathcal{O}) \smallsetminus D_2) \\
&= (-1)^{\mathrm{d}} \left(\chi(\mathbb{P}(N_{D_2} \oplus \mathcal{O})) - \chi(D_2) \right) \\
&= (-1)^{\mathrm{d}} \chi(D_2) \\
&= -\chi(h_2^{-1}(t_2), h^{-1}(t)).
\end{aligned}$$

Finally, this gives

$$\begin{aligned}
&\chi(\bar{X}^\vee, \bar{W}^{-1}(t)) \\
=&\chi(X^\vee, h_1^{-1}(t_1)) + \chi(P^\vee, h_{p,1}^{-1}(t_2)) \\
=&\chi(X^\vee, h^{-1}(t)) - \chi(h_1^{-1}(t_1), h^{-1}(t)) - \chi(h_2^{-1}(t_2), h^{-1}(t)) \\
=&\chi(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2)).
\end{aligned}$$

$\square$

This is exactly what we expected. Therefore, our definition is compatible with the mirror duality between Euler characteristics when there is a proper LG model.

Now we return to the degeneration of simple normal crossing pairs. We consider the case when D has two irreducible components. The case when D has more than two irreducible components works similarly. We consider the degeneration of $(X, D_1 + D_2)$ into

$$(X_1, D_{11} + D_{12}) \cup_{(D_0, D_0 \cap D_{11} + D_0 \cap D_{12})} (X_2, D_{21} + D_{22}).$$

The gluing picture of Section 2 can be generalized to this case directly. Furthermore, the following relation between Euler characteristics still holds.

Theorem 8.4. *Let X_1 and X_2 be* d-*dimensional quasi-Fano varieties which contain the same quasi-Fano hypersurface D_0, such that*

$$K_{X_1}|_{D_0} = -K_{X_2}|_{D_0}.$$

Let $D_0 + D_{11} + D_{12} \in |-K_{X_1}|$ and $D_0 + D_{21} + D_{22} \in |-K_{X_2}|$ such that $D_{11} \cap D_0 = D_{21} \cap D_0$, $D_{12} \cap D_0 = D_{22} \cap D_0$ and

$$K_{D_{11}}|_{D_0 \cap D_{11}} = -K_{D_{21}}|_{D_0 \cap D_{21}}, \quad K_{D_{12}}|_{D_0 \cap D_{12}} = -K_{D_{22}}|_{D_0 \cap D_{22}},$$

and

$$K_{D_{11} \cap D_{12}}|_{D_0 \cap D_{11} \cap D_{12}} = -K_{D_{21} \cap D_{22}}|_{D_0 \cap D_{21} \cap D_{22}}.$$

Let $(X_1^\vee, (h_{10}, h_{11}, h_{12}))$ and $(X_2^\vee, (h_{20}, h_{21}, h_{22}))$ be the rank 3 LG models of $(X_1, D_0 + D_{11} + D_{12} \in |-K_{X_1}|)$ and $(X_2, D_0 + D_{21} + D_{22} \in |-K_{X_2}|)$. Suppose that the fibres of (h_{10}, h_{11}, h_{12}) and (h_{20}, h_{21}, h_{22}) are topologically the same Calabi-Yau manifold, which is topologically mirror to $D_{012} = D_0 \cap D_{11} \cap D_{12} = D_0 \cap D_{21} \cap D_{22}$. Let X be a quasi-Fano variety obtained from $X_1 \cup_{D_0} X_2$ by smoothing and let $D_1 \subset X$ be a quasi-Fano variety obtained from $D_{11} \cup_{D_{012}} D_{21}$ by smoothing and $D_2 \subset X$ be a quasi-Fano variety obtained from $D_{12} \cup_{D_{012}} D_{22}$ by smoothing. Let $(X^\vee, h =$

