CALABI-YAU MANIFOLDS REALIZING SYMPLECTICALLY RIGID MONODROMY TUPLES

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Abstract. We define an iterative construction that produces families of elliptically fibered Calabi-Yau n-folds with section from families of elliptic Calabi-Yau varieties of one dimension lower. Parallel to the geometric construction, we iteratively obtain for each family its Picard-Fuchs operator and a period that is holomorphic near a point of maximal unipotent monodromy through a generalization of the classical Euler transform for hypergeometric functions. In particular, our construction yields one-parameter families of elliptically fibered Calabi-Yau manifolds with section whose Picard-Fuchs operators realize all symplectically rigid Calabi-Yau type differential operators that were classified by Bogner and Reiter. We also demonstrate how our construction computes iteratively the holomorphic period integrals and monodromy matrices for the mirror families of the one-parameter families of deformed Fermat hypersurfaces.

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1. Introduction

The study of Calabi-Yau manifolds, i.e., compact Kähler manifolds with trivial canonical bundle, has been a very active field in algebraic geometry and mathematical physics ever since their christening by Candelas et al. [17] in 1985. For every positive integer n, the zero set of a non-singular homogeneous degree $n+2$ polynomial in the complex projective space

2010 *Mathematics Subject Classification.* 14D0x, 14J32, 32G20, 33Cxx.
Key words and phrases. Calabi-Yau manifolds, elliptic surfaces, Picard-Fuchs equations, Variations of Hodge Structure, Euler's integral transform, Special Functions.

$\mathbb{P}^{n+1}$ is a compact Calabi-Yau manifold of n complex dimensions or a Calabi-Yau n-fold. The construction yields for $n = 1$ an elliptic curve, while for $n = 2$ one obtains a K3 surface. In two complex dimensions K3 surfaces are the only simply connected Calabi-Yau manifolds. In three complex dimensions, the classification of Calabi-Yau manifolds, i.e., Calabi-Yau threefolds, remains an open problem. The results of tremendous ongoing activity in physics that included systematic computer searches have given us a better idea of the landscape of Calabi-Yau threefolds [47, 68]. For example, the work has impressively demonstrated the near omnipresence of elliptic fibrations on Calabi-Yau threefolds. Unfortunately, it has also revealed that it is generally quite difficult to find examples of families of Calabi-Yau threefolds with small Hodge number $h^{2,1}$ by specialization from multi-parameter families.

In contrast, the original quintic-mirror family that was constructed by Candelas, de la Ossa, Green, and Parkes [16] in 1991 is a family of Calabi-Yau threefolds defined over the base $\mathbb{P}^1\backslash\{0, 1, \infty\}$ which has a large structure limit and $h^{2,1} = 1$ on its generic fibers; the family has played a crucial role in the spectacular computations which suggested that mirror symmetry could be used to solve long-standing problems in enumerative geometry. These remarkable observations led to an enormous mathematical activity which both tried to explain the observed phenomena and to establish similar results for other families of Calabi-Yau threefolds. For example, Batyrev described in [6] a way to construct the mirror of Calabi-Yau hypersurfaces in toric varieties via dual reflexive polytopes, and a proof of the so called mirror theorem was later given by Givental in [35] and Lian, Liu, and Yau in [50]. The quintic-mirror family gives rise to a variation of Hodge structure and an associated Picard-Fuchs differential equation. In turn, the solutions to the differential equation, called periods, determine the variation of Hodge structure. The Picard-Fuchs equations of other families of Calabi-Yau threefolds were constructed by Batyrev, van Straten and others in [9] and [8]. Calabi-Yau threefolds with $h^{2,1} = 1$ have since emerged in a pivotal role for both mathematics and physics, where classical geometric, toric, and analytical methods as well as ideas from mirror symmetry are used in their investigation.

Doran and Morgan classified certain one-parameter variations of Hodge structure in [31] which arise from families of Calabi-Yau threefolds with one-dimensional rational deformation space, i.e., families of Calabi-Yau threefolds resembling the quintic-mirror family. It is worth pointing out that their result was achieved not by constructing families of Calabi-Yau threefolds, but by classifying all integral weight-three variations of Hodge structure which can underlie a family of Calabi-Yau threefolds over the thrice-punctured sphere $\mathbb{P}^1\backslash\{0, 1, \infty\}$ – subject to conditions on monodromy coming from mirror symmetry – through the irreducible monodromy representation generated by the local monodromies around the punctures of the base space. The monodromy representations turned out to be identical with the monodromies of the local systems associated with the univariate generalized hypergeometric functions of type ${}_4F_3$ with certain rational coefficients. Each case was later realized as a family of, possibly singular, Calabi-Yau threefolds constructed as hypersurfaces or complete intersections in a Gorenstein toric Fano variety as shown by Clingher et al. [20] – where a non-generic geometric transition was needed in one of the cases. Another important lesson of this “14th case” of Doran et al. in [29] was the conclusion that each of the Calabi-Yau threefolds admitted fibrations by K3 surfaces of high Picard rank. Moreover, in this geometric approach no direct connection to the computation of the families’ periods and their Picard-Fuchs differential operators was provided.

Unfortunately, the classification of one-parameter variations of Hodge structure carried out in [31] does not easily generalize to include the cases where the monodromy representation is non-linearly rigid.[1] However, the construction of the 14 cases lead to the notion of a Calabi-Yau type differential operator by Almkvist, van Enckevort, van Straten and Zudilin, i.e., differential operators that are irreducible, self-dual, Fuchsian operators with regular singular points and only quasi-unipotent monodromies whose local solutions satisfy further integrality conditions [4]. A list of currently over 550 Calabi-Yau type differential operators is available in electronic form.[2] Despite the name the definition of a Calabi-Yau type operator makes no reference to geometry; the majority of known examples was found by computer searches and is – from a geometric point of view – still poorly understood. In fact, it is not known which differential operators are the Picard-Fuchs operator of a geometric family or if there is a methodical way of constructing geometric families from the differential operators of Calabi-Yau type. Dettweiler and Reiter proved [25] using results of Katz [48] that those differential operators of Calabi-Yau type inducing a rigid monodromy representation can be constructed by a series of tensor- and so-called middle Hadamard products from rank-one local systems.

All this research raises the question whether there can be a construction for families of Calabi-Yau threefolds with $h^{2,1} = 1$ on its generic fiber that uses the structures of an elliptic fibration with section and of a K3 fibration with high-Picard rank as a tool in their systematic construction rather than an *ex-post* observation. The existence of such fibrations should add structure that simplifies the analysis and classification of such geometries. Moreover, such a construction should also use *ex-ante* the conditions on the monodromy coming from mirror symmetry rather than trying to specialize multi-parameter families. Since the classification of one-parameter variations of Hodge structure does not generalize to non-linearly rigid monodromy representations, the notion of Calabi-Yau type differential operator will play a central role in this construction. Finally, the aforementioned decomposition result for suitably rigid monodromy representations raises the question whether such a construction can be carried out by an iterative procedure from lower dimensions. This article provides such a construction.

Our iterative construction can be observed directly for the mirror families of the famous one-parameter "Dwork family", i.e., the one-parameter family of deformed Fermat hypersurface in $\mathbb{P}^n$ given by

$$X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1} + (n+1)\,\lambda\, X_0 X_1 \cdots X_n = 0\,. \tag{1.1}$$

For each integer $n \in \mathbb{N}$, Equation (1.1) constitutes a family of $(n-1)$-dimensional Calabi-Yau hypersurfaces $X_\lambda^{(n-1)}$. For $n = 4$ Equation (1.1) is the aforementioned quintic family of Candelas et al. [16]. A second family of special hypersurfaces $Y_t^{(n-1)}$ is defined in terms of the affine variables $x_1, \dots, x_n$ by the equation

$$x_1 \cdots x_n \Big(x_1 + \cdots + x_n + 1 \Big) + \frac{(-1)^{n+1}\, t}{(n+1)^{n+1}} = 0\,. \tag{1.2}$$

[1] A monodromy representation is *linearly rigid* iff the representation is irreducible and already determined by the individual conjugacy classes of all its local quasi-unipotent monodromies [48].

[2] http://www.mathematik.uni-mainz.de/CYequations/db/

It was proved in [7] that the family of special Calabi-Yau hypersurfaces $Y_t^{(n-1)}$ is the mirror family of $X_\lambda^{(n-1)}$. For $n \geq 2$ (cf. Lemma 10.1) the family of hypersurfaces $Y_{\tilde{t}}^{(n-1)}$ carries a fibration over $\mathbb{P}^1$ by hypersurfaces $Y_t^{(n-2)}$ with x_n as the affine base coordinate and

$$t = -\frac{n^n}{(n+1)^{n+1} x_n \, (x_n+1)^n} \tilde{t} \,. \tag{1.3}$$

The rational function on the right hand side of Equation (1.3) relating t to $\tilde{t}$ has a characteristic form determined by the exponents $(1, n)$. The exponents constitute the main part of what we will later refer to as a generalized functional invariant. This generalized functional invariant captures all key features of the one-parameter mirror families necessary to iteratively compute holomorphic period integrals and monodromy matrices. We will show that the converse is true for Picard-Fuchs operators with suitably rigid monodromy. Relations among the holomorphic periods – given in the form of tensor products or middle convolutions – determine generalized functional invariants which in turn allow us to build up iteratively Calabi-Yau manifolds by fibrations of Calabi-Yau manifolds of lower dimension. Remarkably, the starting point of our iterative construction is universal and remains the same as in the case of the mirror families. For $n = 1$ the family $Y_t^{(0)}$ of 2-points in $\mathbb{P}^1$ is given in an affine chart by the equation

$$x_1 \left(x_1 + 1 \right) + \frac{t}{4} = 0 \tag{1.4}$$

for $t \in \mathbb{P} \backslash \{0, 1, \infty\}$. For $n = 1$ the deformed quadratic Fermat pencil X_λ, Y_t in Equation (1.4), and the *general* quadric in $\mathbb{P}^1$ are all equivalent. In this sense, the family in Equation (1.4) is a self-mirror family, i.e., $X_\lambda^{(0)} \cong Y_t^{(0)}$ with $t = 1/\lambda^2$. It will be convenient to set $x_1 = (Y-1)/2$ in Equation (1.4) to obtain the simpler equation $Y^2 = 1 - t$. We consider the function $1/Y$ as the holomorphic period of the family in a neighborhood of $t = 0$ and equals the simplest hypergeometric function, i.e., $\frac{1}{Y} = {}_1F_0\left(\frac{1}{2} \middle| t \right) = (1-t)^{-\frac{1}{2}}$.

2. Summary of results

The main result of this article is an *iterative construction* that produces projective families $\pi : X \to B = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ of elliptically fibered Calabi-Yau n-folds $X_t = \pi^{-1}(t)$ with $t \in B = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ with section from families of elliptic Calabi-Yau varieties with the same properties in one dimension lower for $n = 1, 2, 3, 4$. In this paper we restrict ourselves to elliptic fibrations with sections, so-called Jacobian elliptic fibrations. These families are then presented as Weierstrass models which are ubiquitous in the description of families of elliptic curves and K3 surfaces. Gross proved in [43] that there are only a finite number of distinct topological types of elliptically fibered Calabi-Yau threefolds up to birational equivalence. For an elliptically fibered Calabi-Yau threefold, the existence of a global section makes it then possible to find an explicit presentation as a Weierstrass model as well [59]. For example, our iterative procedure constructs from extremal families of elliptic curves[3] with rational total space, families of Jacobian elliptic K3 surfaces of Picard rank 19 and 18, and in turn from these families of K3 surfaces elliptically fibered Calabi-Yau threefolds with $h^{2,1} = 1$, and can

[3]An elliptic fibration $\pi : X \to \mathbb{P}^1$ is called extremal if and only for the group of sections we have rank $\mathrm{MW}(\pi) = 0$ and the associated elliptic surface has maximal Picard number.

be continued further on. Moreover, all families will be iteratively constructed from a *single* geometric object, the mirror family of Fermat quadrics in $\mathbb{P}^1$ given by

$$X_0^2 + X_1^2 + 2\,t\,X_0\,X_1 = 0\ . \tag{2.1}$$

The broad range of families of Jacobian elliptic Calabi-Yau manifolds obtained by our iterative construction includes the following noteworthy families with generic fibers of dimension n:

- [$n = 1$] the universal families of elliptic curves over the rational modular curves for $\Gamma_0(k)$ with $k = 1, 2, 3, 4$ (cf. Section 5),
- [$n = 2$] families of M_k-polarized K3 surfaces (Picard rank 19) over the rational modular curves for $\Gamma_0(k)^+$ for $k = 1, 2, 3, 4$, and families of Jacobian M-polarized K3 surfaces (Picard rank 18) (cf. Section 6),
- [$n = 3$] families of Calabi-Yau threefolds with $h^{2,1} = 1$ realizing all 14 one-parameter variations of Hodge structure classified by Doran and Morgan in [31] (cf. Section 7),
- [$n = 4$] families of Calabi-Yau fourfolds realizing all 14 one-parameter variations of Hodge structure of weight four and type $(1, 1, 1, 1, 1)$ over a one-dimensional rational deformation space of the so-called odd case (cf. Section 8),
- [$n = 1, 2, 3$] mirror families of the Fermat pencils in $\mathbb{P}^{n+1}$ (cf. Section 10).

Each fiber $X_t = \pi^{-1}(t)$ in the projective families is a compact complex n-fold with trivial canonical bundle $\boldsymbol{\omega}_{X_t} := \wedge^{\mathrm{top}} T^* X_t$. There is a holomorphic sub-bundle $\mathcal{H} \to B$ of the holomorphic vector bundle $R^n\pi_*\mathbb{C}_{X_t} \to B$ whose fiber is $H^0(\boldsymbol{\omega}_{X_t}) \subset H^n(X_t, \mathbb{C})$. The vector bundle $R^n\pi_*\mathbb{C}_{X_t} \to B$ carries a canonical flat connection, called the the *Gauss-Manin connection* ∇. Representing each family as a family of Weierstrass models will allow us to determine an explicit formula for a section $\eta \in \Gamma(B, \mathcal{H})$ such that η_t is represented by a closed differential form on X_t with $\nabla\eta = 0$. Simultaneously, we will construct for each family of Calabi-Yau n-folds a local parallel section of the dual bundle $\mathcal{H}^*$ represented by a closed transcendental n-cycle Σ_t on each fiber X_t. We construct this cycle iteratively, as the warped product of a Pochhammer contour with the corresponding transcendental cycle in dimension $n-1$. We can then construct the local analytic function $\omega : t \mapsto \oint_{\Sigma_t} \eta_t$, called a *period integral*. The period integral is a section of the period sheaf $\Pi(\mathcal{H}, \eta) \to B$ whose stalks are generated by local analytic functions of the form $f : t \mapsto \oint_\Sigma \eta_t$. The Gauss-Manin connection ∇ – more precisely the associated covariant derivative ∇_t along the vector field d/dt on B – of the given family then reduces to a rank-n linear differential equation $L_t^{(n)}\omega(t) = 0$, called the *Picard-Fuchs equation* [42, Sec. 4, 21]. Our construction will always produce families of varieties with Picard-Fuchs equations that are irreducible Fuchsian operators with three regular singularities located at $0, 1$, and ∞ with quasi-unipotent monodromy, and a point of maximally unipotent monodromy at $t = 0$. In particular, all Picard-Fuchs operators we obtain are irreducible Calabi-Yau type operator in the sense of [5] and upon integration over the special cycle Σ_t produce a period that is holomorphic on the unit disk about the point $t = 0$. In this way, our constructed geometries can be considered strong B-model realizations [69].

Within the class of irreducible differential operators of Calabi-Yau type the *symplectically rigid* differential operators constitute an important subclass with three regular singularities and were classified by Bogner and Reiter [15]. In addition to the 14 linearly rigid examples from [31] this class includes all operators whose associated monodromy representation is symplectically rigid. Following the results of Deligne [24] and Katz [48], there is a necessary and sufficient arithmetic criterion for the generalized rigidity of a monodromy tuple within any irreducible reductive algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$: choosing the subgroup to be $\mathrm{GL}_n(\mathbb{C})$ returns the aforementioned notion of linear rigidity, whereas choosing $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ provides us with the more general notion of symplectic rigidity. Bogner and Reiter showed that all of these symplectically rigid monodromy tuples admit a decomposition by a series of middle convolutions and tensor products into Kummer sheaves of rank one [15]. This will find a correspondence in that fact that our iterative constructions builds up all Picard-Fuchs operators from the rank-one Picard-Fuchs operator of the family in Equation (2.1). All 60 symplectically rigid differential operators are available in the database of Almkvist et al. [4], or AESZ database for short. The main result of this article is the following:

Theorem 2.1. *All symplectically rigid Calabi-Yau type operators can be realized as Picard-Fuchs operators of families of Jacobian elliptic Calabi-Yau varieties obtained by the iterative procedure applied to the quadric Fermat pencil (2.1). In particular, for each symplectically rigid differential operator L_t there is a family of Jacobian elliptic Calabi-Yau varieties $\pi : X \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$ equipped with a transcendental cycle Σ_t and a holomorphic top-form η_t on each fiber $X_t = \pi^{-1}(t)$ such that*

(1) *the period $\omega = \int_{\Sigma_t} \eta_t$ is holomorphic on the unit disk about $t = 0$,*
(2) *the period ω satisfies the Picard-Fuchs equation $L_t \omega = 0$.*

We would like to point out that the restriction to symplectically rigid differential operators of Calabi-Yau type in Theorem 2.1 was chosen purely to simplify this exposition. Generally speaking, in all examples that we have considered our iterative construction has shown great potential for generalization to families with n-pointed base curves. For example, our iterative construction already provides a geometric realization of all fourth-order non-symplectically rigid Calabi-Yau operators with four regular singular points that were found in [14].

As we will show, the simplest examples of rank-four Fuchsian differential operators in Theorem 2.1 are Picard-Fuchs operators for families of Weierstrass models obtained by an iteration of quadratic twist, or *pure-twist* as we will call it, from the pencil in Equation (2.1). However, it turns out that only looking at quadratic twists is too restrictive. This is why we give a more general construction that combines quadratic twists with carefully tuned rational base transformations of Weierstrass models. We call such a construction, utilizing a combination of base transformation and twist, the *mixed-twist construction*. A mixed-twist will be determined by a *generalized functional invariant* (i, j, α). It contains two positive integers (i, j) determining a ramified branched cover $u \mapsto c_{ij}/[u^i(u+1)^j]$ of the rational deformation space of a one-parameter family. This map has degree $i + j$ and is totally ramified over 0, has a simple ramification point over 1, and is ramified to degrees i and j over ∞, and for convenience we have chosen $c_{ij} = (-1)^i \, i^i \, j^j/(i + j)^{i+j}$.

The mixed-twist construction works as follows: assume that $\pi_t : X_t \to S$ is a family of n-dimensional Jacobian elliptic fibrations already obtained by our iterative construction such that for each $t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ we have a family of minimal normal Weierstrass models $p_t : X_t \to S$

with $u = (u_1, \dots, u_{n-1}) \in S$ given by

$$Y^2 = 4\,X^3 - g_2\,(t,u)\;X - g_3\,(t,u)\,. \tag{2.2}$$

A closed differential form on X_t with $\nabla\eta = 0$ is given by $\eta_t = du_1 \wedge \cdots \wedge du_{n-1} \wedge dX/Y$. Using a generalized functional invariant (i, j, α), a new family $\tilde{\pi}_t : \tilde{X}_{\tilde{t}} \to \mathbb{P} \times S$ of $(n+1)$-dimensional Jacobian elliptic fibrations is constructed from the family of Weierstrass models $\tilde{p}_{\tilde{t}} : \tilde{W}_{\tilde{t}} \to \mathbb{P}^1 \times S$ with $\tilde{t} \in \mathbb{P}^1\backslash\{0, 1, \infty\}$ and $(\tilde{u}, u_1, \dots, u_{n-1}) \in \mathbb{P}^1 \times S$ given by

$$Y^2 = 4\,X^3 - g_2\left(\frac{c_{ij}\,\tilde{t}}{\tilde{u}^i(\tilde{u}+1)^j}, u\right)\,\Big(\tilde{u}\,(\tilde{u}+1)^\alpha\Big)^4\,X - g_3\left(\frac{c_{ij}\,\tilde{t}}{\tilde{u}^i(\tilde{u}+1)^j}, u\right)\,\Big(\tilde{u}\,(\tilde{u}+1)^\alpha\Big)^6. \tag{2.3}$$

Only the generalized functional invariants are considered that produce minimal normal Weierstrass models and trivial dualizing sheaf $\boldsymbol{\omega}_{\tilde{W}}$. The parameter $\alpha \in \{\frac{1}{2}, 1\}$ in Equation (2.3) allows for an additional quadratic twist. A closed differential form on $\tilde{X}_{\tilde{t}}$ with $\tilde{\nabla}\tilde{\eta} = 0$ is given by $\tilde{\eta}_{\tilde{t}} = d\tilde{u} \wedge \eta_t$ and $t = c_{ij}\tilde{t}/[\tilde{u}^i(\tilde{u}+1)^j]$ As we will show, a key component in the proof of Theorem 2.1 is to understand the role that the generalized functional invariant plays in the period computation. This will allow us to generalize many classically known formulas of the Euler integral transform to include a wider variety of hypergeometric functions.