$(h_1, h_2))$ *be the rank* 2 *LG model obtained by gluing rank* 3 *LG models* $(X_1^\vee, (h_{10}, h_{11}, h_{12}))$ *and* $(X_2^\vee, (h_{20}, h_{21}, h_{22}))$ *along the first factor. Then*

$$\chi(X) = (-1)^{\mathrm{d}}(\chi(X^\vee, h_1^{-1}(t_1) \cup h_2^{-1}(t_2))),$$

$$\chi(D_1) = (-1)^{\mathrm{d}-1}\chi(h_1^{-1}(t_1), h_2^{-1}(t_2)),$$

$$\chi(D_2) = (-1)^{\mathrm{d}-1}\chi(h_2^{-1}(t_2), h_1^{-1}(t_1)),$$

and

$$\chi(D_1 \cap D_2) = (-1)^{\mathrm{d}-2}\chi(h^{-1}(t))$$

for $t = (t_1, t_2)$ *a regular value of* h*, where* χ *is the Euler number.*

Proof. The last three identities follow from Theorem 2.8 and [DHT17, Theorem 2.3]. We compute the RHS of the first identity.

By relative Mayer-Vietoris sequence

$$\cdots \to H^j(X^\vee, (h_1, h_2)^{-1}(t); \mathbb{C}) \to H^j(X_1^\vee, (h_{11}, h_{12})^{-1}(t_1); \mathbb{C}) \oplus H^j(X_2^\vee, (h_{21}, h_{22})^{-1}(t_2); \mathbb{C})$$
$$\to H^j(X_1^\vee \cap X_2^\vee, (h_{11}, h_{12})^{-1}(t_1) \cap (h_{21}, h_{22})^{-1}(t_2); \mathbb{C}) \to \cdots,$$

similar to the proof of Theorem 2.8, we have

(29) $\qquad \chi(X^\vee, (h_1, h_2)^{-1}(t)) = \chi(X_1^\vee, (h_{11}, h_{12})^{-1}(t_1)) + \chi(X_2^\vee, (h_{21}, h_{22})^{-1}(t_2)).$

We can write

(30)
$$\begin{aligned}
\chi(X_1^\vee, (h_{11}, h_{12})^{-1}(t_1)) &= \chi(X_1^\vee, h_{11}^{-1}(t_{11})) + \chi(h_{11}^{-1}(t_{11}), (h_{11}, h_{12})^{-1}(t_1)) \\
&= (-1)^{\mathrm{d}}\chi(X_1 \smallsetminus (D_0 + D_{12})) + (-1)^{\mathrm{d}-1}\chi(D_{11} \smallsetminus (D_0 \cap D_{11})) \\
&= (-1)^{\mathrm{d}}\left(\chi(X_1) - \chi(D_0) - \chi(D_{12}) + \chi(D_0 \cap D_{12}) - \chi(D_{11}) + \chi(D_0 \cap D_{11})\right).
\end{aligned}$$

Similarly,

(31)
$$\chi(X_2^\vee, (h_{21}, h_{22})^{-1}(t_2)) = (-1)^{\mathrm{d}}\left(\chi(X_2) - \chi(D_0) - \chi(D_{22}) + \chi(D_0 \cap D_{22}) - \chi(D_{21}) + \chi(D_0 \cap D_{21})\right).$$

Recall that $X_1 \cup_{D_0} X_2$ is smoothable to X, $D_{11} \cup_{D_0 \cap D_{11} = D_0 \cap D_{21}} D_{12}$ is smoothable to D_1 and $D_{12} \cup_{D_0 \cap D_{12} = D_0 \cap D_{22}} D_{22}$ is smoothable to D_2. Therefore, by [Lee06, Proposition IV. 6], we have

(32) $$\chi(X) = \chi(X_1) + \chi(X_2) - 2\chi(D_0),$$

(33) $$\chi(D_1) = \chi(D_{11}) + \chi(D_{21}) - 2\chi(D_0 \cap D_{11}),$$

and

(34) $$\chi(D_2) = \chi(D_{12}) + \chi(D_{22}) - 2\chi(D_0 \cap D_{12}).$$

Hence,

$$\begin{aligned}
\chi(X^\vee, (h_1, h_2)^{-1}(t)) &= \chi(X_1^\vee, (h_{11}, h_{12})^{-1}(t_1)) + \chi(X_2^\vee, (h_{21}, h_{22})^{-1}(t_2)) \\
&= (-1)^{\mathrm{d}}\left(\chi(X_1) - \chi(D_0) - \chi(D_{12}) + \chi(D_0 \cap D_{12}) - \chi(D_{11}) + \chi(D_0 \cap D_{11})\right) \\
&\quad + (-1)^{\mathrm{d}}\left(\chi(X_2) - \chi(D_0) - \chi(D_{22}) + \chi(D_0 \cap D_{22}) - \chi(D_{21}) + \chi(D_0 \cap D_{21})\right) \\
&= (-1)^{\mathrm{d}}\left(\chi(X) - \chi(D_1) - \chi(D_2)\right),
\end{aligned}$$

where the first line is (29); the second line follows from (30) and (31); the last line follows from (32), (33) and (34). This proves the first equality. $\qquad\square$

For toric complete intersections, the same computation in Section 5 also applies to the degeneration of simple normal crossing pairs. We again have the product relation among generalized functional invariants and the Hadamard product relation among relative periods and bases of solutions of the corresponding PDEs.

9. Gromov-Witten invariants

Classical mirror symmetry relates absolute Gromov-Witten invariants with periods (see, for example, [Giv98], [LLY97]). Recently, relative mirror symmetry has been proved in [FTY19] to relate relative Gromov-Witten invariants of a smooth pair with relative periods of its mirror. In [DKY19], we explain that the relation between periods gives a relation between absolute and relative Gromov-Witten invariants of a Tyurin degeneration of a Calabi-Yau variety. In this paper, we consider gluing of rank $(n+1)$ LG models to an rank n LG model and gluing relative periods of rank $(n+1)$ LG models to a relative period of a rank n LG model. Higher rank LG models are mirror to simple normal crossings pairs. It is natural to ask if relative periods of higher rank LG models also mirror to certain A-model invariants. In this paper, we consider the type of invariants associated to a simple normal crossing pairs defined in [TY20b] which fits well into our context. In [TY20a], a mirror theorem has been proved to relate the formal Gromov-Witten invariants of infinite root stacks, which are invariants associated to simple normal crossing pairs and defined in [TY20b], and periods.

Remark 9.1. Another well-known invariant for simple normal crossing pairs is (punctured) logarithmic Gromov-Witten invariants of [AC14], [Che14], [GS13], [ACGS20]. Punctured invariants are essential for the intrinsic mirror symmetry construction in the Gross-Siebert program [GS19]. An approach of using the Gross-Siebert program to prove the Doran-Harder-Thompson conjecture has appeared in [BD21]. It would be interesting to relate the period calculation in this paper and in [DKY19] to the approach via the Gross-Siebert program. It will be studied elsewhere.

We will briefly review the formal Gromov-Witten theory of infinite root stacks and its mirror theorem.

9.1. **Orbifold Gromov-Witten invariants.** Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack such that its coarse moduli space X is projective. We consider the moduli space $\overline{\mathcal{M}}_{0,l}(\mathcal{X},\beta)$ of l-pointed genus zero degree $\beta \in H_2(X)$ stable maps to $\mathcal{X}$.

The genus-zero orbifold Gromov-Witten invariants of $\mathcal{X}$ are defined as follows

$$(35) \qquad \left\langle \prod_{i=1}^{l} \tau_{a_i}(\gamma_i) \right\rangle_{0,l,\beta}^{\mathcal{X}} := \int_{[\overline{\mathcal{M}}_{0,l}(\mathcal{X},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{l} (\mathrm{ev}_i^* \, \gamma_i) \bar{\psi}_i^{a_i},$$

where,

- $[\overline{\mathcal{M}}_{0,l}(\mathcal{X},\beta)]^{\mathrm{vir}}$ is the virtual fundamental class.
- for $i = 1, 2, \ldots, l$,
$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,l}(\mathcal{X},\beta) \to I\mathcal{X}$$
is the evaluation map and $I\mathcal{X}$ is the inertia stack of $\mathcal{X}$.