The outline of this paper is as follows: in Section 3, we will determine the properties that Weierstrass models will have to fulfill to give rise to the class of families of elliptically fibered K3 surfaces and Calabi-Yau threefolds which will be used in the remainder of this article. In Section 4, we will present two basic examples for how our iterative construction with trivial generalized functional invariant produces geometric families realizing up to rank-four Calabi-Yau type operators as Picard-Fuchs operators. In Sections 5, 6, 7, 8, we use our iterative construction with non-trivial generalized functional invariants to produce all families of elliptically fibered Calabi-Yau n-folds for $n = 1, 2, 3, 4$ needed for the proof of Theorem 2.1. Unfortunately, it turns out to be quite technical to construct a family of closed transcendental n-cycles in each case which upon integration with the holomorphic n-form gives a period that is holomorphic on the unit disk about a point of maximally unipotent monodromy. Having completed this necessary but somewhat tedious analytic and geometric work, the proof of Theorem 2.1 will be completed in Section 9. In Section 10 we demonstrate that a sequence of generalized functional invariants captures all key features of the one-parameter mirror families necessary to compute the holomorphic period integrals and monodromy matrices. The generalized functional invariants also allow us to derive explicit Weierstrass models for the mirror families and relations among the holomorphic periods given in the form of middle convolutions. In Section 11 we provide an outlook on already completed and directly related computations that will be described in a forthcoming article.

Acknowledgments

We would like to thank Matt Kerr, Rick Miranda, Dave Morrison, and Noriko Yui for helpful discussions. The first-named author acknowledges support from the National Sciences and Engineering Research Council, the Pacific Institute for Mathematical Sciences, and a McCalla Professorship at the University of Alberta. The second author acknowledges the generous support of the Kavli Institute for Theoretical Physics and of the University of Alberta's Faculty of Science Visiting Scholar Program.

3. Elliptic fibrations and Weierstrass models

In this section we will recall basic facts of complex algebraic geometry in order to set up notation that will be used in the rest of the article; we refer to the papers [18, 59, 26] for details. We also adopt the definition of terminal, canonical, and log-terminal singularity of a projective variety from aforementioned articles. In particular, the relation between elliptic curves/surfaces/threefolds and their Weierstrass models as well as the existence of the canonical bundle formula (3.1) will be of crucial importance in our construction as all families of Calabi-Yau varieties will be constructed as Weierstrass models.

3.1. **Elliptic fibrations.** In this article, we define an elliptic fibration $\pi : X \to S$ to be a proper surjective morphism with connected fibers between normal complex varieties X and S whose general fibers are nonsingular elliptic curves. If π is smooth over an open subset S^0 whose complement is a divisor with only normal crossings, then the local system $H^i_0 := R^i\pi_*\mathbb{Z}_X\,|_{S^0}$ forms a variation of Hodge structure over S^0. There is a canonical bundle formula for the elliptic fibration: in fact, $\mathcal{L} := R^1\pi_*\mathcal{O}_X$ is an invertible sheaf, and a formula for the dualizing sheaf is given by

$$\boldsymbol{\omega}_X \cong \pi^*\Big(\boldsymbol{\omega}_S \otimes \mathcal{L}^{-1}\Big) \otimes \mathcal{O}_X(D) \tag{3.1}$$

where D is a certain effective divisor on X that only depends on divisors of S over which π has multiple fibers and divisors of X that give (-1)-curves in the fibers of π. The existence of a section for the elliptic fibration $\pi : X \to S$ always prevents the presence of multiple fibers. And the presence of (-1)-curves in the fibers can be avoided by imposing a minimality criterion. In the case of an elliptic surface, we say that the fibration is relatively minimal if there are no (-1)-curves in the fibers of π. In the case of an elliptic threefold, we say that the fibration is relatively minimal if no contraction of a surface in X is compatible with the fibration. Moreover, it is known [38] that for any elliptic fibration on a Calabi-Yau threefold, the base surface can have at worst log-terminal orbifold singularities. In this article we will take the base surface S always to be a blow-up of $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_k$.

It is a well-known that for any elliptic fibration $\pi : X \to S$ with section $\sigma : S \to X$, there always exists a Weierstrass model W over S, i.e., there is a complex variety W and a proper flat surjective morphism $p : W \to S$ with canonical section whose fibers are irreducible cubic curves in $\mathbb{P}^2$ together with a birational map from X to W that maps σ to the canonical section of the Weierstrass model. The map from X to W blows down all components of fibers which do not intersect $\sigma(S)$. It is also known that for a relatively minimal elliptic fibration with section, the morphism on the Weierstrass model is in fact a resolution of the singularities of W. However, the converse is not true in general. We now describe Weierstrass models in more detail.

3.2. **Weierstrass models.** Let $\mathcal{L}$ be a line bundle on S, and g_2 and g_3 sections of $\mathcal{L}^{-4}$ and $\mathcal{L}^{-6}$, respectively, such that the discriminant $\Delta = g_2^3 - 27\,g_3^2$ is not identically zero. Define $\mathbf{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ and let $p : \mathbf{P} \to S$ be the natural projection and $\mathcal{O}_{\mathbf{P}(1)}$ be the tautological line bundle. We denote by x, y, and z the sections of $\mathcal{O}_{\mathbf{P}}(1)\otimes\mathcal{L}^{-2}$, $\mathcal{O}_{\mathbf{P}}(1)\otimes\mathcal{L}^{-3}$, and $\mathcal{O}_{\mathbf{P}}(1)$, respectively, which correspond to the natural injections of $\mathcal{L}^{-2}$, $\mathcal{L}^{-3}$, and $\mathcal{O}_S$ into $\pi_*\mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_S \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$. We often use the affine coordinates $X = x/z$ and $Y = y/z$ as well. We denote by W the projective variety in $\mathbf{P}$ defined by the equation

$$y^2\, z = 4\, x^3 - g_2\, x\, z^2 - g_3\, z^3\ . \tag{3.2}$$

Its canonical section $\sigma : S \to W$ is the section at infinity and given by the point $[x : y : z] = [0 : 1 : 0]$ such that $\sigma(S) \subset W$ is a Cartier divisor on W, and its conormal bundle is isomorphic to $\mathcal{L}$, i.e., $p_*\mathcal{O}_{\sigma(S)}(\sigma(S)) \cong \mathcal{L}$. It then follows that W is normal if S is normal; and W is Gorenstein if S is and the formula (3.1) for the dualizing sheaf becomes

$$\boldsymbol{\omega}_W = \pi^*\Big(\boldsymbol{\omega}_S \otimes \mathcal{L}^{-1}\Big)\ . \tag{3.3}$$

For the Weierstrass model $p : W \to S$ of an elliptic fibration $\pi : X \to S$ with section we can compare the discriminant locus $\Delta(\pi)$, i.e., the points over which X_p is singular, with the vanishing locus of $\Delta(p) = g_2^3 - 27\, g_3^2$. We then have $\Delta(p) \subset \Delta(\pi)$. In fact, the morphism from X to W is a resolution of singularities if and only if $\Delta(p) = \Delta(\pi)$ in which case it is also a small resolution.

We call a Weierstrass model minimal if there is no prime divisor D on S such that $\mathrm{div}(g_2) \geq 4\, D$ and $\mathrm{div}(g_3) \geq 6\, D$. We call two minimal Weierstrass models that are smooth over an open subset $S_0 \subset S$ equivalent if there is an isomorphism of the Weierstrass models over S_0 which preserves the canonical sections. Every Weierstrass fibration is birationally isomorphic to a minimal Weierstrass fibration. A criterion for W having only rational singularities can then be stated as follows: If the reduced discriminant divisor $(\Delta)_{\mathrm{red}}$ has only normal crossings, then W has only rational Gorenstein singularities if and only if the Weierstrass model is minimal.

4. First examples from the pure-twist construction

In this section we will explain two basic series of examples for our iterative construction: one starts with a family of points and produces a family of elliptic curves whose total space is a rational surface by quadratic twist. One then continues by constructing a family of Jacobian elliptic K3 surfaces of Picard rank 19 from the Jacobian rational elliptic surface, and in turn, a family of elliptic Calabi-Yau threefolds with $h^{2,1} = 1$ from the family of Jacobian elliptic K3 surfaces by two more quadratic twists. One can also allow for the Picard rank of the K3 surfaces in the intermediate step to drop from 19 down to 18 and construct a closely related family of elliptic Calabi-Yau threefolds with $h^{2,1} = 1$. The Picard-Fuchs operators for the two families of Calabi-Yau threefolds realize the two simplest rank-four and degree-one Calabi-Yau operators in the AESZ database [4].

The construction of these examples was motivated by physics, in particular the embedding of gauge theory into F-theory [53, 54, 55]. An interpretation of these families from the point of view of variations of Hodge structure was provided in [39, 40, 49, 23]. The idea of using a quadratic twist to construct an isomorphism between two types of moduli problems also appeared in the work of Besser and Livné in [10]: the quadratic twist-construction was used to relate the classification of elliptic surfaces over the projective line with five specified singular fibers, of which four are fixed and one gives the parameter, to the classification of K3 surfaces with a specified isogeny to an abelian surface with quaternionic multiplication. The quadratic twist also appeared in [37] in the study of rigid Calabi-Yau threefolds.

4.1. **A sequence of quadratic twists.** We start with the ramified family of two points $\pm y_0$ solving the equation

$$y_0^2 = 1 - t \tag{4.1}$$

for $t \in \mathbb{C}$. The family (4.1) is isomorphic to the quadric pencil (2.1) by a change of variables. We define the period of this family to be the function with branch cut $1/y_0$. If we arrange the branch cut to coincide with $\{\, t \,|\, 1 \leq t \leq \infty\}$ then $1/y_0$ is a function that is holomorphic in a punctured unit disc about $t = 0$ with

$$\frac{1}{y_0} = {}_1F_0\left(\frac{1}{2}\middle|\, t\right) = (1-t)^{-\frac{1}{2}} \ . \tag{4.2}$$

The period is annihilated by the first-order and degree-one Picard-Fuchs operator

$$L_1 = \theta - t\left(\theta + \frac{1}{2}\right) , \tag{4.3}$$

where $\theta = t\,\frac{d}{dt}$. The Picard-Fuchs operator gives rise to a monodromy tuple of rank one, with monodromy $1, -1$, and -1 over the points $t = 0, 1, \infty$.

To go from the family of points in Equation (4.1) to a family of elliptic curves, one promotes the family parameter t to an additional complex variable x and carries out the quadratic twist $y_0^2 \mapsto -y_1^2/[x(x-t)]$ in Equation (4.1). One obtains the Legendre pencil of elliptic curves X_t given by

$$y_1^2 = x\,(x-1)\,(x-t) , \tag{4.4}$$

where t is a complex parameter in $\mathbb{P}^1\backslash\{0, 1, \infty\}$ and we will denote the projection by $\pi : X_t \to \mathbb{P}^1$. The cohomology groups of the fibers $H^1(X_t, \mathbb{C})$ form the flat vector bundle $\mathbb{V} = R^1\pi_*\mathbb{C}$ of middle cohomology over $\mathbb{P}^1$. The polarized Hodge filtration on $H_\mathbb{Z} = H^1(X_t, \mathbb{Z})$ of the elliptic curve X_t has two steps, F^0 and F^1 defining a pure Hodge structure of weight one and type $(1, 1)$ in

$$\left\{H_\mathbb{Z}, Q, F^1 \subset F^0 = H_\mathbb{Z} \otimes \mathbb{C}\right\} \ .$$

Here, F^0 is the entire cohomology group, and F^1 is $H^{1,0}(X_t)$, the one-dimensional space of holomorphic harmonic 1-forms spanned by the holomorphic one-form dx/y_1. The polarization Q is the natural non-degenerate integer bilinear form on $H_\mathbb{Z}$ derived from the cup product and varies holomorphically in t. The homology group of the elliptic curve is free of rank two, and the periods of dx/y_1 over the two one-cycles satisfy the second-order differential equation of the hypergeometric function ${}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1 \big|\, t\big)$. In fact, the holomorphic solution near $t = 0$ is obtained as period integral over the so-called A-cycle of the elliptic curve which we denote by Σ_1, i.e.,

$$\oint_{\Sigma_1} \frac{dx}{y_1} = \int_0^t \frac{dx}{\sqrt{x\,(x-1)\,(x-t)}} = (\pi i)\; {}_2F_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array}\middle|\, t\right) . \tag{4.5}$$

This can be understood as follows: Equation (4.4) defines a double branched cover of $\mathbb{P}^1$ branched at the four points $x = 0, 1, t, \infty$. The A-cycle Σ_1 is the closed cycle encircling the branching points at $x = 0$ and $x = t$ and can be deformed into the cycle that follows the branch cut from $x = 0$ to $x = t$ on one y-sheet and returns on the second. It is well-known that the period (4.5) is annihilated by the second-order and degree-one[4] Picard-Fuchs operator

$$L_2 = \theta^2 - t\left(\theta + \frac{1}{2}\right)^2 . \tag{4.6}$$

The Hadamard product of two power series $f(t) = \sum_{n\geq 0} f_n\, t^n$ and $g(t) = \sum_{n\geq 0} g_n\, t^n$ is defined by $(f \star g)(t) := \sum_{n\geq 0} f_n\, g_n\, t^n$. The quadratic twist has turned Equation (4.1) into (4.4) and, similarly, the Hadamard product with the function ${}_1F_0$ turns the holomorphic period in Equation (4.2) into the holomorphic period in Equation (4.5), i.e.,

$${}_1F_0\left(\frac{1}{2}\middle| t\right) \star {}_1F_0\left(\frac{1}{2}\middle| t\right) = {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\middle| t\right) .$$

There is a corresponding notion of the Hadamard product (cf. [15, Def. 4.11]) for the Picard-Fuchs differential operators involved: the Hadamard product of the differential operator annihilating ${}_1F_0$ with the Picard-Fuchs operator annihilating the period (4.2) yields the Picard-Fuchs operator annihilating the period (4.5), i.e., $\mathcal{H}_{1/2}(L_1) = L_2$.

To move from the family of elliptic curves in Equation (4.4) to a family of K3 surfaces, one again promotes the family parameter t to an additional complex variable u and carries out the quadratic twist $y_1^2 \mapsto y_2^2/[u(u-t)]$ in Equation (4.4). One obtains the twisted Legendre pencil given by

$$y_2^2 = x\,(x-1)\,(x-u)\,u\,(u-t) . \tag{4.7}$$

Equation (4.16) defines the Néron model for a family of elliptically fibered K3 surfaces X_t of Picard rank 19 with section over $\mathbb{P}^1$, which we will denote by $\pi : X_t \to \mathbb{P}^1$. Hoyt [45] and Endo [33] extended arguments of Shimura and Eichler [32] to show that on the parabolic cohomology group $H_\mathbb{Z} \cong \mathbb{Z}^3$ associated with $\pi : X_t \to \mathbb{P}^1$ there is a natural polarized Hodge filtration given by

$$\left\{H_\mathbb{Z}, Q, F^2 \subset F^1 \subset H_\mathbb{Z} \otimes \mathbb{C}\right\} .$$

Here, $H_\mathbb{Z} \otimes \mathbb{C}$ consists of cohomology classes spanned by suitable meromorphic differentials of the second kind, and Q is a non degenerate $\mathbb{Q}$-valued bilinear form determined by period relations of the holomorphic two-form $du \wedge dx/y_2$ [46]. The linear subspace F^2 is defined as cone $Q = 0$ and F^1 is the tangent plane to this cone along F^2. In the situation of Equation (4.7), it follows $Q = 2z_1^2 + 2z_2^2 - 2z_3^2$. As a consequence, the local system $\mathbb{V} = R^2\pi_*\mathbb{C}$ of middle cohomology is irreducible and the cohomology group $H_\mathbb{Z}$ carries a pure Hodge structure of weight two and type $(1, 1, 1)$ with

$$H_\mathbb{Z} \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} .$$

Moreover, a basis of transcendental cycles can be constructed from cycles in the elliptic fiber and carefully chosen curves in the base connecting the cusps $0, 1, t, \infty$ [45]. In fact, for suitable

[4] *degree* refers to the highest power in t

continuously varying one-cycles $\Sigma_1', \check{\Sigma}_1'$, that form a basis of the first homology of the elliptic fiber, the expression

$$\int_t^u du \begin{pmatrix} \int_{\Sigma_1'} \frac{dx}{y_2} \\ \int_{\check{\Sigma}_1'} \frac{dx}{y_2} \end{pmatrix} \tag{4.8}$$

defines a vector-valued holomorphic function that converges as u approaches any of the cusps at $u = 0, 1, \infty$. In particular, periods of the holomorphic two form $du \wedge dx/y_2$ over these transcendental cycles satisfy the third-order differential equation of the hypergeometric function ${}_3F_2\big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1 \big| t\big)$. This is a direct consequence of the so-called Euler integral transform formula: through an integral transform with parameter, (the solution of) a second-order Fuchsian differential equation with three regular singularities is related to a third-order Fuchsian differential equation with three regular singularities. In fact, after choosing particular branches the holomorphic solutions are related by the Euler integral formula

$$\int_0^t \frac{du}{\sqrt{u\,(u-t)}}\; {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| u \right) = (\pi i)\; {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| t \right) . \tag{4.9}$$

Since the singular fiber is of Kodaira-type I_2^* over $u = 0$ and of Kodaira-type I_0^* over $u = t$ – the latter having monodromy $-\mathbb{I}$ independent of the chosen homological invariant – there is a unique A-cycle that is transformed into itself as a path encircles the cusps at $u = 0$ or $u = t$. One obtains a transcendental two-cycle Σ_2 on X_t by following the A-cycle in the fiber and the line segment between the cusps at $u = 0$ and $u = t$ in the base. Concretely, if one integrates the holomorphic two-form $du \wedge dx/y_2$ over this two-cycle Σ_2 one obtains for the holomorphic period

$$\oiint_{\Sigma_2} du \wedge \frac{dx}{y_2} = (\pi i)^2\; {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| t \right) . \tag{4.10}$$

The period is annihilated by the third-order and degree-one Picard-Fuchs operator

$$L_3 = \theta^3 - t \left(\theta + \frac{1}{2} \right)^3 . \tag{4.11}$$

The quadratic twist has turned Equation (4.4) into (4.7) and, similarly, the Hadamard product with the function ${}_1F_0$ turns the holomorphic period in Equation (4.5) into the holomorphic period in Equation (4.10), i.e.,

$$ {}_1F_0\left(\left. \frac{1}{2} \right| t \right) \star {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t \right) = {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| t \right) .$$

Moreover, the Hadamard product of the differential operator annihilating ${}_1F_0$ with the Picard-Fuchs operator annihilating the period (4.5) again yields the Picard-Fuchs operator annihilating the period (4.10), i.e., $\mathcal{H}_{1/2}(L_2) = L_3$.

To transgress from the family of K3 surfaces in Equation (4.7) to a family of Calabi-Yau threefolds one promotes the family parameter t to an additional complex variable s and carries out yet another quadratic twist $y_2^2 \mapsto y_3^2/[s(s-t)]$ in Equation (4.7). One obtains

$$y_3^2 = x\,(x-1)\,(x-u)\,u\,(u-s)\,s\,(s-t) . \tag{4.12}$$

It is easy to check that this family constitute a pencil of elliptically fibered Calabi-Yau threefolds, which we will again denote by $\pi : X_t \to \mathbb{P}^1$. Each member of the family is elliptically fibered over $\mathbb{P}^2$ with affine coordinates u, s and also fibered by K3 surfaces of Picard rank 19 over $\mathbb{P}^1$ with affine coordinates u. As a consequence, the local system $\mathbb{V} = R^3\pi_*\mathbb{C}$ of middle cohomology is irreducible and the transcendental cohomology group $H_{\mathbb{Z}}$ carries a pure Hodge structure of weight three and type $(1, 1, 1, 1)$ with

$$H_{\mathbb{Z}} \otimes \mathbb{C} = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \ .$$

As before, the natural Hodge structure on the parabolic cohomology group of X_t can be described in terms of periods and period relations of the holomorphic three-form $ds \wedge du \wedge dx/y_3$. In particular, the periods satisfy the fourth-order differential equation of the hypergeometric function ${}_4F_2\big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1\big| t\big)$. Again, this is a direct consequence of the Euler integral transform formula

$$\int_0^t \frac{ds}{\sqrt{s\,(s-t)}}\ {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| s \right) = (\pi i)\ {}_4F_3\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{array} \right| t \right) \ . \tag{4.13}$$

We construct a transcendental three-cycle Σ_3 on each X_t as Lefschetz cycle by tracing out the two-cycle Σ_2 over each point of the line segment between the cusps $s = 0$ and $s = t$. If one integrates the holomorphic three-form $ds \wedge du \wedge dx/y_2$ over the cycle Σ_3 one obtains for the holomorphic period

$$\oiiint_{\Sigma_3} ds \wedge du \wedge \frac{dx}{y_3} = (\pi i)^3\ {}_4F_3\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{array} \right| t \right) \ . \tag{4.14}$$

The period is annihilated by the fourth-order and degree-one Picard-Fuchs operator

$$L_4 = \theta^4 - t \left(\theta + \frac{1}{2} \right)^4 \ . \tag{4.15}$$

The Picard-Fuchs operator (4.15) is one of the 14 original Calabi-Yau operators mentioned in the introduction and was labelled "(3)" in the AESZ database [4]. A quadratic twist has turned Equation (4.7) into (4.12) and, similarly, a Hadamard product turned the holomorphic period in Equation (4.10) into the holomorphic period in Equation (4.14), i.e.,

$${}_1F_0\left(\left. \frac{1}{2} \right| t \right) \star {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| t \right) = {}_4F_3\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{array} \right| t \right) \ ,$$

and a similar statement holds on the level of the Picard-Fuchs operators, i.e., $\mathcal{H}_{1/2}(L_3) = L_4$.

4.2. **Closely related examples.** To go from the family of elliptic curves in Equation (4.4) to a family of K3 surfaces of Picard rank 18, one again promotes the family parameter t to an additional complex variable u and carries out the quadratic twist $y_1^2 \mapsto y_2^2/[u^2 - t^2]$ in Equation (4.4). One obtains the twisted Legendre pencil given by

$$y_2^2 = x\,(x-1)\,(x-u)\,(u^2 - t^2) \ . \tag{4.16}$$

It defines the Weierstrass model for a family of elliptically fibered K3 surfaces X_t of Picard rank 18 with section over $\mathbb{P}^1$, denoted by $\pi : X_t \to \mathbb{P}^1$. We first derive an analogue of the Euler integral transform formula (4.9) that can be used in the Picard rank 18 case. Assuming $|t| < 1$, it follows

$$\begin{aligned} & \int_{-t}^{t} \frac{du}{\sqrt{u^2-t^2}}\; {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| u \right) = \int_{-1}^{1} \frac{du}{\sqrt{u^2-1}}\; {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| u\, t \right) \\ & = \frac{i}{2} \int_0^1 \frac{d\tilde{u}}{\sqrt{\tilde{u}(1-\tilde{u})}} \left({}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t\, \tilde{u}^{1/2} \right) + {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| -t\, \tilde{u}^{1/2} \right) \right) . \end{aligned} \tag{4.17}$$

Since for $|t| < 1$ the series expansion of the hypergeometric function converges absolutely and uniformly, Equation (4.17) can be simplified as follows:

$$\begin{aligned} & i \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}\, t^n \int_0^1 d\tilde{u}\, (\tilde{u})^{\frac{n+1}{2}-1} (1-\tilde{u})^{\frac{1}{2}-1} = i\, \Gamma\left(\frac{1}{2}\right)^2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{2n}^2 \left(\frac{1}{2}\right)_n}{[(2n)!]^2\, n!}\, t^{2n} \\ & = (\pi i) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n^2 \left(\frac{3}{4}\right)_n^2}{\left(\frac{1}{2}\right)_n (n!)^3}\, t^{2n} = (\pi i)\; {}_4F_3\left(\left. \begin{array}{c} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) . \end{aligned} \tag{4.18}$$

Therefore, periods for the holomorphic two form $du \wedge dx/y_2$ over transcendental cycles satisfy a fourth-order differential equation. We define a vanishing two-cycle $\hat{\Sigma}_2$ by following the line segment between the cusps $u = -t$ and $u = t$ (avoiding $u = 0$ by using a portion of a small circle) and the aforementioned A-cycle in the fiber. We obtain for the period integral

$$\oiint_{\hat{\Sigma}_2} du \wedge \frac{dx}{y_2} = (\pi i)^2\; {}_4F_3\left(\left. \begin{array}{c} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) . \tag{4.19}$$

The period is annihilated by the third-order and degree-two Picard-Fuchs operator

$$\hat{L}_3 = \theta^3 (\theta - 1) - \frac{t^2}{16} (2\theta + 3)^2 (1 + 2\theta)^2 . \tag{4.20}$$

A quadratic twist has turned Equation (4.4) into (4.16) and, similarly, a Hadamard product turns the holomorphic period in Equation (4.5) into the holomorphic period in Equation (4.19), i.e.,

$${}_1F_0\left(\left. \frac{1}{2} \right| t^2 \right) \star {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t \right) = {}_4F_3\left(\left. \begin{array}{c} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) .$$

To go from the family of K3 surfaces in Equation (4.16) to a family of Calabi-Yau threefolds one promotes the family parameter t to an additional complex variable s and carries out yet another quadratic twist $y_2^2 \mapsto y_3^2/[s^2 - t^2]$ in Equation (4.16). One obtains

$$y_3^2 = x\,(x-1)\,(x-u)\,(u^2 - s^2)\,(s^2 - t^2) . \tag{4.21}$$

This family constitute a non-trivial pencils of elliptically fibered Calabi-Yau threefolds, which we will again denote by $\pi : X_t \to \mathbb{P}^1$. As before, the natural Hodge structure on the parabolic cohomology group of X_t can be described in terms of periods and period relations of the holomorphic three-form $ds \wedge du \wedge dx/y_3$. In particular, the periods satisfy the fourth-order differential equation of the hypergeometric function ${}_4F_3\big(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};1,1,1\big|\,t\big)$. Again, this is a direct consequence of the Euler integral transform formula

$$\int_{-t}^{t}\frac{ds}{\sqrt{s^2-t^2}}\;{}_4F_3\left(\begin{array}{c}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,\frac{1}{2}\end{array}\middle|\,s^2\right)=(\pi i)\;{}_4F_3\left(\begin{array}{c}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{array}\middle|\,t^2\right)\,. \tag{4.22}$$

We construct a transcendental three-cycle $\hat{\Sigma}_3$ on X_t as Lefschetz cycle by tracing out the cycle $\hat{\Sigma}_2$ over each point of the line segment between the cusps $s=-t$ and $s=t$ (avoiding $s=0$ by using a portion of a small circle). If one integrates the holomorphic three-form $ds\wedge du\wedge dx/y_2$ over the vanishing cycle $\hat{\Sigma}_3$ one obtains for the holomorphic period

$$\oiiint_{\hat{\Sigma}_3} ds\wedge du\wedge\frac{dx}{y_3}=(\pi\,i)^3\;{}_4F_3\left(\begin{array}{c}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{array}\middle|\,t^2\right)\,. \tag{4.23}$$

The period is annihilated by the fourth-order and degree-two Picard-Fuchs operator

$$\hat{L}_4=\theta^4-\frac{t^2}{16}\,(2\,\theta+1)^2\,(2\,\theta+3)^2\;. \tag{4.24}$$

The Picard-Fuchs operator (4.24) is another of the 14 original Calabi-Yau operators mentioned in the introduction and was labelled "(10)" in the AESZ database [4]. A quadratic twist has turned Equation (4.7) into (4.12) and, similarly, a Hadamard product turns the holomorphic period in Equation (4.10) into the holomorphic period in Equation (4.14), i.e.,

$${}_1F_0\left(\frac{1}{2}\middle|\,t^2\right)\star{}_4F_3\left(\begin{array}{c}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,\frac{1}{2}\end{array}\middle|\,t^2\right)={}_4F_3\left(\begin{array}{c}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{array}\middle|\,t^2\right)\,.$$

5. Step 1: From the family of points to families of elliptic curves

In this section, we describe the first step of our iterative construction that produces families of elliptic curves whose total spaces are extremal rational elliptic surfaces. All these families of Jacobian rational elliptic surfaces are obtained from a single geometric object, the family in Equation (4.1) by applying either a quadratic twist or a quadratic twist followed by a rational transformation of the base. We call the former the *pure-twist construction* and the latter the *mixed-twist construction* at Step 1. At the same time, we will compute the (fiberwise) Picard-Fuchs operators by application of the classical Euler integral transform for hypergeometric functions. Notice that the pure-twist construction at Step 1 is subsumed by the introductory example from Section 4.1. In contrast, the mixed-twist construction will produce a wider variety of extremal families of elliptic curves by using a generalized functional invariant (i,j,α). In particular, the iterative construction at Step 1 will produce the elliptic modular surfaces for $\Gamma_0(n)$ with $n=1,2,3,4$.

5.1. **Pure-twist construction.** We start by describing the *pure-twist construction* that produces a family of elliptic curves from the branched family of points given by

$$y^2=1-t\,. \tag{5.1}$$

The pure twist of Equation (5.1) is obtained by replacing t by tx and twisting the equation by the factor $x(x-1)$. We have the following lemma:

Lemma 5.1. *The Weierstrass equation*

$$y^2 = (1 - t\,x)\,x\,(x-1) \tag{5.2}$$

defines an extremal family of elliptic curves with singular fibers of Kodaira-type I_2^* over $t = \infty$ and I_2 over $t = 0, 1$.

Proof. Setting $y = i\,Y/(2t)$ and $X = x + (t+1)/3$ in equation (5.1), we obtain the Weierstrass equation

$$Y^2 = 4\,X^3 - \underbrace{\frac{4}{3}\Big(t^2 - t + 1\Big)}_{=:\,g_2(t)}\,X - \underbrace{\frac{4}{27}\Big(t+1\Big)\Big(t-2\Big)\Big(2t-1\Big)}_{=:\,g_3(t)}\;. \tag{5.3}$$

The statement of the lemma can then be easily checked using the Weierstrass equation. □

We define a family of closed one-cycles Σ_1 in the total space of Equation (5.2) as follows: for each t contained in the open punctured disc of radius 1 about the origin, i.e., $t \in D_1^*(0)$, one follows the positively oriented Pochhammer contour[5] around $x = 0$ and $x = 1$ in the affine x-plane such that $0 < |xt| < 1$. One then chooses for y a solution of Equation (5.2) continuous in x in the affine y-plane. For $t \in D_1^*(0)$ we obtain a continuously varying family of closed one-cycles that we denote by Σ_1. To simplify our exposition, we will not distinguish between the cycle Σ_1 and its projection onto the x-plane.

We now evaluate the period integral of the holomorphic one-form $dx/(2y)$ over the one-cycle Σ_1, i.e.,

$$\omega_1 = \oint_{\Sigma_1} \frac{dx}{2y}\;, \tag{5.4}$$

where a factor of 2 has been inserted to account for the fact that a Pochhammer contour covers the branch cut between $x = 1$ and $x = 1$ twice. We then have the following lemma:

Lemma 5.2. *For $t \in D_1^*(0)$ the period of the holomorphic one-form $dx/(2y)$ over the one-cycle Σ_1 for the family of elliptic curves described in Lemma 5.1 is given by*

$$\omega_1 = (2\pi i)\;{}_2F_1\!\left(\left.\begin{array}{c}\frac{1}{2},\frac{1}{2}\\ 1\end{array}\right|t\right)\;. \tag{5.5}$$

Proof. Recalling that $|xt| < 1$ for all $x \in \Sigma_1$, the series expansion of the hypergeometric function converges absolutely and uniformly. For $|t| < 1$ one then obtains for the period integral

$$i\oint_{\Sigma_1} \frac{dx}{\sqrt{x\,(1-x)}}\;{}_1F_0\!\left(\left.\frac{1}{2}\right|xt\right) = (4\pi i)\;{}_1F_0\!\left(\left.\frac{1}{2}\right|t\right) \star {}_1F_0\!\left(\left.\frac{1}{2}\right|t\right)\;. \tag{5.6}$$

□

[5] A Pochhammer contour is a cycle in the x-plane with punctures at $x = 0$ and $x = 1$ that is homologous to zero but not homotopic to zero.

5.2. **Mixed-twist construction.** Given two positive integers (i, j), we define a ramified map $x \mapsto c_{ij}/[x^i(x+1)^j]$ of degree $\deg = i + j$ from $\mathbb{P}^1 \to \mathbb{P}^1$ with $c_{ij} = (-1)^i\, i^i\, j^j/(i+j)^{i+j}$. The map is totally ramified over 0, has a simple ramification point over 1, and is ramified to degrees i and j over ∞. The mixed twist of Equation (5.1) is obtained by replacing $t \mapsto t\, c_{ij}/[x^i(x+1)^j]$ followed by a quadratic twist. We then have the following lemma:

Lemma 5.3. *For (i, j, α) with $1 \le i \le 2$, $1 \le j \le 2\,\alpha$ and $\alpha \in \{\frac{1}{2}, 1\}$, the equation*

$$y^2 = \left(1 - \frac{t\, c_{ij}}{x^i(x+1)^j}\right) x^2\,(x+1)^{2\alpha}\ , \tag{5.7}$$

defines a family of genus-one curves where x and y are affine complex coordinates and t is the family parameter. The family of elliptic curves has a singular fiber of Kodaira-type I_n over $t = 0$ with $n \in \{1, 2, 3, 4\}$.

□

We call this construction of families in Equation (5.7) the *mixed-twist construction* at Step 1 with generalized functional invariant (i, j, α). We define a family of closed one-cycles Σ'_1 in the total space of Equation (5.7) as follows: for each $t \in D^*_1(0)$ one follows the positively oriented circle $C_r(0) = \{x : |x| = r\}$ in the affine x-plane where r as a function of t is given in Table 1, and one chooses for y a solution of Equation (5.7) continuous in x. For $t \in D^*_1(0)$ we obtain a continuously varying family of closed one-cycles that we denote by Σ'_1. We will not distinguish between the cycle Σ'_1 and its projection onto the x-plane.

Lemma 5.4. *For every $t \in D^*_1(0)$ the closed one-cycle $\Sigma'_1 = \Sigma'_1(t)$ constitutes a non-trivial class, called A-cycle, in the first homology of the elliptic curve in Equation (5.7). Moreover, we have $|c_{ij}\, t/(x^i(x+1)^j)| < 1$ for all $x \in \Sigma'_1$ and $t \in D^*_1(0)$.*

Proof. It is easy to show that the four roots $x_1(t), \dots, x_4(t)$ of Equation (5.7) (allowing ∞ for x_4) satisfy

$$|x_1| \le |x_2| < r < |x_3| \le |x_4|$$

for all $t \in D^*_1(0)$ where r as a function of t is given in Table 1. Hence, Σ'_1 is an A-cycle of the elliptic curve in Equation (5.7) encircling exactly to branching points x_1, x_2 in the affine x-plane. The second statement follows by explicit computation. Notice that there is a non-intersecting but equivalent second cycle $\Sigma^\#_1$ encircling exactly to branching points x_3, x_4 in the x-plane. □

We now evaluate the period of the holomorphic one-form dx/y over the one-cycle Σ'_1, i.e.,

$$\omega_1 = \oint_{\Sigma'_1} \frac{dx}{y} = \oint_{\Sigma'_1} \frac{dx}{x(x+1)^\alpha}\ {}_1F_0\left(\frac{1}{2}\middle|\, \frac{t\, c_{ij}}{x^i(x+1)^j}\right)\ . \tag{5.8}$$

We have the following lemma:

Lemma 5.5. *For $t \in D^*_1(0)$ the period of the holomorphic one-form dx/y over the one-cycle Σ'_1 for the family of elliptic curves defined in Equation (5.7) is given by the Hadamard product*

$$\omega_1 = (2\pi i)\ {}_{i+j}F_{i+j-1}\left(\begin{array}{c} \frac{\alpha}{i+j},\quad \dots \quad \frac{\alpha+i+j-1}{i+j} \\ \frac{1}{i}, \dots, \frac{i-1}{i}, \frac{\alpha}{j}, \dots, \frac{\alpha+j-1}{j} \end{array}\middle|\, t\right) \star {}_1F_0\left(\frac{1}{2}\middle|\, t\right)\ . \tag{5.9}$$

Proof. For $|\frac{c_{ij}\,t}{x^i(x+1)^j}| < 1$ the hypergeometric series converges absolutely and uniformly, and we can evaluate the period (4.10) by term-by-term integration. We decompose the integral as follows

$$\begin{aligned}&\oint_{\Sigma_1'} \frac{dx}{x(x+1)^\alpha}\; {}_1F_0\left(\frac{1}{2}\middle|\, \frac{c_{ij}\,t}{x^i(x+1)^j}\right)\\ &= \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)_n}{n!}\,(c_{ij}\,t)^n \oint_{|x|=r} dx\;\; x^{-in-1}\,(1+x)^{1-\alpha-jn-1}\;.\end{aligned}$$

Using $r<1$ and the formula $(1+z)^{-k} = \sum_{n\geq 0} \frac{\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(-z)^n$, we obtain from a residue computation

$$\begin{aligned}&\sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)_n}{n!}\,(c_{ij}\,t)^n \oint_{C_r(0)} dx\;\; x^{-in-1}\,(1+x)^{-\alpha-jn}\\ &= (2\pi i) \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)_n \Gamma((i+j)\,n+\alpha))}{\Gamma(in+1)\,\Gamma(jn+\alpha)\,n!}\left((-1)^i c_{ij}\,t\right)^n\;.\end{aligned}\tag{5.10}$$

Since we also have $|t|<1$, Equation (5.2) can be simplified using Gamma-function identities and the Hadamard product. We obtain

$$\begin{aligned}&(2\pi i) \sum_{n\geq 0} \frac{\left(\frac{1}{2}\right)_n \Gamma((i+j)\,n+\alpha))}{\Gamma(in+1)\,\Gamma(jn+\alpha)\,n!}\left((-1)^i c_{ij}\,t\right)^n\\ &= (2\pi i)\; {}_{i+j}F_{i+j-1}\left(\begin{array}{c} \frac{\alpha}{i+j},\quad \ldots \quad \frac{\alpha+i+j-1}{i+j}\\ \frac{1}{i},\ldots,\frac{i-1}{i},\frac{\alpha}{j},\ldots,\frac{\alpha+j-1}{j}\end{array}\middle|\, t\right) \star {}_1F_0\left(\frac{1}{2}\middle|\, t\right)\;.\end{aligned}\tag{5.11}$$

□

Checking when cancellations for the rational parameters in the hypergeometric function occur yields the following lemma:

Lemma 5.6. *The period integral in Equation (5.9) reduces to the hypergeometric function holomorphic near $t=0$ given by*

$$\omega_1^{(\mu)} = (2\pi i)\; {}_2F_1\left(\begin{array}{c}\mu,\,1-\mu\\ 1\end{array}\middle|\, t\right)\tag{5.12}$$

for the values and relations between (i,j,α) and μ given in Table 1. The period is annihilated by the second-order and degree-one Picard-Fuchs operator

$$L_2^{(\mu)} = \theta^2 - t\,(\theta+\mu)\,(\theta+1-\mu)\;.\tag{5.13}$$

The families of elliptic curves defined by Equation (5.7) have singular fibers of the Kodaira-types listed in the last column of Table 1.

□

Remark. Using the Hadamard product we can decompose the rank-two local systems of the hypergeometric function in Equation (5.12) into two rank-one systems as follows:

$$ {}_2F_1\left(\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\middle| t\right) = {}_1F_0(\mu|\,t) \star {}_1F_0(1-\mu|\,t) \; . \tag{5.14}$$

This is the kind of decomposition used in [15]. Instead we are using the decomposition formula (5.9) that allows to build all families in Lemma 5.3 from a *single* geometric object, the family in Equation (4.1).