- $\gamma_i \in H^*_{\mathrm{CR}}(\mathcal{X})$ are cohomological classes of the Chen-Ruan orbifold cohomology $H^*_{\mathrm{CR}}(\mathcal{X})$.
- $a_i \in \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq l$.
- $\bar{\psi}_i \in H^2(\overline{\mathcal{M}}_{0,l}(\mathcal{X}, \beta), \mathbb{Q})$ is the descendant class.

In the context of mirror theorems, we need to consider the following generating function of genus zero invariants, called the J-function:

$$J_{\mathcal{X}}(t, z) := z + t + \sum_{\beta \in \mathrm{NE}(X)} \sum_{l \geq 0} \sum_{\alpha} \frac{q^{\beta}}{l!} \left\langle \frac{\phi_{\alpha}}{z - \bar{\psi}}, t, \ldots, t \right\rangle^{\mathcal{X}}_{0, l+1, \beta} \phi^{\alpha},$$

where

- $\mathrm{NE}(X) \subset H_2(X, \mathbb{R})$ is the cone generated by effective curves and $\mathrm{NE}(X)_{\mathbb{Z}} := \mathrm{NE}(X) \cap (H_2(X, \mathbb{Z}) / \mathrm{tors})$.
- $\{\phi_{\alpha}\}, \{\phi^{\alpha}\} \subset H^*_{\mathrm{CR}}(\mathcal{X})$ are additive bases dual to each other under orbifold Poincaré pairing,
- $t = \sum_{\alpha} t^{\alpha} \phi_{\alpha} \in H^*_{\mathrm{CR}}(\mathcal{X})$.

The J-function is a slice of Givental's Lagrangian cone. We refer to [Tse10] and [CG07] for more details. One can decompose the J-function according to the degree of curves

$$J_{\mathcal{X}}(t, z) = \sum_{\beta} J_{\mathcal{X}, \beta}(t, z) q^{\beta}.$$

9.2. **The formal Gromov-Witten invariants of infinite root stacks.** Let X be a smooth projective variety and let

$$D_1, \ldots, D_n \subset X$$

be smooth irreducible divisors. We assume that

$$D = D_1 + \ldots + D_n$$

is simple normal crossings. For any index set $I \subseteq \{1, \ldots, n\}$, we define

$$D_I := \cap_{i \in I} D_i.$$

In particular, we write

$$D_{\varnothing} := X.$$

Let

$$\vec{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n.$$

We define

$$I_{\vec{s}} := \{i : s_i \neq 0\} \subseteq \{1, \ldots, n\}.$$

We consider the multi-root stack

$$X_{D, \vec{r}} := X_{(D_1, r_1), \ldots, (D_n, r_n)},$$

where $\vec{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}_{\geq 0})^n$ and r_i's are pairwise coprime. For the purpose of this paper, we only consider genus zero invariants. By [TY20b, Corollary 16], genus zero orbifold Gromov-Witten invariants of $X_{D, \vec{r}}$, after multiplying suitable powers of r_i, is independent of r_i for r_i sufficiently large. Following [TY20b, Definition 18], the formal Gromov-Witten invariants of $X_{D, \infty}$ is defined as a limit of the corresponding genus zero orbifold Gromov-Witten invariants of $X_{D, \vec{r}}$.

Let

- $\gamma_j \in H^*(D_{I_{\vec{s}j}})$, for $j \in \{1, 2, \ldots, m\}$;
- $a_j \in \mathbb{Z}_{\geq 0}$, for $j \in \{1, 2, \ldots, m\}$.