(i,j,α)	μ	singular fibers	r
$(1,1,1)$	$\frac{1}{2}$	I_4, I_1, I_1^*	$\frac{1}{4}(\lvert t\rvert+1)$
$(2,1,1)$	$\frac{1}{3}$	I_3, I_1, IV^*	$\frac{2}{3(1+3\sqrt{3})}(\lvert t\rvert+3\sqrt{3})$
$(1,1,\frac{1}{2})$	$\frac{1}{4}$	I_2, I_1, III^*	$\frac{1}{4}(\lvert t\rvert+1)$
$(2,1,\frac{1}{2})$	$\frac{1}{6}$	I_1, I_1, II^*	$\frac{2}{3(1+3\sqrt{3})}(\lvert t\rvert+3\sqrt{3})$

Table 1. Relation between twist parameters

5.3. **Weierstrass normal forms.** We describe the fibrations (5.7) by exhibiting their Weierstrass normal form (by moving one Weierstrass point to infinity if necessary)

$$ Y^2 = 4\,X^3 - g_2(t)\,X - g_3(t)\;, \tag{5.15}$$

where g_2 and g_3 are polynomials of degree at most 4 and 6, respectively, in the affine coordinate t with $[t:1]\in\mathbb{P}^1$. We denote the discriminant by $\Delta = g_2^3 - 27\,g_3^2$ and the functional invariant by $J = g_2^3/\Delta$. All Weierstrass equation are given in Table 3. In Table 3, g_2, g_3, Δ, J are the Weierstrass coefficients, discriminant, and J-function; the ramification points of J and the Kodaira-types of the fibers over the ramification points are given, as well as the sections that generate the Mordell-Weil group and the monodromy group in $\mathrm{PSL}(2,\mathbb{Z})$. For the families of Weierstrass models in Equation (5.15) we will use dX/Y as the holomorphic one-form on each regular fiber. It is well-known (cf. [66]) that the Picard-Fuchs equation for the Weierstrass elliptic surface in Equation (5.15) is given by the Fuchsian system

$$ \frac{d}{dt}\left(\begin{array}{c}\omega_1\\ \eta_1\end{array}\right) = \left(\begin{array}{cc} -\frac{1}{12}\frac{d\ln\Delta}{dt} & \frac{3\,\delta}{2\,\Delta}\\ -\frac{g_2\,\delta}{8\,\Delta} & \frac{1}{12}\frac{d\ln\Delta}{dt}\end{array}\right)\cdot\left(\begin{array}{c}\omega_1\\ \eta_1\end{array}\right)\;, \tag{5.16}$$

where $\omega_1 = \oint_{\Sigma_1'}\frac{dX}{Y}$ and $\eta_1 = \oint_{\Sigma_1'}\frac{X\,dX}{Y}$ for each one-cycle Σ_1' and with $\delta = 3\,g_3\,g_2' - 2\,g_2\,g_3'$. We have the following lemma:

Lemma 5.7. *For $(n,\mu)\in\left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$, the Weierstrass normal forms of the families of elliptic curves given in Table 3 are extremal Jacobian rational elliptic surfaces with a fiber of Kodaira-type I_n over $t=0$. Moreover, for the cycle Σ_1' in (X,Y) the period integral ω_1 in Equation (5.16) coincides with $\omega_1^{(\mu)}$ in Equation (5.12).*

Proof. From the degrees of g_2 and g_3 it follows that the total space of the elliptic fibration is rational. From the rank of the Kodaira-type fibers is follows that the families of elliptic curves

(n,μ)	(i,j,α)	transformation	
$\left(4,\frac{1}{2}\right)$	$(1,1,1)$	$x = -\frac{3t}{3X+t+4}$	$y = \frac{9\,i\,t\,Y}{4\,(3X+t+4)^2}$
$\left(3,\frac{1}{3}\right)$	$(2,1,1)$	$x = \frac{4t-6X-9}{6X+9}$	$y = \frac{4\sqrt{2}\,i\,t\,Y}{9\,(2X+3)^2}$
$\left(2,\frac{1}{4}\right)$	$\left(1,1,\frac{1}{2}\right)$	$x = \frac{1}{2}X - \frac{1}{3}$	$y = \frac{\sqrt{2}}{8}Y$
$\left(1,\frac{1}{6}\right)$	$\left(2,1,\frac{1}{2}\right)$	$x = \frac{2}{3}X - \frac{1}{3}$	$y = \frac{\sqrt{6}}{9}Y$

TABLE 2. Extremal rational fibrations via twist

are extremal. The torsion groups can be read off from their classification of Miranda and Persson [57]. The remainder of this first statement then follows from explicit computation. Moreover, it is easily checked that Equation (5.16) implies that the period ω is annihilated by the operator $L_2^{(\mu)}$ in Equation (5.13). The affine coordinate t on the base in Equation (5.15) is chosen in such a way that the singular fiber of type I_n is located over $t = 0$. For $n > 0$, the local monodromy matrix then specifies an A-cycle in the elliptic fiber that is transformed into itself as one closely encircles the singular fiber at $t = 0$. It follows that in a unit disc about $t = 0$ the period integral over this A-cycle is the unique holomorphic solution ${}_2F_1(\mu, 1-\mu; 1|\, t)$. The conversion of the twist models in Equation (5.7) into Weierstrass normal forms is given by Table 2. □

Remark. The points of maximal unipotent monodromy in the base curve of an elliptic surface is the support of the semi-stable elliptic fibers, i.e., these are the points over which there is a singular fiber of type I_n or I_n^* for $n \geq 1$ [28, Cor. 1].

Remark. The names of the Jacobian elliptic surfaces in Table 3 coincide with the one used by Herfurtner [44]. We will often use μ or n to identify the surface.

Let us denote by $X_0(n)$ the genus-zero modular curves for the standard subgroups $\Gamma_0(n)$ if $n = 2, 3, 4$ and the unique normal subgroup of index 2 in $\mathrm{PSL}(2,\mathbb{Z})$ (which we will still denote by $\Gamma_0(1)$) if $n = 1$. That is, $X_0(n)$ is the compactification of $\mathbb{H}/\Gamma_0(n)$ and the moduli space of elliptic curves together with cyclic n-isogeny. It is well-known that the Weierstrass models described in Lemma 5.7 are the elliptic modular surfaces over genus-zero modular curves $X_0(n)$ for $n = 1, 2, 3, 4$. The following proposition summarizes the main result of the iterative construction at Step 1:

Proposition 5.8. *The iterative construction at Step 1 constructs the elliptic modular surfaces over the genus-zero modular curves $X_0(n)$ for $n = 1, 2, 3, 4$ from the family given in Equation (4.1).*

Remark. The family parameter t is the corresponding Hauptmodul of the modular curve and generates the function field The Fricke involution acts by $\tau \mapsto -1/(n\tau)$ on the elliptic τ-parameter, and exchanges the cusps at $t = 0$ and $t = 1$ and fixes the elliptic point (or cusp if $n = 4$) at $t = \infty$. Thus, the Fricke involution acts by $t \mapsto 1 - t$.

name	G / MW(π)	g_2, g_3, Δ, J / sections	ramification of J and singular fibers: t	J	$m(J)$	fiber
X_{141}	$\Gamma_0(4)$	$g_2 = \frac{1}{3}\left(4t^2 - 64\,t + 64\right)$	$8 \pm 4\sqrt{3}$	0	3	smooth
$\mu = \frac{1}{2}$		$g_3 = \frac{8}{27}(2-t)(32-32t-t^2)$	$-16 \pm 12\sqrt{2}, 2$	1	2	smooth
		$\Delta = -256\,t^4\,(t-1)$	0	∞	4	I_4 (A_3)
		$J = -\frac{(t^2-16t+16)^3}{108t^4(t-1)}$	1	∞	1	I_1
	$\mathbb{Z}/4\mathbb{Z}$	$(X,Y)_1 = (\frac{2}{3}t - \frac{4}{3}, 0)$	∞	∞	1	I_1^* (D_5)
		$(X,Y)_{2,3} = (-\frac{1}{3}t - \frac{4}{3}, \pm 4\,i\,t)$				
X_{431}	$\Gamma_0(3)$	$g_2 = 27 - 24\,t$	∞	0	1	IV^* (E_6)
$\mu = \frac{1}{3}$		$g_3 = 27 - 36\,t + 8\,t^2$	$\frac{9}{8}$	0	3	smooth
		$\Delta = -1728\,t^3\,(t-1)$	$\frac{9}{4} \pm \frac{3\sqrt{3}}{4}$	1	2	smooth
		$J = \frac{(-9+8\,t)^3}{64\,t^3(t-1)}$	0	∞	3	I_3 (A_2)
	$\mathbb{Z}/3\mathbb{Z}$	$(X,Y)_{1,2} = (-\frac{3}{2}, \pm 2\sqrt{2}\,i\,t)$	1	∞	1	I_1
X_{321}	$\Gamma_0(2)$	$g_2 = \frac{16}{3} - 4\,t$	$\frac{4}{3}$	0	3	smooth
$\mu = \frac{1}{4}$		$g_3 = -\frac{64}{27} + \frac{8}{3}\,t$	$\frac{8}{9}$	1	2	smooth
		$\Delta = -64\,t^2\,(t-1)$	∞	1	1	III^* (E_7)
		$J = \frac{(-4+3t)^3}{27\,t^2(t-1)}$	0	∞	2	I_2 (A_1)
	$\mathbb{Z}/2\mathbb{Z}$	$(X,Y) = (\frac{2}{3}, 0)$	1	∞	1	I_1
X_{211}	$\Gamma_0(1)$	$g_2 = 3$	∞	0	2	II^* (E_8)
$\mu = \frac{1}{6}$		$g_3 = -1 + 2\,t$	$\frac{1}{2}$	1	2	smooth
		$\Delta = -108\,t\,(t-1)$	0	∞	1	I_1
	$\{\mathbb{I}\}$	$J = -\frac{1}{4\,t\,(t-1)}$	1	∞	1	I_1

TABLE 3. Extremal rational fibrations

5.4. **Rational parameter changes.** We can also apply transformations to the rational deformation spaces of the family of elliptic curves. In particular, if we apply a linear transformation to the parameter $t \mapsto \frac{t}{t-1}$ in the families of elliptic curves in Lemma 5.7 and combine it with a twist, the transformation

$$
\begin{aligned}
t &\mapsto \frac{t}{t-1}\,, \\
g_2(t) &\mapsto \tilde{g}_2(t) = g_2\left(\frac{t}{t-1}\right)(1-t)^4\,, \\
g_3(t) &\mapsto \tilde{g}_3(t) = g_3\left(\frac{t}{t-1}\right)(1-t)^6
\end{aligned}
\tag{5.17}
$$

is obtained. This transformation produces families of elliptic curves that will be denoted by $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, and $\tilde{X}_{211}$. They have the same number and type of singular fibers and the same Mordell-Weil groups. In comparison with the families X_{141}, X_{431}, X_{321}, and X_{211}, the singular fibers over $t = 1$ and $t = \infty$ have been interchanged. Since we will keep the position of the ramification points fixed when further applying our iterative procedure, these families will give rise to different Picard-Fuchs operators at subsequent steps. In Table 5 we have summarized the defining data for all rational Jacobian elliptic surfaces discussed so far. For $t \in D_1^*(0)$ let $\tilde{\Sigma}_1'$ be the family of closed one-cycles defined the same way Σ_1' was in the previous section, but with t replaced by $\frac{t}{t-1}$ and (X,Y) replaced by $((1-t)^2\,X, (1-t)^3\,Y)$ in its definition to account for the rational transformation and the twist. The proof of Lemma 5.7 then carries over to the new families. Combined with a well-known identity for the Gauss hypergeometric function, we obtain the following lemma:

Lemma 5.9. *For $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2\left(\frac{t}{t-1}\right)(1-t)^4}_{:=\tilde{g}_2(t)}\,X - \underbrace{g_3\left(\frac{t}{t-1}\right)(1-t)^6}_{:=\tilde{g}_3(t)} \tag{5.18}$$

with polynomials $\tilde{g}_2, \tilde{g}_3$ given in Table 5 define families of elliptic curves with a fiber of Kodaira-type I_n over $t=0$. For $t \in D_1^*(0)$, the period integral of dX/Y over the family of one-cycles $\tilde{\Sigma}_1'$ is given by

$$\tilde{\omega}_1^{(\mu)} = \frac{2\pi i}{1-t}\;{}_2F_1\!\left(\left.\begin{array}{c}\mu,1-\mu\\1\end{array}\right|\frac{t}{t-1}\right) = \frac{2\pi i}{(1-t)^{1-\mu}}\;{}_2F_1\!\left(\left.\begin{array}{c}\mu,\mu\\1\end{array}\right|\frac{t}{t-1}\right). \tag{5.19}$$

The holomorphic period is annihilated by the second-order and degree-two Picard-Fuchs operator

$$\tilde{L}_2^{(\mu)} = \theta^2 - t\,(2\,\theta^2 + 2\,\theta + \mu^2 - \mu + 1) + t^2\,(\theta+1)^2\,. \tag{5.20}$$

□

The torsion sections of the families in Equation (5.18) are listed in Table 4.

name	torsion	sections
$\tilde{X}_{141}$	[4]	$(X,Y)_1 = \left(-\frac{2}{3}\,(t-1)(-2+t)\,,0\right)$ $(X,Y)_{2,3} = \left(-\frac{1}{3}\,(t-1)(5\,t-4)\,,\pm 4\,i\,(t-1)^2\,t\right)$
$\tilde{X}_{431}$	[3]	$(X,Y)_{1,2} = \left(-\frac{3}{2}\,(t-1)^2\,,\pm 2\,i\,(t-1)^2\,t\,\sqrt{2}\right)$
$\tilde{X}_{321}$	[2]	$(X,Y) = \left(\frac{2}{3}\,(t-1)^2\,,0\right)$
$\tilde{X}_{211}$	[1]	–

TABLE 4. Torsion sections of Equation (5.18)

$\dim_{\mathbb{C}}$	geometry	g_2 or $\tilde{g}_2$	g_3 or $\tilde{g}_3$	Δ	name	μ
2	rational rk(MW) = 0	$\frac{4}{3}\,t^2 - \frac{64}{3}\,t + \frac{64}{3}$	$\frac{8}{27}\,(2-t)\left(-t^2-32\,t+32\right)$	$-256\,t^4\,(t-1)$	X_{141}	$\frac{1}{2}$
2	rational rk(MW) = 0	$\frac{4}{3}\left(t^2-16\,t+16\right)(t-1)^2$	$-\frac{8}{27}\,(-2+t)\left(t^2+32\,t-32\right)(t-1)^3$	$-256\,t^4\,(t-1)^7$	$\tilde{X}_{141}$	$\frac{1}{2}$
2	rational rk(MW) = 0	$-24\,t+27$	$8\,t^2-36\,t+27$	$-1728\,t^3\,(t-1)$	X_{431}	$\frac{1}{3}$
2	rational rk(MW) = 0	$3\,(t-1)^3\,(t-9)$	$-\left(t^2+18\,t-27\right)(t-1)^4$	$-1728\,t^3\,(t-1)^8$	$\tilde{X}_{431}$	$\frac{1}{3}$
2	rational rk(MW) = 0	$\frac{16}{3}-4\,t$	$-\frac{64}{27}+\frac{8}{3}\,t$	$-64\,t^2\,(t-1)$	X_{321}	$\frac{1}{4}$
2	rational rk(MW) = 0	$\frac{4}{3}\,(t-1)^3\,(t-4)$	$\frac{8}{27}\,(t-1)^5\,(t+8)$	$-64\,t^2\,(t-1)^9$	$\tilde{X}_{321}$	$\frac{1}{4}$
2	rational rk(MW) = 0	3	$2\,t-1$	$-108\,t\,(t-1)$	X_{221}	$\frac{1}{6}$
2	rational rk(MW) = 0	$3\,(t-1)^4$	$(t-1)^5\,(t+1)$	$-108\,t\,(t-1)^{10}$	$\tilde{X}_{221}$	$\frac{1}{6}$

TABLE 5. Weierstrass coefficients of rational elliptic surfaces

6. Step 2: From families of elliptic curves to families of K3 surfaces

In this section we will describe the second step in our iterative construction that produces families of Jacobian elliptic K3 surfaces of Picard rank 19 and 18. All families of Jacobian elliptic K3 surfaces will be obtained from the extremal Jacobian rational elliptic surfaces in Proposition 5.8 by applying a quadratic twist or a quadratic twist followed by a rational transformation of the base. We call the former construction the pure-twist construction and the latter the mixed-twist construction at Step 2. We will also compute the Picard-Fuchs operators and holomorphic periods for all families by application of the Euler integral transform for generalized hypergeometric functions. In particular, the iterative construction at Step 2 will produce families of Jacobian elliptic K3 surfaces with M_n-polarization $n = 1, 2, 3, 4$ and M polarization.

Every family of Jacobian elliptic K3 surfaces over $\mathbb{P}^1$ will be presented by exhibiting its Weierstrass normal form

$$Y^2 = 4\,X^3 - G_2(t,u)\,X - G_3(t,u)\;, \tag{6.1}$$

where G_2 and G_3 are polynomials of degree at most 8 and 12, respectively, in the affine coordinate $u \in \mathbb{C}$. G_2 and G_3 will also be polynomials in the family parameter t.

6.1. K3 families of Picard rank 19.

The following lemmas describe the construction of K3 surfaces of Picard rank 19 from the rational elliptic surfaces constructed in Lemmas 5.7 and 5.9.

6.1.1. *Pure-twist construction.* We first describe the *pure-twist construction* of Jacobian elliptic K3 surfaces of Picard rank 19. To do so we start with the Jacobian rational elliptic surfaces defined by $g_2(t)$ and $g_3(t)$ in Lemma 5.7 with singular fibers of Kodaira-type I_n over $t = 0$, I_1 over $t = 1$ and II^*, III^*, IV^*, or I_1^* over $t = \infty$. It then follows:

Lemma 6.1. *For* $(n,\mu) \in \Big\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\Big\}$ *the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2(t\,u)\Big(u\,(u-1)\Big)^2}_{=:\,G_2(t,u)}\,X - \underbrace{g_3(t\,u)\Big(u\,(u-1)\Big)^3}_{=:\,G_3(t,u)} \tag{6.2}$$

with g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces with singular fibers of Kodaira-type I_n^* over $u = 0$ and of Kodaira-type I_0^* over $u = 1$, of Kodaira-type I_1 over $u = 1/t$ and one of type II^*, III^*, IV^*, or I_1^* over $u = \infty$. The families are families of L_n-lattice polarized K3 surfaces of Picard rank 19 with $L_n = H \oplus D_4 \oplus D_{4+n} \oplus R_{9-n}/G_n$ where (R_{9-n}, G_n) denotes $(E_8, 1)$, $(E_7, \mathbb{Z}_2)$, $(E_6, 1)$, and $(D_5, \mathbb{Z}_2)$ for $n = 1, \ldots, 4$.

Proof. The proof amounts to checking the Kodaira-types of singular fibers from G_2, G_3, Δ and comparing with the list of all Jacobian elliptic surfaces given in the Shimada classification [63] of Jacobian elliptic K3 surfaces. Moreover, it is easy to see that exactly the $\mathbb{Z}_2$-torsion survives the pure-twist construction. All torsion sections are listed in Table 6. □

Λ	torsion	sections
L_4	[2]	$(X,Y) \;=\; \left(\frac{2}{3}\,(ut-2)\,u\,(u-1)\,,0\right)$
L_3	[1]	–
L_2	[2]	$(X,Y) \;=\; \left(\frac{2}{3}\,u\,(u-1)\,,0\right)$
L_1	[1]	–

TABLE 6. Torsion sections of Equation (6.2)

Remark. Setting $u \mapsto t\,u$ and $(X,Y) \mapsto (X/t^2, Y/t^3)$ in Equation (6.6) we obtain the Weierstrass equation

$$Y^2 = 4\,X^3 - g_2(u)\Big(u\,(u-t)\Big)^2\,X - g_3(u)\Big(u\,(u-t)\Big)^3\;. \tag{6.3}$$

The four families in Equation (6.3) for $\mu \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$ were considered by Hoyt in [46].

The configuration of singular fibers, the Mordell-Weil group, the determinants of the discriminant groups, and lattice polarizations of the constructed families of K3 surfaces are summarized in Table 7.

derived from		ρ	Configuration of singular fibers				MW(π)	Discriminant Q	Λ
Srfc	μ		$u=0$	$u=1/t$	$u=\infty$	$u=1$			
X_{141}	$\frac{1}{2}$	19	$I_4^*\ (D_8)$	I_1	$I_1^*\ (D_5)$	$I_0^*\ (D_4)$	[2]	$2^4 \;=\; 4\cdot(2\cdot 2)^2/2^2$	L_4
X_{431}	$\frac{1}{3}$	19	$I_3^*\ (D_7)$	I_1	$IV^*\ (E_6)$	$I_0^*\ (D_4)$	[1]	$2^4\cdot 3 \;=\; 4\cdot 3\cdot(2\cdot 2)^2/1$	L_3
X_{321}	$\frac{1}{4}$	19	$I_2^*\ (D_6)$	I_1	$III^*\ (E_7)$	$I_0^*\ (D_4)$	[2]	$2^3 \;=\; 2\cdot(2\cdot 2)^2/2^2$	L_2
X_{211}	$\frac{1}{6}$	19	$I_1^*\ (D_5)$	I_1	$II^*\ (E_8)$	$I_0^*\ (D_4)$	[1]	$2^4 \;=\; 4\cdot(2\cdot 2)/1$	L_1

TABLE 7. K3 surfaces from extremal rational surfaces

Remark. The points of maximal unipotent monodromy in the base curve of an elliptic surface is the support of the semi-stable elliptic fibers, i.e., the points with a singular fiber of type I_n or I_n^* for $n \geq 1$ [28, Cor. 1].

6.1.2. *Mixed-twist construction.* We now describe the *mixed-twist construction* of Jacobian elliptic K3 surfaces of Picard rank 19. We again start with the rational elliptic surfaces defined by $g_2(t)$ and $g_3(t)$ in Lemma 5.7 with singular fibers I_n at $t=0$, I_1 at $t=1$ and II^*, III^*, IV^*, or I_1^* at $t=\infty$. We then apply the rational map $u \mapsto -\frac{1}{4u(u+1)}$ of degree 2 from $\mathbb{P}^1 \to \mathbb{P}^1$. The map is totally ramified over 0 and 1, and is ramified to degrees 1 and 1 over ∞. Following Section 5, we call this construction the mixed-twist construction with generalized functional invariant $(1,1,1)$. It then follows:

Lemma 6.2. *For* $(n,\mu) \in \left\{(1,\frac{1}{6}), (2,\frac{1}{4}), (3,\frac{1}{3}), (4,\frac{1}{2})\right\}$ *the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2\left(-\frac{t}{4u(u+1)}\right)\Big(u\,(u+1)\Big)^4}_{=:\,G_2(t,u)}\,X - \underbrace{g_3\left(-\frac{t}{4u(u+1)}\right)\Big(u\,(u+1)\Big)^6}_{=:\,G_3(t,u)} \tag{6.4}$$

with g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces of Picard rank 19 with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u=0$ and $u=-1$, a fiber of

Kodaira-type I_{2n} over $u = \infty$, and two singular fibers of type I_1 over $u = -1/2 \pm \sqrt{1-t}/2$. The families are families of M_n-lattice polarized K3 surfaces with $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ for $n = 1, \ldots, 4$.

Proof. The proof amounts to checking the Kodaira-types of singular fibers from G_2, G_3, Δ and comparing with the list of all Jacobian elliptic surfaces given in the Shimada classification [63] of Jacobian elliptic K3 surfaces. The fact that the constructed families of K3 surfaces are M_n-polarized was explained by Dolgachev in [27]. Moreover, it is easy to see that all torsion sections of the rational elliptic surfaces survive the mixed-twist construction. The torsion sections are listed in Table 8. □

Λ	torsion	sections
M_4	[4]	$(X,Y)_1 = \left(-\frac{1}{6}(u+1)\,u\left(8u^2+t+8u\right), 0\right)$ $(X,Y)_{2,3} = \left(\frac{1}{12}(u+1)\,u\left(-16u^2+t-16u\right), \pm itu^2(u+1)^2\right)$
M_3	[3]	$(X,Y)_{1,2} = \left(-\frac{3}{2}u^2(u+1)^2, \pm\frac{i}{2}\sqrt{2}tu^2(u+1)^2\right)$
M_2	[2]	$(X,Y) = \left(\frac{2}{3}u^2(u+1)^2, 0\right)$
M_1	[1]	–

TABLE 8. Torsions sections of Equation (6.4)

Remark. A rational transformation relates the families obtained in Lemma 6.1 and 6.2. Applying the transformation

$$u \mapsto -\frac{1}{4u(u+1)}\,, \qquad (X,Y) \mapsto \left(\frac{X\,(2u+1)^2}{16\,(u(u+1))^4}, \frac{Y\,(2u+1)^3}{64\,(u(u+1))^6}\right) \tag{6.5}$$

in Equation (6.2) yields Equation (6.4). The transformation (6.5) also maps the holomorphic two-form $du \wedge dX/Y$ of each family to $du \wedge dX/Y$ of the respective other family.

The configurations of singular fibers, the Mordell-Weil groups, the determinants of the discriminant groups, and the lattice polarizations are summarized in Table 9.

derived from		ρ	Configuration of singular fibers			MW(π)	Discriminant Q	Λ
Srfc	μ		$u=\infty$	$u^2+u+t/4$	$u=0,-1$			
X_{141}	$\frac{1}{2}$	19	$I_8\ (A_7)$	$2\,I_1$	$2\,I_1^*\ (2\,D_5)$	[4]	$2^3 = 8\cdot 4^2/4^2$	M_4
X_{431}	$\frac{1}{3}$	19	$I_6\ (A_5)$	$2\,I_1$	$2\,IV^*\ (2\,E_6)$	[3]	$2\cdot 3 = 6\cdot 3^2/3^2$	M_3
X_{321}	$\frac{1}{4}$	19	$I_4\ (A_3)$	$2\,I_1$	$2\,III^*\ (2\,E_7)$	[2]	$2^2 = 4\cdot 2^2/2^2$	M_2
X_{211}	$\frac{1}{6}$	19	$I_2\ (A_1)$	$2\,I_1$	$2\,II^*\ (2\,E_8)$	[1]	$2 = 2/1$	M_1

TABLE 9. K3 surfaces from extremal rational surfaces

Let us denote by $\Gamma_0(n)^+$ the modular group $\Gamma_0(n)$ extended by the Fricke involution, i.e., the element corresponding to $\tau \mapsto -1/(n\tau)$. It is a classical result due to Dolgachev that the moduli spaces of pseudo-ample M_n-polarized K3 surfaces are isomorphic to the rational modular curves that are the compactification of the curves $\mathbb{H}/\Gamma_0(n)^+$ [27]. We therefore have the following corollary summarizing the results of the mixed-twist construction at Step 2 analogous to Proposition 5.8:

Corollary 6.3. *The iterative construction with generalized functional invariant* $(i, j, \alpha) = (1, 1, 1)$ *at Step 2 constructs families of Jacobian elliptic* M_n*-polarized K3 surfaces over the rational modular curves* $\mathbb{H}/\Gamma_0(n)^+$ *for* $n = 1, 2, 3, 4$.

Proof. The proof follows directly by checking that the singular fibers and Mordell-Weil groups for the families constructed in Lemma 6.2 agree with ones given by Dolgachev in [27]. □

6.2. **Period computation for families of K3 surfaces of Picard rank** 19**.** In this section, we construct the Picard-Fuchs operators for all families of K3 surfaces obtained in the previous section. Using Pochhammer contours we also construct families of two-cycles for all families of K3 surfaces that upon integration with the holomorphic two-from yield periods that are holomorphic near the point of maximal unipotent monodromy.

6.2.1. *Pure-twist construction.* We will first describe the computation of the holomorphic period for the Jacobian elliptic K3 surfaces of Picard rank 19 constructed in Section 6.1.1.

In the previous section, we constructed the pure-twist family of K3 surfaces of Picard rank 19 given by

$$Y^2 = 4\,X^3 - g_2(t\,u)\Big(u\,(u-1)\Big)^2 X - g_3(t\,u)\Big(u\,(u-1)\Big)^3 \tag{6.6}$$

with singular fibers of Kodaira-type I_n^* over $u = 0$ and of Kodaira-type I_0^* over $u = 1$, of Kodaira-type I_1 at $u = 1/t$ and one of of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u = \infty$. Since the singular fiber is of type I_n^* over $u = 0$ and of type I_0^* over $u = 1$ there is an A-cycle that is transformed into itself when encircling the singular fibers over $u = 0$ and $u = 1$.

We define a family of closed two-cycles Σ_2 in the total space defined by Equation (6.6) as follows: for each $t \in D_1^*(0)$ we follow a Pochhammer contour $\Sigma_1 = \Sigma_1(t)$ around $u = 0$ and $u = 1$ in the u-plane such that $0 < |ut| < 1$. For each $u \in \Sigma_1(t)$, we trace out the cycle in the (X, Y)-coordinates of the elliptic fiber that is obtained from the A-cycle $\Sigma_1'(ut)$ defined in Lemma (5.4) after rescaling $(X, Y) \to (u\,(u-1)\,X, (u\,(u-1))^{3/2}\,Y)$ to account for the quadratic twist. Therefore, for $t \in D_1^*(0)$ we obtain a continuously varying family of closed two-cycles that we denote as warped product $\Sigma_2 = \Sigma_1 \times_u \Sigma_1'$.

A typical example of such a Pochhammer contour Σ_1 is depicted in Figure 1. The locations of the singular fibers of Kodaira-type I_n^* over $u = 0$ and of Kodaira-type I_0^* over $u = 1$ are shown as black dots in Figure 1, and the singular fiber of Kodaira-type I_1 over $u = 1/t$ is shown as red dot.

We now evaluate the period integral for the holomorphic two $du \wedge dX/Y$ over the two-cycle Σ_2, i.e.,

$$\omega_2 = \frac{1}{2} \oiint_{\Sigma_2} du \wedge \frac{dX}{Y}\;. \tag{6.7}$$

We then have the following lemma:

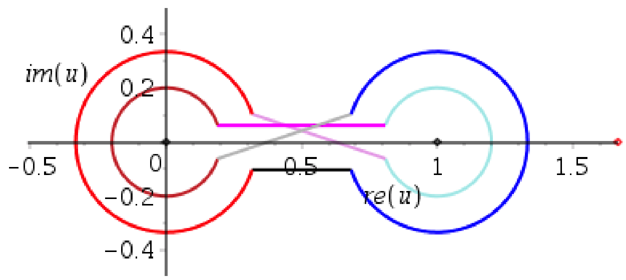


Figure 1. Pochhammer contour in u-plane

Lemma 6.4. *For $t \in D_1^*(0)$ the period of the holomorphic two-form $du/2 \wedge dX/Y$ over the two-cycle Σ_2 for the families of K3 surfaces in Lemma 6.1 is given by*

$$\omega_2^{(\mu)} = (2\pi i)^2 \ {}_3F_2\left(\left.\begin{matrix}\mu, \frac{1}{2}, 1-\mu \\ 1, 1\end{matrix}\right| t\right) . \tag{6.8}$$

Proof. To integrate the holomorphic two-form over the two-cycle Σ_2 we use Fubini's theorem. Integrating over the A-cycle first, we obtain

$$\frac{1}{2\pi i} \oiint_{\Sigma_2} du \wedge \frac{dX}{Y} = i \oint_{\Sigma_1} \frac{du}{\sqrt{u\,(1-u)}} \ {}_2F_1\left(\left.\begin{matrix}\mu, 1-\mu \\ 1\end{matrix}\right| ut\right) . \tag{6.9}$$

Recalling that $|ut| < 1$ for all $u \in \Sigma_1$, the series expansion of the hypergeometric function converges absolutely and uniformly. For $|t| < 1$ one then obtains for the period integral

$$\begin{aligned} & i \sum_{n\geq 0} \frac{(\mu)_n (1-\mu)_n}{n!^2}\, t^n \oint_{\Sigma_1} du \;\; u^{(n+\frac{1}{2})-1}\,(1-u)^{\frac{1}{2}-1} \\ & = 4\, i\, \Gamma\left(\frac{1}{2}\right)^2 \sum_{n\geq 0} \frac{(\mu)_n (1-\mu)_n \left(\frac{1}{2}\right)_n}{n!^3}\, t^n \\ & = (4\pi i)\ {}_1F_0\left(\left.\frac{1}{2}\right| t\right) \star {}_2F_1\left(\left.\begin{matrix}\mu, 1-\mu \\ 1\end{matrix}\right| t\right) . \end{aligned} \tag{6.10}$$

□

Notice that the requirement of $|ut| < 1$ in the construction of the two-cycle Σ_2 is needed for uniform convergence of the hypergeometric function and also implies that the fiber of Kodaira-type I_1 at $u = 1/t$ is always located outside of the Pochhammer contour. A well-known result for hypergeometric functions then yields the following corollary.

Corollary 6.5. *The period $\omega_2^{(\mu)}$ in Equation (6.8) is the solution of the differential equation $L_3^{(\mu)}\, \omega_2^{(\mu)} = 0$ that is holomorphic near $t = 0$ with*

$$L_3^{(\mu)} = \theta^3 - t\left(\theta + \frac{1}{2}\right)\left(\theta + \mu\right)\left(\theta + 1 - \mu\right) . \tag{6.11}$$

□

6.2.2. *Mixed-twist construction.* In this section, we explain how the period computation from Section 6.2.1 generalizes to the mixed-twist construction.

In the previous section, we applied the rational map $u \mapsto -\frac{1}{4u(u+1)}$ of degree two to construct a second family of K3 surfaces of Picard rank 19 which we called the mixed-twist family. It was given by the equation

$$Y^2 = 4\,X^3 - g_2\left(-\frac{t}{4\,u(u+1)}\right)\Big(u\,(u+1)\Big)^4 X - g_3\left(-\frac{t}{4\,u(u+1)}\right)\Big(u\,(u+1)\Big)^6 \tag{6.12}$$

with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u = 0$ and $u = -1$, a fiber of Kodaira-type I_{2n} over $u = \infty$, and two singular fibers of Kodaira-type I_1 over $u_\pm = -1/2 \pm \sqrt{1-t}/2$.

We set $C = \mathbb{P}^1 \backslash \{u_-, -1, 0, u_+, \infty\}$ and denote the fundamental group of C based at any regular point $u_0 \in C$ by $\Gamma = \pi_1(C, u_0)$. Generators for Γ are suitable simple loops γ_u around points $u \in \{u_-, -1, 0, u_+, \infty\}$. We denote homotopy equivalence classes by [.] and the group multiplication of loops up to homotopy by $*$. We can then fix a homological invariant for the family of Jacobian elliptic K3 surfaces in Equation (6.12) by defining a locally constant sheaf over C whose generic stalk is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A monodromy representation

$$M : \Gamma \to \mathrm{SL}(2, \mathbb{Z}) \tag{6.13}$$

defines the transition functions for this sheaf. It is easy to see that there is a homological invariant such that the monodromy $M(\gamma'_{u_+,0})$ for a loop $\gamma'_{u_+,0}$ around the singular fibers at $u = 0$ and $u = u_+$ equals the monodromy $M(\gamma'_{-1,u_-})$ for a loop γ'_{-1,u_-} around the singular fibers at $u = -1$ and $u = u_-$, i.e., $M(\gamma'_{u_+,0}) = M(\gamma'_{-1,u_-}) =: M$. Because of the singular fiber of Kodaira-type I_{2n} at $u = \infty$, it is always possible to find a solution for the homological invariant satisfying

$$\underbrace{\left(\begin{array}{cc} 1 & 2n \\ 0 & 1 \end{array}\right)}_{=M(\gamma'_\infty)} M^2 = \mathbb{I}\,,$$

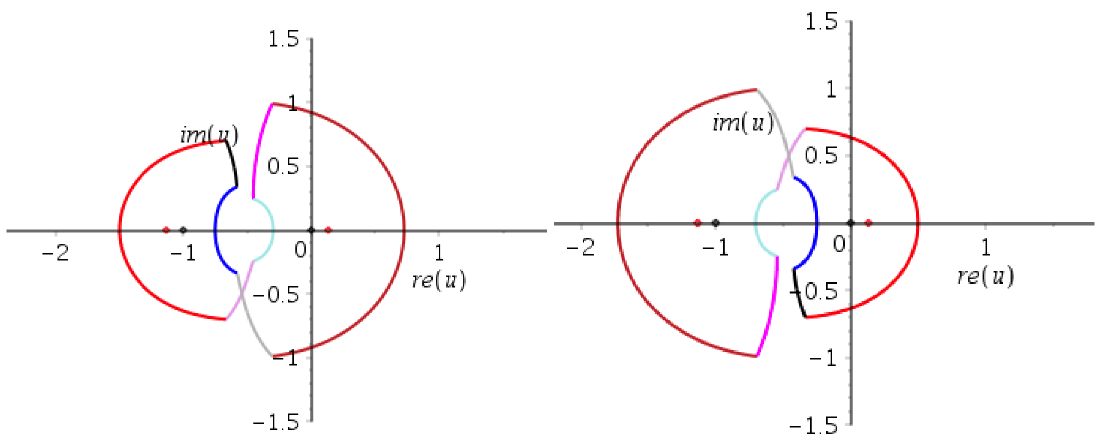
namely, the matrices

$$M(\gamma'_\infty) = \left(\begin{array}{cc} 1 & 2n \\ 0 & 1 \end{array}\right), \qquad M(\gamma'_{u_+,0}) = M(\gamma'_{-1,u_-}) = \left(\begin{array}{cc} 1 & -n \\ 0 & 1 \end{array}\right). \tag{6.14}$$

The matrices (6.14) fix the homological invariant for the family in Equation (6.12). Since the matrices in Equation (6.14) determine a homological invariant – not just the conjugacy classes for the monodromy matrices – for the families in Equation (6.12), there is an A-cycle in the elliptic fiber that is transformed into itself under the action of the loops $\gamma'_{u_+,0}$ and γ'_{-1,u_-}, and any small loop around $u = \infty$.

We define a family of closed two-cycles Σ''_2 for the families of Jacobian elliptic K3 surfaces in Equation (6.12) as follows: for each $t \in D_1^*(0)$, we follow a loop $\Sigma''_1 = \Sigma''_1(t)$ in the affine u-plane that covers the Pochhammer contour Σ_1 from Section 6.2.1 under the map $u \mapsto -\frac{1}{4u(u+1)}$ such that $0 < |t| < 4\,|u(u+1)|$. Notice that there are two choices for such a cycle Σ''_1 both of them being homologous to the class of the figure-eight cycle $[\gamma'_{u_+,0} * \gamma'^{-1}_{-1,u_-}]$. For each $u \in \Sigma''_1(t)$, we trace out the cycle in the (X, Y)-coordinates of the elliptic fiber that is obtained from the A-cycle

$\Sigma'_1(-t/(4u(u+1)))$ defined in Lemma (5.4) after rescaling $(X,Y) \to (u^2\,(u+1)^2\,X, u\,(u+1)\,Y)$ to account for the twist. For $t \in D_1^*(0)$ we obtain a continuously varying family of closed two-cycles that we denote as warped product $\Sigma''_2 = \Sigma''_1 \times_u \Sigma'_1$. The two choices for the path Σ''_1 in



(A) First pre-image

(B) Second pre-image

Figure 2. Pre-images of Pochhammer contour

the u-plane that cover the example of a Pochhammer contour Σ_1 from Figure 1 are depicted in Figure 2a and Figure 2b. The black dots indicate the positions of the fibers of type II^*, III^*, IV^*, or I_1^*, and the red dots the positions of the fibers of type I_1. We then have the following lemma:

Lemma 6.6. *For $t \in D_1^*(0)$ the period of the holomorphic two-form $du/2 \wedge dX/Y$ over the two-cycle Σ''_2 for the families of K3 surfaces in Lemma 6.2 is given by*

$$\omega_2^{(\mu)} = (2\pi i)^2\ {}_3F_2\!\left(\left.\begin{matrix}\mu, \frac{1}{2}, 1-\mu\\ 1, 1\end{matrix}\right| t\right) . \tag{6.15}$$

Proof. The period integral has a holomorphic integrand for all $u \neq 0, -1$. This means that the contour of the period integral can be deformed into an integral over $\gamma'_{u_+,0}$ and γ'^{-1}_{-1,u_-}. We decompose the period integral as follows

$$\begin{aligned}\frac{1}{2\pi i}\oiint_{\Sigma''_2} du \wedge \frac{dX}{Y} &= \oint_{[\gamma'_{u_+,0}\gamma'^{-1}_{-1,u_-}]} \frac{du}{u\,(u+1)}\ {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| -\frac{t}{4\,u(u+1)}\right)\\ &= \left(\oint_{[\gamma'_{u_+,0}]} - \oint_{[\gamma'_{-1,u_-}]}\right)\frac{du}{u\,(u+1)}\ {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| -\frac{t}{4\,u(u+1)}\right) .\end{aligned} \tag{6.16}$$

The loops $\gamma'_{u_+,0}$ and γ'_{-1,u_-} are homologous to circles $C_r(0)$ given by $|u| = r$ and $C_r(-1)$ given by $|u+1| = r$, respectively, such that $|u_+| < r$ and $|u_- + 1| < r$. It is then sufficient to choose $r = (1+|t|)/4$ to ensure $|t| < 4\,|u(u+1)|$, $|u_+| < r < 1/2$, and $|u_- + 1| < r$ for all $u \in C_r(0), C_r(-1)$ and $t \in D_1^*(0)$. Using the variable transformation $-\tilde{u} = u + 1$ and $u = -(\tilde{u}+1)$ such that $\tilde{u}_\pm = u_\mp$, we obtain

$$\begin{aligned}(6.17)\quad &\left(\oint_{C_r(0)} - \oint_{C_r(-1)}\right) \frac{du}{u\,(u+1)}\; {}_2F_1\!\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| -\frac{t}{4\,u(u+1)}\right)\\ &= \oint_{C_r(0)} \frac{du}{u\,(u+1)}\; {}_2F_1\!\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| -\frac{t}{4\,u(u+1)}\right)\\ &\quad + \oint_{\tilde{C}_r(0)} \frac{d\tilde{u}}{\tilde{u}\,(\tilde{u}+1)}\; {}_2F_1\!\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| -\frac{t}{4\,\tilde{u}(\tilde{u}+1)}\right)\\ &= 2\oint_{C_r(0)} \frac{du}{u\,(u+1)}\; {}_2F_1\!\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| -\frac{t}{4\,u(u+1)}\right).\end{aligned}$$

Recalling that $|t| < 4\,|u(u+1)|$, the series expansion of the hypergeometric function converges absolutely and uniformly. We obtain

$$\begin{aligned}(6.18)\quad & 2\oint_{C_r(0)} \frac{du}{u\,(u+1)}\; {}_2F_1\!\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| -\frac{t}{4\,u(u+1)}\right)\\ &= 2\sum_{n\geq 0} \frac{(\mu)_n(1-\mu)_n}{n!^2}\left(\frac{-t}{4}\right)^n \sum_{k\geq 0}\frac{(n+k)!}{n!\,k!}\oint_{|u|=\epsilon}\frac{du}{u^{n+1}}\,(-u)^k\,.\end{aligned}$$

Using $(2n)! = (\frac{1}{2})_n\, n!\, 2^{2n}$ and $|t| < 1$, we can complete the period computation by using

$$(6.19)\qquad 4\pi i \sum_{n\geq 0} \frac{(\mu)_n(1-\mu)_n(2n)!}{n!^4}\left(\frac{t}{4}\right)^n = (4\pi i)\; {}_3F_2\!\left(\left.\begin{array}{c}\mu, \frac{1}{2}, 1-\mu\\ 1, 1\end{array}\right| t\right).$$

□

Remark. By Clausen's identity, we can write each holomorphic period $\omega_2^{(\mu)}$ as the following perfect square

$$(6.20)\qquad {}_3F_2\!\left(\left.\begin{array}{c}\mu, \frac{1}{2}, 1-\mu\\ 1, 1\end{array}\right| t\right) = \left[{}_2F_1\!\left(\left.\begin{array}{c}\frac{\mu}{2}, \frac{1-\mu}{2}\\ 1\end{array}\right| t\right)\right]^2.$$

The ratios of solutions of the hypergeometric differential equation with holomorphic solution ${}_2F_1(\frac{\mu}{2}, \frac{1-\mu}{2}; 1\big|\, t)$ are so-called Schwarzian s-maps for so-called triangle groups. The triangle groups that arise the different μ-values are listed in Table 10 and are precisely the modular groups $\Gamma_0(n)^+$ that arise in the construction of the moduli spaces of M_n-polarized K3 surfaces in Corollary 6.3.

n	μ	triangle group
n	μ	$(2, \frac{2}{1-2\mu}, \infty)$
1	$\frac{1}{6}$	$(2, 3, \infty)$
2	$\frac{1}{4}$	$(2, 4, \infty)$
3	$\frac{1}{3}$	$(2, 6, \infty)$
4	$\frac{1}{2}$	$(2, \infty, \infty)$

TABLE 10. Triangle groups

The Kummer identity relates the hypergeometric function in the symmetric square back to the original period of the fiber, i.e., if t equals the uniformizing variable $t = 4\,T\,(1-T)$ that is invariant under $T \mapsto 1-T$, it follows

$$ {}_2F_1\left(\left. \begin{matrix} \frac{\mu}{2}, \frac{1-\mu}{2} \\ 1 \end{matrix} \right| t \right) = {}_2F_1\left(\left. \begin{matrix} \mu, 1-\mu \\ 1 \end{matrix} \right| T \right) . \tag{6.21} $$

The geometric origin of Equation (6.20) combined with Equation (6.21) is the fact that an M_n-polarized K3 surface admits a Shioda-Inose structure relating it to a Kummer surface associated to products of elliptic curves related by isogeny. The study of threefolds fibered by Kummer surfaces associated to products of elliptic curves was conducted in [30].