The formal genus zero Gromov-Witten invariants of $X_{D,\infty}$ are defined as

$$\left\langle [\gamma_1]_{\vec{s}^1} \bar{\psi}^{a_1}, \ldots, [\gamma_m]_{\vec{s}^m} \bar{\psi}^{a_m} \right\rangle_{0, \{\vec{s}^j\}_{j=1}^m, \beta}^{X_{D,\infty}} := \left(\prod_{i=1}^n r_i^{s_{i,-}} \right) \left\langle \gamma_1 \bar{\psi}^{a_1}, \ldots, \gamma_m \bar{\psi}^{a_m} \right\rangle_{0, \{\vec{s}^j\}_{j=1}^m, \beta}^{X_{D,\vec{r}}}$$

for sufficiently large $\vec{r}$, where

- the vectors
$$\vec{s}^j = (s_1^j, \ldots, s_n^j) \in (\mathbb{Z})^n, \text{ for } j = 1, 2, \ldots, m,$$
  satisfy the following condition:
$$\sum_{j=1}^m s_i^j = \int_\beta [D_i], \text{ for } i \in \{1, \ldots, n\}.$$

  These vectors are used to record contact orders of markings with respect to divisors $D_1, \ldots, D_n$. For orbifold Gromov-Witten theory of multi-root stacks $X_{D,\vec{r}}$, they record ages of twisted sectors of the inertial stacks $IX_{D,\vec{r}}$. We refer to [TY20b] for more details.
- $s_{i,-}$ is the number of markings that have negative contact order with the divisor D_i. In other words,
$$s_{i,-} := \#\{j : s_i^j < 0\}, \text{ for } i = 1, 2, \ldots, n.$$

Remark 9.2. When the divisor D is smooth, the formal Gromov-Witten invariants of the infinite root stack $X_{D,\infty}$ are simply relative Gromov-Witten invariants of the smooth pairs (X, D) following the work of [ACW17], [TY18] and [FWY19].

9.3. **Mirror theorems.** A Givental style mirror theorem can be stated as an equality between the J-function and the I-function via mirror maps. More generally, a mirror theorem can be stated using Givental's formalism (Givental's Lagrangian cone etc.). A mirror theorem for the formal Gromov-Witten invariants of infinite root stacks is proved in [TY20a] as a limit of the mirror theorem for multi-root stacks.

The state space $\mathfrak{H}$ for the Gromov-Witten theory of $X_{D,\infty}$ is defined as follows:

$$\mathfrak{H} := \bigoplus_{\vec{s} \in \mathbb{Z}^n} \mathfrak{H}_{\vec{s}},$$

where

$$\mathfrak{H}_{\vec{s}} := H^*(D_{I_{\vec{s}}}).$$

Each $\mathfrak{H}_{\vec{s}}$ naturally embeds into $\mathfrak{H}$. For an element $\gamma \in \mathfrak{H}_{\vec{s}}$, we write $[\gamma]_{\vec{s}}$ for its image in $\mathfrak{H}$. The pairing on $\mathfrak{H}$

$$(-,-) : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$$

is defined as follows: for $[\alpha]_{\vec{s}}$ and $[\beta]_{\vec{s}'}$, define

$$(36) \qquad ([\alpha]_{\vec{s}}, [\beta]_{\vec{s}'}) = \begin{cases} \int_{D_{I_{\vec{s}}}} \alpha \cup \beta, & \text{if } \vec{s} = -\vec{s}'; \\ 0, & \text{otherwise.} \end{cases}$$

The pairing on the rest of the classes is generated by linearity.

When D_i are nef, the non-extended I-function is

$$(37) \qquad I_{X_D,\infty}(q,t,z) := \sum_{\beta \in \mathrm{NE}(X)} J_{X,\beta}(t,z) q^\beta \prod_{i=1}^{n} \prod_{0<a<d_i} (D_i + az)[\mathbf{1}]_{(-d_1,-d_2,\ldots,-d_n)}.$$

A mirror theorem for infinite root stacks is proved in [TY20a, Section 4] (see also [TY20b, Theorem 29]) which states that the I-function $I_{X_D,\infty}$ lies in Givental's Lagrangian cone for $X_{D,\infty}$. When $-K_X - D$ is nef, the I-function equals to the J-function via a change of variable called the mirror map. When $n = 1$, this specializes to the mirror theorem for relative Gromov-Witten invariants of smooth pairs (X, D) proved in [FTY19, Theorem 1.4]. One can also write down the extended I-function which is more complicated. Again, by [TY20b, Theorem 29] the extended I-function lies in Givental's Lagrangian cone. For our propose, we only write down the part of the extended I-function of $X_{D,\infty}$,denoted by $I_{X_D,\infty,0}(q,x,t,z)$, that takes value in $\mathfrak{H}_{(0,\ldots,0)} := H^*(X)$:

$$I_{X_D,\infty,0}(q,x,t,z) := \sum_{\substack{\beta \in \mathrm{NE}(X),(k_{i1},\ldots,k_{im}) \in (\mathbb{Z}_{\geq 0})^m \\ \sum_{j=1}^{m} jk_{ij}=d_i,1\leq i\leq n}} J_{X,\beta}(t,z) q^\beta \frac{\prod_{i=1}^{n} \prod_{j=1}^{m} x_{ij}^{k_{ij}}}{z^{\sum_{i=1}^{n} \sum_{j=1}^{m} k_{ij}} \prod_{i=1}^{n} \prod_{j=1}^{m}(k_{ij}!)}$$

$$\cdot \left(\prod_{i=1}^{n} \prod_{0<a\leq d_i} (D_i + az) \right),$$

where variables x_{ij}'s are used to record tangency conditions: contact order j along the divisor D_i.

We can specialize it to the toric complete intersections that we considered in Section 5. Let X be a complete intersection in a toric variety Y defined by a generic section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s$, where each L_l is a nef line bundle. Let $\rho_l = c_1(L_l)$ and $\mathbf{D}_j$, $1 \leq j \leq m$, be toric divisors. Let $D := D_1 + \ldots + D_n \in |-K_X|$ and assume that D is simple normal crossings and D_i are nef for $1 \leq i \leq n$. Then the I-function for (X, D) can be written as

$$(38) \quad I_{X_D,\infty,0}(q,x,t,z) := e^{\sum_{i=0}^{r} p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \prod_{j=1}^{m} \left(\frac{\prod_{k=-\infty}^{0}(\mathbf{D}_j + kz)}{\prod_{k=-\infty}^{\langle \mathbf{D}_j,d \rangle}(\mathbf{D}_j + kz)} \right) \left(\prod_{l=0}^{s} \prod_{k=1}^{\langle \rho_l,d \rangle} (\rho_l + kz) \right) q^d$$

$$\frac{\prod_{i=1}^{n} \prod_{j=1}^{m} x_{ij}^{k_{ij}}}{z^{\sum_{i=1}^{n} \sum_{j=1}^{m} k_{ij}} \prod_{i=1}^{n} \prod_{j=1}^{m}(k_{ij}!)} \cdot \left(\prod_{i=1}^{n} \prod_{0<a\leq d_i} (D_i + az) \right).$$

Note that the nefness assumption on D_i can be removed if D_i is coming from a toric divisor of Y.

Similar to the discussion in [DKY19, Section 6.4], the information of periods in Section 5.3 can be extracted from the I-function (38). By relative mirror theorem [TY20a, Theorem 29], these periods compute genus zero Gromov-Witten invariants of (X, D).

Recall that, we consider the degeneration of (X, D) into $(X_1, D_1) \cup_{(D_0,D_{12})} (\tilde{X}_2, \tilde{D}_2)$, where $X_1 \cap \tilde{X}_2 = D_0$ and $D_1 \cap \tilde{D}_2 = D_{12}$. Then $I^X, I^{X_1}, I^{\tilde{X}_2}$ and I^{D_0} in Section 5.3 are extracted from the I-functions of $(X, D), (X_1, D_0 + D_1), (\tilde{X}_2, D_0 + \tilde{D}_2)$ and (D_0, D_{12}) respectively.

As mentioned in Remark 5.6, the gluing formula still holds if we remove the common factor involving ρ_{s+1} from the I-functions. In here, it means removing the factors involving D_i, for $1 \leq i \leq n$. Then we obtain a relation among I-functions of $X, (X_1, D_1), (\tilde{X}_2, \tilde{D}_2)$ and D_0.

Through mirror symmetry, the gluing formulae among periods provides a relation among Gromov-Witten invariants. On the A-model side, there are degeneration formulae relating Gromov-Witten invariants. In future work, we will study how these formulae from A-model and B-model are related.

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