6.2.3. *Rational parameter changes.* For each of the one-parameter families of M_n-polarized K3 surfaces in Lemma 6.2 with holomorphic period

$$ (2\pi i)^2\ {}_3F_2\left(\left. \begin{matrix} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{matrix} \right| t \right) \tag{6.22} $$

for $(n,\mu) \in \left\{ (1,\frac{1}{6}), (2,\frac{1}{4}), (3,\frac{1}{3}), (4,\frac{1}{2}) \right\}$, we will construct five new families of one-parameter families of K3 surfaces by employing a linear or quadratic transformation to their rational parameter space. By construction, these new families will remain families of M_n-polarized K3 surfaces.

The first series of families is obtained from the M_n-polarized families in Equation (6.4) by the transformation

$$ \begin{aligned} t &\mapsto t\ , \\ G_2(t,u) &\mapsto G_2^{(1)}(t,u) = G_2\,(t,u)\ (1-t)^2\ , \\ G_3(t,u) &\mapsto G_3^{(1)}(t,u) = G_3\,(t,u)\ (1-t)^3\ . \end{aligned} \tag{6.23} $$

The second series of families is obtained from the M_n-polarized families in Equation (6.4) by the transformation

$$ \begin{aligned} t &\mapsto \frac{t}{t-1}\ , \\ G_2(t,u) &\mapsto G_2^{(2)}(t,u) = G_2\left(\frac{t}{t-1}, u\right)\ (1-t)^2\ , \\ G_3(t,u) &\mapsto G_3^{(2)}(t,u) = G_3\left(\frac{t}{t-1}, u\right)\ (1-t)^3\ . \end{aligned} \tag{6.24} $$

The third series of families is obtained from the M_n-polarized families in Equation (6.4) by the transformation

$$ \begin{aligned} t &\mapsto 4\,t\,(1-t)\ , \\ G_2(t,u) &\mapsto G_2^{(3)}(t,u) = G_2(4\,t\,(1-t), u)\ , \\ G_3(t,u) &\mapsto G_3^{(3)}(t,u) = G_3(4\,t\,(1-t), u)\ . \end{aligned} \tag{6.25} $$

The fourth series of families is obtained from the M_n-polarized families in Equation (6.4) by the transformation

$$
\begin{aligned}
t &\mapsto \frac{t^2}{4(t-1)}\;, \\
G_2(t,u) &\mapsto G_2^{(4)}(t,u) = G_2\left(\frac{t^2}{4(t-1)}, u\right)(1-t)^2\;, \\
G_3(t,u) &\mapsto G_3^{(4)}(t,u) = G_3\left(\frac{t^2}{4(t-1)}, u\right)(1-t)^3\;.
\end{aligned} \tag{6.26}
$$

The fifth series of families is obtained from the M_n-polarized families in Equation (6.4) by the transformation

$$
\begin{aligned}
t &\mapsto -\frac{4\,t}{(1-t)^2}\;, \\
G_2(t,u) &\mapsto G_2^{(5)}(t,u) = G_2\left(-\frac{4\,t}{(1-t)^2}, u\right)(1-t)^4\;, \\
G_3(t,u) &\mapsto G_3^{(5)}(t,u) = G_3\left(-\frac{4\,t}{(1-t)^2}, u\right)(1-t)^6\;.
\end{aligned} \tag{6.27}
$$

It is easily checked that $G_2^{(\tilde{m})}(t,u)$ and $G_3^{(\tilde{m})}(t,u)$ with $\tilde{m}=1,\dots,5$ taken from Equations (6.23)–(6.27) define families of minimal Weierstrass equations. We have the following lemma:

Lemma 6.7. *For $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ the Weierstrass equations*

$$
Y^2 = 4\,X^3 - G_2^{(\tilde{m})}(t,u)\,X - G_3^{(\tilde{m})}(t,u) \tag{6.28}
$$

with $G_2^{(\tilde{m})}(t,u)$ and $G_3^{(\tilde{m})}(t,u)$ for $1 \le \tilde{m} \le 5$ taken from Equations (6.23)–(6.27) and g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces of Picard rank 19 with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u=0$ and $u=-1$, a fiber of Kodaira-type I_{2n} over $u=\infty$, and two singular fibers of Kodaira-type I_1. The families are families of M_n-lattice polarized K3 surfaces with $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ for $n=1,\dots,4$.

For $t \in D_1^*(0)$ we define a family of closed two-cycles $\tilde{\Sigma}_2''^{(\tilde{m})}$ the same way we defined the family of two-cycles Σ_2'' in Section 6.2.2, but with t replaced by its transform on the right hand side of Equations (6.23)–(6.27) and (X,Y) replaced by $\left((1-t)^{\tilde{\alpha}}\,X,(1-t)^{\tilde{\alpha}}\,Y\right)$ for $\tilde{\alpha} \in \{0,\frac{1}{2},1\}$ in its definition to account for the rational transformation and twist. Notice that since $t \in D_1^*(0)$, we can always arrange the branch cut of the function $(1-t)^{\tilde{\alpha}}$ in a way that does not intersect the open punctured disc $D_1^*(0)$. Thus, for $t \in D_1^*(0)$ we obtain a continuously varying family of closed two-cycles $\tilde{\Sigma}_2''^{(\tilde{m})}$ for $1 \le \tilde{m} \le 5$. It then follows:

Lemma 6.8. *For $t \in D_1^*(0)$ the period of the holomorphic two-form $du/2 \wedge dX/Y$ over the two-cycle $\tilde{\Sigma}_2''^{(\tilde{m})}$ for the families obtained in Lemma 6.7 is given by*

$$
\tilde{\omega}_2^{(\tilde{p},\tilde{q})} = (2\pi i)^2\,\left(\frac{1}{(1-t)^{\frac{1-\tilde{p}-\tilde{q}}{2}}}\;{}_2F_1\!\left(\begin{matrix}\tilde{p},\tilde{q}\\1\end{matrix}\middle|\,t\right)\right)^2 \tag{6.29}
$$

and listed in Table 11 along with the relation between $(\mu, \tilde{m})$ and $(\tilde{p}, \tilde{q})$. The period is the solution of the differential equation $\tilde{L}_3^{(\tilde{p},\tilde{q})}\, \tilde{\omega}_2^{(\tilde{p},\tilde{q})} = 0$ that is holomorphic near $t = 0$ with

$$\begin{aligned} \tilde{L}_3^{(\tilde{p},\tilde{q})} &= \theta^3 - t\,(2\,\theta + 1)\Big(\theta^2 + \theta + 2\,pq - p - q + 1\Big) \\ &\quad + t^2\,(\theta + 1)\,(\theta + 1 - q + p)\,(\theta + 1 - q + p)\ . \end{aligned} \tag{6.30}$$

Proof. The proof of this lemma amounts to checking some classical and well-known hypergeometric function identities listed in Table 11. The identities allow us to write each period as a symmetric square. The differential operators can then be obtained as a symmetric square. □

$\tilde{m}$	$(\tilde{p}, \tilde{q})$	$\tilde{\omega}_2^{(\tilde{p},\tilde{q})}/(2\pi i)^2$	$\tilde{L}_3^{(\tilde{p},\tilde{q})}$
1	$\left(\frac{\mu}{2}, \frac{1-\mu}{2}\right)$	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, t\right)$ $= \left(\frac{1}{(1-t)^{\frac{1-\mu/2-(1-\mu)/2}{2}}}\, {}_2F_1\left(\frac{\mu}{2}, \frac{1-\mu}{2}; 1 \middle\vert\, t\right)\right)^2$	$\theta^3 - t\,(\theta + \frac{1}{2})\,(2\,\theta^2 + 2\,\theta - \mu^2 + \mu + 1)$ $+ t^2\,(\theta + 1)\,(\theta + \mu + \frac{1}{2})\,(\theta - \mu + \frac{3}{2})$
2	$\left(\frac{\mu}{2}, \frac{1+\mu}{2}\right)$	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, \frac{t}{t-1}\right)$ $= \left(\frac{1}{(1-t)^{\frac{1-\mu/2-(1+\mu)/2}{2}}}\, {}_2F_1\left(\frac{\mu}{2}, \frac{1+\mu}{2}; 1 \middle\vert\, t\right)\right)^2$	$\theta^3 - t\,(\theta + \frac{1}{2})\,(2\,\theta^2 + 2\,\theta + \mu^2 - \mu + 1)$ $+ t^2\,(\theta + 1)\,(\theta + \frac{1}{2})\,(\theta + \frac{3}{2})$
3	$(\mu, 1-\mu)$	${}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, 4\,t\,(1-t)\right)$ $= \left(\frac{1}{(1-t)^{\frac{1-\mu-(1-\mu)}{2}}}\, {}_2F_1\left(\mu, 1-\mu; 1 \middle\vert\, t\right)\right)^2$	$\theta^3 - 2\,t\,(\theta + \frac{1}{2})\,(\theta^2 + \theta - 2\,\mu^2 - 2\,\mu)$ $+ t^2\,(\theta + 1)\,(\theta + 2\,\mu)\,(\theta + 2\,(1-\mu))$
4	$\left(\mu, \frac{1}{2}\right)$	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, \frac{t^2}{4(t-1)}\right)$ $= \left(\frac{1}{(1-t)^{\frac{1-\mu-1/2}{2}}}\, {}_2F_1\left(\mu, \frac{1}{2}; 1 \middle\vert\, t\right)\right)^2$	$\theta^3 - t\,(\theta + \frac{1}{2})\,(2\,\theta^2 + 2\,\theta + 1)$ $+ t^2\,(\theta + 1)\,(\theta + \mu + \frac{1}{2})\,(\theta - \mu + \frac{3}{2})$
5	(μ, μ)	$\frac{1}{1-t}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, -\frac{4\,t}{(1-t)^2}\right)$ $= \left(\frac{1}{(1-t)^{\frac{1-\mu-\mu}{2}}}\, {}_2F_1\left(\mu, \mu; 1 \middle\vert\, t\right)\right)^2$	$\theta^3 - 2\,t\,(\theta + \frac{1}{2})\,(\theta^2 + \theta + 2\,\mu^2 - 2\,\mu + 1)$ $+ t^2\,(\theta + 1)^3$

TABLE 11. Rational transformations of family parameter and periods

6.2.4. *Connection to the middle convolution.* We observe that the period computation in Equation (6.10) can be recast as follows

$$\begin{aligned} 2\,\omega_2^{(\mu)} &= \oiint_{\Sigma_2} du \wedge \frac{dX}{Y} = (2\pi i)\oint_{[\gamma_1^{-1}\gamma_0^{-1}\gamma_1\gamma_0]} \frac{du}{\sqrt{u\,(u-1)}}\ {}_2F_1\!\left(\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\middle\vert\, ut\right) \\ &= (2\pi i)\oint_{[\gamma_1^{-1}\gamma_0^{-1}\gamma_1\gamma_0]} \frac{du}{u}\ \left(1 - \frac{1}{u}\right)^{-\frac{1}{2}}\ {}_2F_1\!\left(\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\middle\vert\, ut\right) \\ &= (2\pi i)\oint_{[\gamma_t^{-1}\gamma_0^{-1}\gamma_t\gamma_0]} \frac{dU}{U}\ \left(1 - \frac{t}{U}\right)^{-\frac{1}{2}}\ {}_2F_1\!\left(\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\middle\vert\, U\right)\ . \end{aligned} \tag{6.31}$$

In the notation of Bogner and Reiter [15], the last line of Equation (6.31) is denoted as the Hadamard product $H^t_{1/2}(\omega_1^{(\mu)})$. Similarly, following the notation of Bogner and Reiter the differential operator in Equation (6.11) satisfies $\mathcal{H}_{1/2}(L_2^{(\mu)}) = L_3^{(\mu)}$, i.e., it can be obtained as formal Hadamard-twist of the differential operator $L_2^{(\mu)}$ defined in Equation (5.13). For later use, we will also denote the monodromy tuple induced by the operator $L_2^{(\mu)}$ as $\mathbf{T}^{(\mu)}$.

The Hadamard product can also be expressed in terms of a convolution operation defined in [15], i.e.,

$$H^t_{1/2}(\omega_1^{(\mu)}) = \exp\left(-i\pi/2\right)\, C^t_{1/2}\left(\frac{\omega_1^{(\mu)}}{\sqrt{t}}\right) = C^t_{1/2}\left(\frac{\omega_1^{(\mu)}}{\sqrt{-t}}\right) .$$

The period $\omega_1^{(\mu)}/\sqrt{-t}$ is the period of the holomorphic one-form dX/Y for the family of elliptic curves given by

$$Y^2 = 4\,X^3 - g_2(t)\,t^2\,X + g_3(t)\,t^3 \tag{6.32}$$

with singular fibers of Kodaira-type I_n^* over $t = 0$, of Kodaira-type I_1 over $t = 1$ and one of Kodaira-type II, III, IV or I_4 over $t = \infty$. The polynomials $g_2(t)$ and $g_3(t)$ were given in Lemma 5.7. The periods of the family (6.32) are annihilated by the second-order and degree-one operator

$$\underline{L}_2^{(\mu)} = \left(\theta + \frac{1}{2}\right)^2 - t\left(\theta + \frac{1}{2} + \mu\right)\left(\theta + \frac{1}{2} + 1 - \mu\right) . \tag{6.33}$$

Similarly, the differential operator in Equation (6.11) can also be written as $\mathcal{C}_{1/2}(\underline{L}_2^{(\mu)}) = L_3^{(\mu)}$ in the notation of Bogner and Reiter. We denote the monodromy tuple induced by the operator $\underline{L}_2^{(\mu)}$ as $\underline{\mathbf{T}}^{(\mu)}$. We have the following corollary:

Corollary 6.9. *In the notation of Bogner and Reiter* [15]*, the Picard-Fuchs operators L_3, $L_2^{(\mu)}$, and $\underline{L}_2^{(\mu)}$ for the families of elliptic curves given in Equation (6.2), in Lemma 5.7, and in Equation (6.32), respectively, are related as follows:*

$$L_3^{(\mu)} = \mathcal{H}_{1/2}(L_2^{(\mu)}) = \mathcal{C}_{1/2}(\,\underline{L}_2^{(\mu)}) .$$

Similarly, their periods are related by

$$2\,\omega_2^{(\mu)} = H^t_{1/2}(\omega_1^{(\mu)}) = C^t_{1/2}\left(\frac{\omega_1^{(\mu)}}{\sqrt{-t}}\right) .$$

The monodromy tuple induced by $L_3^{(\mu)}$ is a sub-factor of the middle Hadamard product denoted by $\mathrm{MH}_{-1}(\mathbf{T}^{(\mu)})$ *in Bogner and Reiter* [15] *or, equivalently, the middle convolution* $\mathrm{MC}_{-1}(\,\underline{\mathbf{T}}^{(\mu)})$ *where $\mathbf{T}^{(\mu)}$ is the monodromy tuple induced by the operator $L_2^{(\mu)}$ and $\underline{\mathbf{T}}^{(\mu)}$ is the monodromy tuple induced by the operator $\underline{L}_2^{(\mu)}$.*

6.3. **K3 families of Picard rank 18.** The following lemmas describe the construction of families of Jacobian elliptic K3 surfaces of Picard rank 18 from the rational elliptic surfaces in Lemmas 5.7 and 5.9.

6.3.1. *Pure-twist construction.* We will first describe the *pure-twist construction* of Jacobian elliptic K3 surfaces of Picard rank 18. We have the following lemma:

Lemma 6.10. *For $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2(t\,u)\left(u^2-1\right)^2}_{=:\,G_2(t,u)}\,X - \underbrace{g_3(t\,u)\left(u^2-1\right)^3}_{=:\,G_3(t,u)} , \tag{6.34}$$

with g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces of Picard rank 18 with singular fibers of Kodaira-type I_n over $u = 0$ and of Kodaira-type I_0^* over $u = \pm 1$, of Kodaira-type I_1 over $u = 1/t$ and one of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u = \infty$. Their configurations of singular fibers, the Mordell-Weil groups, the determinants of the discriminant groups, and lattice polarizations are summarized in Table 12. The lattices have rank 18 and will be denoted by L, $\tilde{L}$, L', and $\tilde{L}'$ for $n = 1, \dots, 4$.

□

derived from		ρ	Configuration of singular fibers				MW(π)	Discriminant Q	Λ
Srfc	μ		$u = 0$	$u = 1/t$	$u = \infty$	$u = \pm 1$			
X_{141}	$\frac{1}{2}$	18	$I_4\,(A_3)$	I_1	$I_1^*\,(D_5)$	$2\,I_0^*\,(2\,D_4)$	[2]	$2^6 = 4 \cdot 4 \cdot (2 \cdot 2)^2/2^2$	$\tilde{L}'$
X_{431}	$\frac{1}{3}$	18	$I_3\,(A_2)$	I_1	$IV^*\,(E_6)$	$2\,I_0^*\,(2\,D_4)$	[1]	$2^4 \cdot 3^2 = 3^2 \cdot (2 \cdot 2)^2/1$	L'
X_{321}	$\frac{1}{4}$	18	$I_2\,(A_1)$	I_1	$III^*\,(E_7)$	$2\,I_0^*\,(2\,D_4)$	[2]	$2^4 = 2^2 \cdot (2 \cdot 2)^2/2^2$	$\tilde{L}$
X_{211}	$\frac{1}{6}$	18	I_1	I_1	$II^*\,(E_8)$	$2\,I_0^*\,(2\,D_4)$	[1]	$2^4 = 4 \cdot (2 \cdot 2)/1$	L

Table 12. K3 surfaces from extremal rational surfaces

The torsion sections for the families in Equation (6.10) are listed in Table 13.

Λ	torsion	sections
$\tilde{L}'$	[2]	$(X, Y) = \left(\frac{2}{3}\,(tu-2)\,(u-1)\,(u+1)\,, 0\right)$
L'	[1]	–
$\tilde{L}$	[2]	$(X, Y) = \left(\frac{2}{3}\,(u+1)\,(u-1)\,, 0\right)$
L	[1]	–

Table 13. Torsions sections of Equation (6.10)

Remark. Setting $u \mapsto t\,u$ and $(X, Y) \mapsto (X/t^2, Y/t^3)$ in Equation (6.6) we obtain the Weierstrass equation

$$Y^2 = 4\,X^3 - g_2(u)\left(u^2 - t^2\right)^2 X - g_3(u)\left(u^2 - t^2\right)^3 . \tag{6.35}$$

This shows that the construction of the introductory example in Section 4.2 arises here as a special case.

6.3.2. *Mixed-twist construction.* We will now describe the *mixed-twist construction* of Jacobian elliptic K3 surfaces of Picard rank 18. To do so we start with the rational elliptic surfaces defined by $g_2(t)$ and $g_3(t)$ in Lemma 5.7 with singular fibers of Kodaira-type I_n over $t = 0$, of Kodaira-type I_1 over $t = 1$, and of Kodaira-type II^*, III^*, IV^*, or I_1^* over $t = \infty$. We then apply the rational map $u \mapsto 1 + \frac{1}{2\,u(u+1)}$ of degree 2 from $\mathbb{P}^1 \to \mathbb{P}^1$. It then follows:

Lemma 6.11. *For $(n, \mu) \in \left\{(1, \frac{1}{6}), (2, \frac{1}{4}), (3, \frac{1}{3}), (4, \frac{1}{2})\right\}$ the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2\left(t\left(1 + \frac{1}{2\,u(u+1)}\right)\right)\left(u\,(u+1)\right)^4}_{=:\,G_2(t,u)} X - \underbrace{g_3\left(t\left(1 + \frac{1}{2\,u(u+1)}\right)\right)\left(u\,(u+1)\right)^6}_{=:\,G_3(t,u)} , \tag{6.36}$$

with g_2 and g_3 taken from Table 5 define families of Jacobian elliptic K3 surfaces of Picard rank 18 with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u = 0$ and $u = -1$, two fibers of Kodaira-type I_n at the roots of $2u^2 + 2u + 1$, and two singular fibers of type I_1 at the roots of $2(t-1)u(u+1) + t$. The configurations of singular fibers, the Mordell-Weil groups, the determinants of the discriminant groups, and lattice polarizations are summarized in Table 14. The lattices have rank 18 and will be denoted by $M = H \oplus E_8 \oplus E_8$, $\tilde{M} = H \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}/\mathbb{Z}_2$, $M' = H \oplus E_6^{\oplus 2} \oplus A_2^{\oplus 2}/\mathbb{Z}_3$, and $\tilde{M}' = H \oplus D_5^{\oplus 2} \oplus A_3^{\oplus 2}/\mathbb{Z}_4$ for $n = 1, \dots, 4$.

□

derived from		ρ	Configuration of singular fibers			MW(π)	Discriminant Q	Λ
Srfc	μ		$2u^2+2u+1$	$(2t-1)u^2+(2t-1)u+t$	$u=0,-1$			
X_{141}	$\frac{1}{2}$	18	$2\,I_4\ (2\,A_3)$	$2\,I_1$	$2\,I_1^*\ (2\,D_5)$	[4]	$2^4 = 4^2 \cdot 4^2/4^2$	$\tilde{M}'$
X_{431}	$\frac{1}{3}$	18	$2\,I_3\ (2\,A_2)$	$2\,I_1$	$2\,IV^*\ (2\,E_6)$	[3]	$3^2 = 3^4/3^2$	M'
X_{321}	$\frac{1}{4}$	18	$2\,I_2\ (2\,A_1)$	$2\,I_1$	$2\,III^*\ (2\,E_7)$	[2]	$2^2 = 2^4/2^2$	$\tilde{M}$
X_{211}	$\frac{1}{6}$	18	$2\,I_1$	$2\,I_1$	$2\,II^*\ (2\,E_8)$	[1]	$1 = 1/1$	M

Table 14. K3 surfaces from extremal rational surfaces

The torsion sections for the families in Equation (6.36) are listed in Table 15.

Λ	torsion	sections
M_4	[4]	$(X,Y)_1 = \left(\frac{1}{3}\,(u+1)\,u\left(2\,tu^2+2\,tu-4\,u^2+t-4\,u\right), 0\right)$ $(X,Y)_{2,3} = \left(-\frac{1}{6}\,(u+1)\,u\left(2\,tu^2+2\,tu+8\,u^2+t+8\,u\right), \pm it\,(2\,u+1+i)\,(-2\,u-1+i)\,u^2\,(u+1)^2\right)$
M_3	[3]	$(X,Y)_{1,2} = \left(-\frac{3}{2}\,u^2\,(u+1)^2, \pm\frac{i}{2}\,\sqrt{2}t\,(2\,u+1+i)\,(-2\,u-1+i)\,u^2\,(u+1)^2\right)$
M_2	[2]	$(X,Y) = \left(\frac{2}{3}\,u^2\,(u+1)^2, 0\right)$
M_1	[1]	–

Table 15. Torsions sections of Equation (6.36)

Remark. A rational transformation relates the families obtained in Lemma 6.10 to the families obtained in Lemma 6.11. In fact, applying the transformation

$$
\begin{aligned}
u &\mapsto 1 + \frac{1}{2u(u+1)}\,, \\
(X,Y) &\mapsto \left(\left(-\frac{(2\,u+1)}{2\,u^2\,(u+1)^2}\right)^2 X,\ \left(-\frac{(2\,u+1)}{2\,u^2\,(u+1)^2}\right)^3 Y \right)
\end{aligned}
\tag{6.37}
$$

in Equation (6.34) yields Equation (6.36). The transformation (6.37) also maps the holomorphic two-form $du \wedge dX/Y$ on the variety given by Equation (6.34) to $du \wedge dX/Y$ on the variety given by Equation (6.36).

6.3.3. *Two-parameter pure-twist construction.* We now describe the construction of two-parameter families of Jacobian elliptic K3 surfaces of Picard rank 18. We again start with the rational elliptic surfaces defined by $g_2(t)$ and $g_3(t)$ in Lemma 5.7:

Lemma 6.12. *For* $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ *the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2(u)\,\big((u-a)\,(u-b)\big)^2}_{=:\,G_2(u,a,b)}\,X - \underbrace{g_3(u)\,\big((u-a)\,(u-b)\big)^3}_{=:\,G_3(u,a,b)} \tag{6.38}$$

with g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces with singular fibers of type I_n over $u=0$ and of Kodaira-type I_0^* over $u=a,b$, of Kodaira-type I_1 over $u=1$ and one of type II^*, III^*, IV^*, or I_1^* over $u=\infty$. The lattices generically have Picard rank 18 and coincide with L, $\tilde{L}$, L', and $\tilde{L}'$ from Lemma 6.10, respectively. Setting $a=0$ and $b=t$ in Equation (6.38) recovers Equation (6.3). Setting $a=-b=t$ in Equation (6.38) recovers Equation (6.35)

□

We then apply the rational map $u \mapsto a + (a-b)/(4\,u\,(u+1))$ of degree 2 from $\mathbb{P}^1 \to \mathbb{P}^1$. The map is totally ramified over a, has a simple ramification point over b, and is ramified to degrees 1 and 1 over ∞. In particular, we have a branching point at b with corresponding ramification point $u=-1/2$ in the pre-image with branch number 1, a branching point at a with corresponding ramification point $u=\infty$ in the pre-image with branch number 1, and a branching point at ∞ with corresponding ramification points $u=0$ and $u=-1$ in the pre-image with branch numbers 0. It then follows:

Lemma 6.13. *For* $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ *the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{g_2\left(a+\frac{a-b}{4\,u\,(u+1)}\right)\Big(u\,(u+1)\Big)^4}_{=:\,G_2(u,a,b)}\,X - \underbrace{g_3\left(a+\frac{a-b}{4\,u\,(u+1)}\right)\Big(u\,(u+1)\Big)^6}_{=:\,G_3(u,a,b)}\,, \tag{6.39}$$

with g_2 and g_3 given in Table 5 define families of Jacobian elliptic K3 surfaces 18 with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u=0$ and $u=-1$, two fibers of Kodaira-type I_n over the roots of $4\,a\,u\,(u+1)+a-b$, and two singular fibers of Kodaira-type I_1 over the roots of $4\,(a-1)\,u\,(u+1)+a-b$. The lattices generically have Picard rank 18 and coincide with M, $\tilde{M}$, M', and $\tilde{M}'$ from Lemma 6.11, respectively. Setting $a=0$ and $b=t$ in Equation (6.39) recovers Equation (6.4). Setting $a=-b=t$ in Equation (6.39) recovers Equation (6.36)

□

Remark. The two families of Jacobian elliptically K3 surfaces in Equation (6.38) and (6.39) are related by the degree-two rational map τ given by

$$\tau:\;\begin{cases} u & \mapsto a + \frac{a-b}{4\,u\,(u+1)}\,,\\ X & \mapsto \left(-\dfrac{(a-b)(2\,u+1)}{4\,u^2\,(u+1)^2}\right)^2 X\,,\\ Y & \mapsto \left(-\dfrac{(a-b)(2\,u+1)}{4\,u^2\,(u+1)^2}\right)^3 Y\,. \end{cases} \tag{6.40}$$

The transformation (6.40) also maps the holomorphic two-form $du \wedge dX/Y$ on the variety given by Equation (6.38) and $du \wedge dX/Y$ on the variety given by Equation (6.39).

Remark. Let us explain the geometric relationship between the pure-twist and mixed-twist construction. If we denote by $f : \mathbb{P}^1 \to \mathbb{P}^1$ the quadratic map $f(u) = a + (a-b)/(4\,u\,(u+1))$, then the pull-back by f of the Jacobian rational elliptic surface S given by $g_2(t)$ and $g_3(t)$ in Lemma 5.7 gives the family of Jacobian elliptic K3 surface $X \to \mathbb{P}^1$ in Equation (6.39), previously obtained by the mixed-twist construction. On X, we have the deck transformation $\imath$ given by $(u, X, Y) \mapsto (-u-1, X, Y)$ and the fiberwise elliptic involution $-\mathrm{id} : (u, X, Y) \mapsto (u, X, -Y)$. The composition $\jmath = -\mathrm{id} \circ \imath$ is a Nikulin involution leaving $du \wedge dX/Y$, and the minimal resolution of the quotient $X/\jmath$ is the Jacobian elliptic K3 surface $X' \to \mathbb{P}^1$ in Equation (6.38) previously obtained by the pure-twist construction. The K3 surface X' is the quadratic twist of S having fibers of Kodaira-type I_0^* over the two ramification points of f. The situation is summarized in Figure 3.

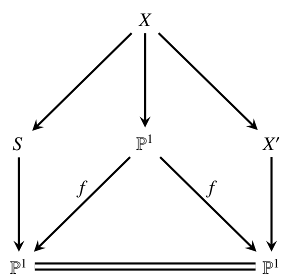


FIGURE 3. Relation between mixed-twist and pure-twist

For each pair of K3 surfaces obtained from the mixed-twist and the pure-twist construction, we can easily check that the determinants of the discriminant groups of their transcendental lattices always differ by a square of the form $(1/2)^{2\alpha}$ with $0 \le \alpha \le 2$. As the transcendental lattices are related by the isometry τ_* given in Equation (6.40) and is generally not a Hodge isometry, this is in perfect agreement with a general result obtained by Mehran [56].

6.4. **Period computation for families of K3 surfaces of Picard rank** 18. In this section, we carry out the period computation for all families of Jacobian elliptic K3 surfaces of Picard rank 18 constructed in the previous section.

6.4.1. *Pure-twist construction.* We define a family of closed two-cycles $\hat{\Sigma}_2$ in the total space of Equation (6.34) as follows: for each $t \in D_1^*(0)$, we follow a Pochhammer contour $\hat{\Sigma}_1$ invariant under $u \mapsto -u$ around $u = -1$ and $u = 1$ in the affine u-plane such that $0 < |ut| < 1$. For each $u \in \hat{\Sigma}_1(t)$, we trace out the cycle in the (X, Y)-coordinates of the elliptic fiber that is obtained from the A-cycle $\Sigma_1'(ut)$ defined in Lemma 5.4 after rescaling $(X, Y) \to \big((u^2-1)\,X, (u^2-1)^{3/2}\,Y\big)$ to account for the quadratic twist. For $t \in D_1^*(0)$ we obtain a continuously varying family of closed two-cycles that we denote as warped product $\hat{\Sigma}_2 = \hat{\Sigma}_1 \times_u \Sigma_1'$. Such a family of cycles has the special property that we can divide up the Pochhammer contour in the affine u-plane into two parts according to $\mathrm{Re}(u) \ge 0$ and $\mathrm{Re}(u) \le 0$. Using the two branches of the square-root function, the substitutions $u = \sqrt{\tilde{u}}$ and $u = -\sqrt{\tilde{u}}$ transform the right half and the left half into the same cycle $\tilde{\Sigma}_1$ which is itself a Pochhammer contour around $\tilde{u} = 0$ and $\tilde{u} = 1$ such that $|\tilde{u}|^{1/2}\,|t| < 1$ for each t and $\tilde{u} \in \tilde{\Sigma}_1$.

An example of such a Pochhammer contour is depicted in Figure 4. Breaking up the the Pochhammer contour in Figure 4 is shown in Figures 5 and 6.

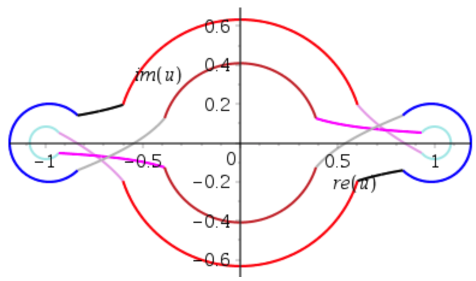
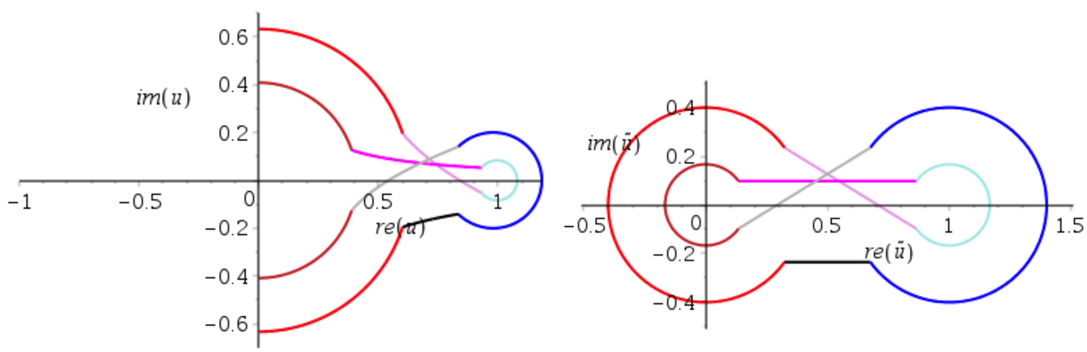
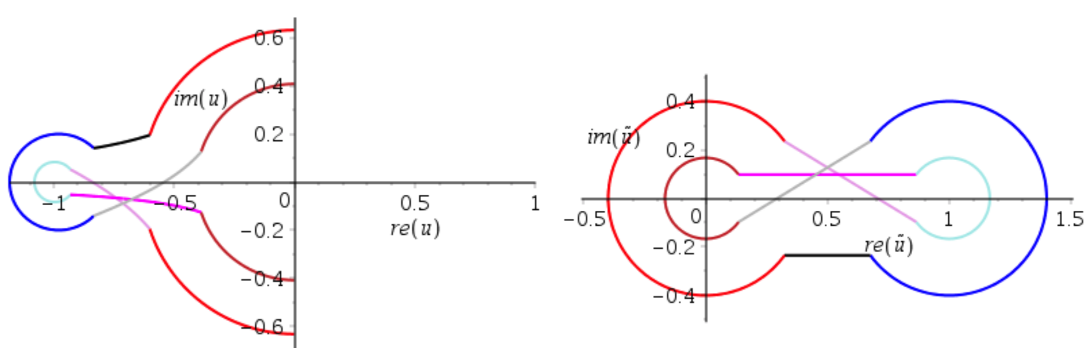


FIGURE 4. Pochhammer contour in u-plane



(A) Contour piece with $\mathrm{Re}(u) \geq 0$ (B) Image under $\tilde{u} = u^2$

FIGURE 5. Half of a Pochhammer contour (I)



(A) Contour piece with $\mathrm{Re}(u) \leq 0$ (B) Image under $\tilde{u} = u^2$

FIGURE 6. Half of a Pochhammer contour (II)

We now evaluate the period integral for the holomorphic two $du/2 \wedge dX/Y$ over the two-cycle $\hat{\Sigma}_2$, i.e.,

$$\hat{\omega}_2 = \frac{1}{2} \oiint_{\hat{\Sigma}_2} du \wedge \frac{dX}{Y} \; . \tag{6.41}$$

We then have the following lemma:

Lemma 6.14. *For $t \in D_1^*(0)$ the period of the holomorphic two-form $du/2 \wedge dX/Y$ over the two-cycle $\hat{\Sigma}_2$ for the families of K3 surfaces in Lemma 6.10 is given by*

$$\hat{\omega}_2^{(\mu)} = (2\pi i)^2 \, {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) \; . \tag{6.42}$$

Proof. To integrate the holomorphic two-form $du \wedge dX/Y$ over the two-cycle $\hat{\Sigma}_2$ we use Fubini's theorem. Integrating over the A-cycle first, we obtain

$$\frac{1}{2\pi i} \oiint_{\hat{\Sigma}_2} du \wedge \frac{dX}{Y} = i \oint_{\hat{\Sigma}_1} \frac{1}{\sqrt{1-u^2}} \, {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| ut \right) \; . \tag{6.43}$$

We break up the Pochhammer contour into two parts with $\mathrm{Re}(u) \geq 0$ and $\mathrm{Re}(u) \leq 0$ using the two branches of the square-root function. We obtain two copies of a Pochhammer contour $\tilde{\Sigma}_1$ around $\tilde{u} = 0, 1$. The period integral for the holomorphic two-form can then be simplified as follows:

$$\begin{aligned} & \frac{1}{2\pi i} \oiint_{\hat{\Sigma}_2} du \wedge \frac{dX}{Y} \\ = & \frac{i}{2} \int_{\Sigma_1} \frac{d\tilde{u}}{\sqrt{\tilde{u}(1-\tilde{u})}} \left({}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| t\tilde{u}^{1/2} \right) + {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| -t\tilde{u}^{1/2} \right) \right) \end{aligned} \tag{6.44}$$

Recalling that $|\tilde{u}^{1/2} t| < 1$ for all $\tilde{u} \in \tilde{\Sigma}_1$, the series expansion of the hypergeometric function converges absolutely and uniformly. One then obtains for the period integral

$$\begin{aligned} & i \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{(\mu)_n \, (1-\mu)_n}{(n!)^2} \, t^n \int_{[\tilde{\gamma}_1^{-1} \tilde{\gamma}_0^{-1} \tilde{\gamma}_1 \tilde{\gamma}_0]} d\tilde{u} \, (\tilde{u})^{\frac{n+1}{2}-1} \, (1-\tilde{u})^{\frac{1}{2}-1} \\ = \, & i \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{(\mu)_n \, (1-\mu)_n}{(n!)^2} \, t^n \, \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \left(1 - e^{2\pi i \, \frac{1}{2}}\right) \left(1 - e^{2\pi i \, \frac{n+1}{2}}\right) \\ = \, & 4i \, \Gamma(\frac{1}{2})^2 \sum_{n=0}^{\infty} \frac{(\mu)_{2n} \, (1-\mu)_{2n} \, (\frac{1}{2})_n}{[(2n)!]^2 \, n!} \, t^{2n} \; . \end{aligned} \tag{6.45}$$

Using the duplication formula $\Gamma(2z) = (2\pi)^{-1/2}\, 2^{2z-1/2}\, \Gamma(z)\, \Gamma(z+\frac{1}{2})$ and $|t| < 1$, we can simplify Equation (6.45) further to obtain

$$\begin{aligned} & 4\pi i \sum_{n=0}^{\infty} \frac{2^{2n}\, (\frac{\mu}{2})_n\, (\frac{1+\mu}{2})_n\, 2^{2n}\, (\frac{1-\mu}{2})_n\, (1-\frac{\mu}{2})_n\, (\frac{1}{2})_n}{[2^{2n}\, (\frac{1}{2})_n\, n!]^2\, n!}\, t^{2n} \\ & = (4\pi i)\ {}_4F_3\!\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) . \end{aligned} \tag{6.46}$$

□

Remark. The holomorphic period $\hat{\omega}_2^{(\mu)}$ in Equation (6.42) can still be expressed as Hadamard product using the hypergeometric function identity

$$ {}_1F_0\!\left(\left. \frac{1}{2} \right| t^2 \right) \star {}_2F_1\!\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| t \right) = {}_4F_3\!\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right) . \tag{6.47}$$

A well-known result for hypergeometric functions then yields the following corollary.

Corollary 6.15. *The period $\hat{\omega}_2^{(\mu)}$ in Equation (6.42) is the solution of the differential equation $\hat{L}_3^{(\mu)}\, \hat{\omega}_2^{(\mu)} = 0$ that is holomorphic near $t = 0$ with*

$$\begin{aligned} \hat{L}_3^{(\mu)} & = \theta^3\, (\theta - 1) - t\, (\theta + \frac{1}{2})\, (\theta + \mu)\, (\theta + 1 - \mu) \\ & - t^2\, (\theta + \mu)\, (\theta + \mu + 1)\, (\theta + 1 - \mu)\, (\theta + 2 - \mu) . \end{aligned} \tag{6.48}$$

□

6.4.2. *Mixed-twist construction.* In this section, we will explain how the period computation from Section 6.4.1 generalizes to the mixed-twist construction.

We applied the rational map $u \mapsto 1 + \frac{1}{2\, u(u+1)}$ of degree two to constructed the family of K3 surfaces in Equation (6.36) which we call the mixed-twist family. It was given by

$$Y^2 = 4\, X^3 - g_2\left(t\left(1 + \frac{1}{2\, u(u+1)}\right)\right) \Big(u\, (u+1)\Big)^4 X - g_3\left(t\left(1 + \frac{1}{2\, u(u+1)}\right)\right) \Big(u\, (u+1)\Big)^6 \tag{6.49}$$

with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* over $u = 0$ and $u = -1$, two fibers of Kodaira-type I_n over the roots $u_{n,\pm} = (-1 \pm i)/2$ of $2u^2 + 2u + 1$, and two singular fibers of Kodaira-type I_1 over the roots $u_{1,\pm} = 1/2\, (-1 \pm \sqrt{(1+t)/(1-t)})$ of $2(t-1)u(u+1) + t$.

We set $C = \mathbb{P}^1 \backslash \{0, -1, u_{1,\pm}, u_{n,\pm}, \infty\}$ and denote the fundamental group of C based at $u_0 \in C$ by $\Gamma = \pi_1(C, u_0)$. Generators for Γ are suitable simple loops γ_u around $u \in \{0, -1, u_{1,\pm}, u_{n,\pm}, \infty\}$ such that the following relation holds

$$[\gamma_{u_{n,+}}] * [\gamma_{1,-}] * [\gamma_{-1}] * [\gamma_{u_{n,-}}] * [\gamma_0] * [\gamma_{u_{1,+}}] = 1 , \tag{6.50}$$

where [.] denotes a homotopy equivalence class and $*$ the group multiplication of loops up to homotopy. We will then fix a homological invariant for families of Jacobian elliptic K3 surfaces in Equation (6.12) by defining a locally constant sheaf over C whose generic stalk is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A monodromy representation

$$M : \Gamma \to \mathrm{SL}(2, \mathbb{Z}) \tag{6.51}$$

defines the transition functions for this sheaf. We fix these monodromy matrices to be

$$(6.52)\quad M_0 = M_{-1} = \begin{cases} ST & \text{for } n=1,\, II^* \\ T^{-1}ST & \text{for } n=2,\, III^* \\ T^{-1}(ST)^2T & \text{for } n=3,\, IV^* \\ (ST^2)^{-1}(-T)ST^2 & \text{for } n=4,\, I_1^* \end{cases}, \qquad M_{u_{1,\pm}} = STS^{-1}\,,\ M_{u_{n,\pm}} = T^n\,,$$

where we write $M_u = M(\gamma_u)$ and used the following $\mathrm{SL}(2,\mathbb{Z})$-generators

$$(6.53)\qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrices satisfy the condition

$$(6.54)\qquad M_{u_{1,-}} \cdot M_{-1} \cdot M_{u_{n,-}} \cdot M_0 \cdot M_{u_{1,+}} \cdot M_{u_{n,+}} = \mathbb{I}\,.$$

From the explicit form of the matrices (6.52) it is easy to see that there is an A-cycle in the fiber that is transformed into itself when simultaneously encircling the singular fibers over the three points $u = u_{1,-}, -1, u_{n,-}$ or the three points $u = u_{n,-}, 0, u_{1,+}$. The same statement holds when swapping the roles of $u_{n,-}$ and $u_{n,+}$.

We define a family of closed two-cycles $\hat{\Sigma}''_2$ for the families of Jacobian elliptic K3 surfaces in Equation (6.49) as follows: for each $t \in D_1^*(0)$ we take a loop $\hat{\Sigma}''_1 = \hat{\Sigma}''_1(t)$ in the affine u-plane that covers the Pochhammer contour $\hat{\Sigma}_1$ from Section 6.4.1 under the map $u \mapsto 1 + \frac{1}{2\,u(u+1)}$ such that $0 < |t|\,|1 + \frac{1}{2\,u(u+1)}| < 1$. Notice that generically there are two choices for such a cycle $\hat{\Sigma}''_1$. Both choices are homologous in C to the class $[\gamma'_{u_{1,+},0,u_{n,\pm}} * \gamma'^{-1}_{u_{n,\pm},-1,u_{1,-}}]$ where the cycles $\gamma'_{u_{1,+},0,u_{n,\pm}}$ and $\gamma'_{u_{n,\pm},-1,u_{1,-}}$ encircle the singular fibers over the three points $u_{1,+}$, 0, $u_{n,\pm}$ and the points $u_{n,\pm}$, -1, $u_{1,-}$, respectively. For each $u \in \hat{\Sigma}''_1(t)$ we trace out the cycle in the elliptic fiber obtained from the A-cycle $\Sigma'_1(t\,(1 + \frac{1}{2\,u(u+1)}))$ defined in Lemma 5.4 by rescaling $(X,Y) \to (u^2\,(u+1)^2\,X, u\,(u+1)\,Y)$ to account for the twist. As a result we obtain a continuously varying family of closed two-cycles that we denote as warped product $\hat{\Sigma}''_2 = \hat{\Sigma}''_1 \times_u \Sigma'_1$ for $t \in D_1^*(0)$.

The two choices for the path $\hat{\Sigma}''_1$ in the u-plane that cover the example of a Pochhammer contour $\hat{\Sigma}_1$ from Figure 7 are depicted in Figure 8a and Figure 8b. In Figure 7, the black dots mark the position of the fibers of Kodaira-type I_0^* over $u = \pm 1$ and the fiber of Kodaira-type I_n over $u = 0$, and the red dot marks the position of the fiber of Kodaira-type I_1 over $u = 1/t$. In Figure 8a and Figure 8b, the black dots mark the positions of the fibers of Kodaira-type

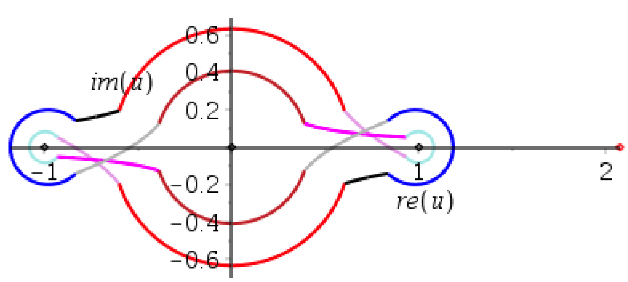
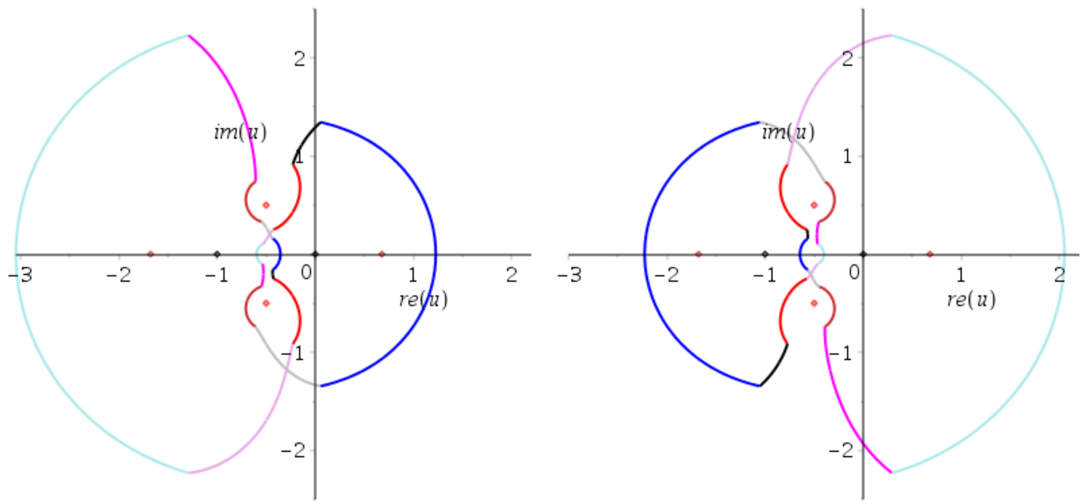


FIGURE 7. Pochhammer contour in u-plane for pure-twist

II^*, III^*, IV^*, or I_1^* over $u = 0$ and $u = -1$, the orange dots mark the positions of the fibers of Kodaira-type I_1 over $u_{1,\pm} = -1/2 \pm i/2$, the red dots mark the positions of the fibers of Kodaira-type I_n over the roots $u_{n,\pm}$ of $2(t-1)\,u(u+1)+t$.



(A) First pre-image (B) Second pre-image

FIGURE 8. Pre-images of Pochhammer contour

Lemma 6.16. *For $t \in D_1^*(0)$, the period of the holomorphic two-form $du/2 \wedge dX/Y$ over the two-cycle $\hat{\Sigma}_2''$ for the families of K3 surfaces in Lemma 6.11 is given by*

$$\hat{\omega}_2^{(\mu)} = (2\pi i)^2\, {}_4F_3\left(\left.\begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array}\right| t^2\right) . \tag{6.55}$$

Proof. The period integral has a holomorphic integrand for all $u \neq 0, -1$. This means that the contour of the period integral can be deformed into an integral over $\gamma'_{u_{1,+},0,u_{n,\pm}}$ and $\gamma'^{-1}_{u_{n,\pm},-1,u_{1,-}}$. Moreover, for the class $[\gamma'_{u_{1,+},0,u_{n,\pm}} * \gamma'^{-1}_{u_{n,\pm},-1,u_{1,-}}]$ the winding number around $u_{n,\pm}$ is zero. Thus, the contour can be further replaced by an integral over $\gamma'_{u_{1,+},0}$ and $\gamma'^{-1}_{-1,u_{1,-}}$. We then obtain for the period integral

$$\frac{1}{2\pi i} \iint_{\hat{\Sigma}_2''} du \wedge \frac{dX}{Y} = \left(\oint_{[\gamma'_{u_{1,+},0}]} - \oint_{[\gamma'_{-1,u_{1,-}}]} \right) \frac{du}{u\,(u+1)}\; {}_2F_1\left(\left.\begin{array}{c} \mu, 1-\mu \\ 1 \end{array}\right| t\left(1 + \frac{1}{2\,u(u+1)}\right)\right) .$$

Furthermore, the loops $\gamma'_{u_{1,+},0}$ and $\gamma'_{-1,u_{1,-}}$ are homologous to circles $C_r^+ : |u - u_{1,+}/2| = r$ and $C_r^- : |u - (u_{1,-} - 1)/2| = r$, respectively, such that $|u_{1,+}|/2 < r < |u_{1,+}|/2 + 1$ or, equivalently, $|u_{1,-} + 1|/2 < r < |u_{1,-} + 1|/2 + 1$. It is then sufficient to choose $r = |u_{1,+}|/2 + 1 - |t|/2$ to ensure $0 < |t|\,|1 + \frac{1}{2\,u(u+1)}| < 1$ for all $u \in C_r^\pm$ and $t \in D_1^*(0)$. Using the variable transformations

$-\tilde{u} = u+1$ and $u = -(\tilde{u}+1)$ such that $\tilde{u}_{\bullet,\pm} = u_{\bullet,\mp}$ we obtain

$$\begin{aligned}
&\left(\oint_{C_r^+} - \oint_{C_r^-}\right) \frac{du}{u\,(u+1)}\, {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| t\left(1+\frac{1}{2\,u(u+1)}\right)\right)\\
&= \oint_{C_r^+} \frac{du}{u\,(u+1)}\, {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| t\left(1+\frac{1}{2\,u(u+1)}\right)\right)\\
&\quad + \oint_{\tilde{C}_r^+} \frac{d\tilde{u}}{\tilde{u}\,(\tilde{u}+1)}\, {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| t\left(1+\frac{1}{2\,\tilde{u}(\tilde{u}+1)}\right)\right)\\
&= 2\oint_{C_r^+} \frac{du}{u\,(u+1)}\, {}_2F_1\!\left(\left.\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\right| t\left(1+\frac{1}{2\,u(u+1)}\right)\right) .
\end{aligned} \tag{6.56}$$

To simplify notation we set ${}_2F_1(\mu, 1-\mu; 1|\,t) = f(t) = \sum_{n\geq 0} f_n\, t^n$ for $|t| < 1$. Recalling that $|t|\,|1+\frac{1}{2\,u(u+1)}| < 1$, the series expansion of the hypergeometric function converges absolutely and uniformly. We obtain

$$\begin{aligned}
&2\oint_{C_r^+} \frac{du}{u\,(u+1)}\, f\left(t\left(1+\frac{1}{2\,u(u+1)}\right)\right)\\
&= 2\sum_{n=0}^{\infty} f_n\, t^n \sum_{l=0}^{n} \binom{n}{l} 2^{-l} \sum_{r=0}^{\infty} \binom{l+r}{r} (-1)^r \oint_{|u-u_{1,+}|=r} \frac{du}{u^{1+l}}\, u^r\\
&= 4\pi i \sum_{n=0}^{\infty} f_n\, t^n \sum_{l=0}^{n} \binom{n}{l}\binom{2l}{l} \frac{(-1)^l}{2^l} .
\end{aligned} \tag{6.57}$$

Using the identity $\sum_{l=0}^n \binom{n}{l}\binom{2l}{l}\frac{(-1)^l}{2^l} = \binom{n}{\frac{n}{2}}\frac{\delta_{n,\text{ev}}}{2^n}$ and $(2n)! = (\frac{1}{2})_n\, n!\, 2^{2n}$, this can be simplified further to obtain

$$\begin{aligned}
&4\pi i \sum_{n=0}^{\infty} f_n\, t^n \binom{n}{\frac{n}{2}} \frac{\delta_{n,\text{ev}}}{2^n} = 4\pi i \sum_{n=0}^{\infty} f_{2n}\, t^{2n} \frac{(2n)!}{2^{2n}\,(n!)^2}\\
&= 4\pi i \sum_{n=0}^{\infty} f_{2n} \frac{\left(\frac{1}{2}\right)_n}{n!}\, t^{2n} = 4\pi i\; {}_1F_0\left(\left.\frac{1}{2}\right| t^2\right) \star f(t) .
\end{aligned} \tag{6.58}$$

The proof is concluded by using Equation (6.47). □

7. Step 3: From families of K3 surfaces to families of Calabi-Yau threefolds

In this section we will describe the third step in our iterative construction that produces families of elliptic Calabi-Yau threefolds with section and $h^{2,1} = 1$ over $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_k$ for $1 \leq k < 6$. The threefolds will be constructed from the families of Jacobian elliptic K3 surfaces of Picard rank 18 and 19 constructed in Section 6 by applying a quadratic twist or a quadratic twist followed by a rational transformation of the base, i.e., the pure-twist or the mixed-twist construction at Step 3. Some new families will also be constructed by a *product-twist construction* from two extremal rational elliptic surfaces constructed in Section 5 directly. We will also compute the Picard-Fuchs operators and holomorphic periods for all families. In particular, the iterative construction at Step 3 will produce families of Calabi-Yau threefolds as minimal Weierstrass models that realize all 14 one-parameter variations of Hodge structure over a one-dimensional rational deformation space classified by Doran and Morgan in [31].

As we construct the Weierstrass models by the pure-twist, mixed-twist, and product-twist construction, we will further divide up each construction into up to three cases which are called the hypergeometric case, the extra case, and the even case. The labeling of the cases will become clear once we compute the Picard-Fuchs operators for these families: the Picard-Fuchs operators will provide a geometric realization for all corresponding fourth-order Calabi-Yau operators classified by Bogner and Reiter [15] matching precisely their nomenclature.

7.1. **Base surface compactification.** First, we will briefly discuss how the families of Weierstrass models that will be constructed in affine charts can always be extended to complex projective surfaces. We will describe families of threefolds by exhibiting their Weierstrass equations

$$Y^2 = 4\,X^3 - \mathcal{G}_2(s,t,u)\,X - \mathcal{G}_3(s,t,u) \tag{7.1}$$

where $\mathcal{G}_2$ and $\mathcal{G}_3$ are polynomials in $s,t,u \in \mathbb{C}$. t is a complex family parameter taking values in $\mathbb{P}^1\backslash\{0,1,\infty\}$ and s and u are affine coordinates on a complex surface. The Weierstrass model in Equation (7.1) can – under certain conditions – be extended from the given affine chart to a compact projective surface. In particular, for the Weierstrass models that we will construct in this section, we will check whether they extend to three types of complex projective surfaces so that s and u in Equation (7.1) become complex affine coordinates, namely $\mathbb{P}^2$ and certain $\mathbb{F}_k$. As a reminder we call $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1})$ with $k \in \mathbb{N}$ a Hirzebruch surface. Then $\mathbb{F}_k$ is the quotient space of $\mathbb{C}^4\backslash\{s_0 = s_1 = u_0 = u_1 = 0\}$ by the action of $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$(\lambda_2, \lambda_3)\,[u_0 : u_1 : s_0 : s_1] = [\lambda_3 u_0 : \lambda_3 u_1 : \lambda_3^k \lambda_2 s_0 : \lambda_2 s_1]\;.$$

It follows that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_1$ is the non-minimal surface that is obtained as the blow-up of $\mathbb{P}^2$ in one point.

The Weierstrass equation (7.1) extends $\mathbb{F}_k$, if we produce a minimal Weierstrass equation when introducing projective variables $[u_0 : u_1 : s_0 : s_1] \in \mathbb{F}_k$ with $k \in \mathbb{N}$, and $[x : y : z] \in \mathbb{WP}_{(2,3,1)}$ and writing each fiber as the hypersurface

$$y^2 z = 4\,x^3 - \mathcal{G}_2\left(\frac{s_0}{s_1\,u_1^k}, t, \frac{u_0}{u_1}\right) s_1^8\,u_1^{4(k+2)}\,x\,z^2 - \mathcal{G}_3\left(\frac{s_0}{s_1\,u_1^k}, t, \frac{u_0}{u_1}\right) s_1^{12}\,u_1^{6(k+2)}\,z^3. \tag{7.2}$$

The coordinates $[x : y : z] \in \mathbb{WP}_{(2,3,1)}$ are sections of line bundles over the base, and we can specify which bundles by means of their transformation properties under the group $(\lambda_2, \lambda_3) \in \mathbb{C}^* \times \mathbb{C}^*$. There is also a third obvious action $\lambda_1 \in \mathbb{C}^*$ in each fiber $\mathbb{WP}_{(2,3,1)}$. If the condition is satisfied that the total weight of Equation (7.2) equals the sum of weights of the defining variables for each acting $\mathbb{C}^*$ in Equation (7.2), then a Calabi-Yau threefold is obtained by removing the loci $\{s_0 = s_1 = 0\}$, $\{u_0 = u_1 = 0\}$, $\{x = y = z = 0\}$ from the solution set of Equation (7.1), and taking the quotient $(\mathbb{C}^*)^3$. The three $\mathbb{C}^*$-group actions on the defining variables in Equation (7.2) are given by the weights listed in Table 16 where *deg* denotes the total weight of Equation (7.2) and *sum* denotes the sum of weights of the defining variables. The details of this construction were explained in [58]. Notice that when re-writing Equation (7.1) in the form (7.2) we insist on obtaining a *minimal* Weierstrass fibration, and this will put restrictions on the admissible polynomials $\mathcal{G}_2, \mathcal{G}_3$ which allow for such an extension. Therefore, not every Weierstrass model will extend to all of the aforementioned surfaces and after constructing a Weierstrass model we will always have to check the admissible complex base surfaces it extends to.[6]

$\mathbb{C}^*$	deg	x	y	z	s_0	s_1	u_0	u_1	sum
λ_1	3	1	1	1	0	0	0	0	3
λ_2	12	4	6	0	1	1	0	0	12
λ_3	$6(k+2)$	$2(k+2)$	$3(k+2)$	0	k	0	1	1	$6(k+2)$

TABLE 16. Weights of variables in Weierstrass equation

7.2. **Elliptic threefolds by the pure-twist construction.** In this section we will describe the *pure-twist construction* for families of elliptic Calabi-Yau threefolds. There will be two cases that we consider, the so-called *hypergeometric case* and the *extra case*.

7.2.1. *The hypergeometric case.* The construction of eight families of elliptic threefolds in the hypergeometric case is achieved by the pure-twist construction applied to the families of Jacobian elliptic K3 surfaces from Section 6.1.1. In particular, for every family of Jacobian elliptic K3 surfaces given in Lemma 6.2, we define a family of Gorenstein threefolds as Weierstrass models by mapping $t \mapsto t\,s$ and carrying out a quadratic twist. Thus, we obtain new Weierstrass coefficients $\mathcal{G}_2$ and $\mathcal{G}_3$ by setting

$$\begin{aligned} t &\mapsto t\,s \\ G_2(t,u) &\mapsto \mathcal{G}_2(s,t,u) = G_2(t\,s,u)\Big[s\,(s-1)\Big]^2 \\ G_3(t,u) &\mapsto \mathcal{G}_3(s,t,u) = G_3(t\,s,u)\Big[s\,(s-1)\Big]^3 \end{aligned} \tag{7.3}$$

in Equation (7.1) where G_2 and G_3 were defined in Equation (6.2). We then have the following lemma:

Lemma 7.1. *For* $(n,\mu) \in \Big\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\Big\}$ *the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{G_2(t\,s,u)\Big[s\,(s-1)\Big]^2}_{=:\,\mathcal{G}_2(s,t,u)} X - \underbrace{G_3(t\,s,u)\Big[s\,(s-1)\Big]^3}_{=:\,\mathcal{G}_3(s,t,u)} \tag{7.4}$$

with $G_2(t,u)$ and $G_3(t,u)$ given in Lemma 6.1 define families of elliptic Calabi-Yau threefolds over projective complex surfaces. The possible base surfaces for each family of Weierstrass models are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_k$ with $1 \le k < 1/\mu$. The torsion part of the Mordell-Weil group is two-torsion for $n = 2,4$ and trivial for $n = 1,3$.

Proof. One can easily check what complex projective surfaces the Weierstrass model (7.4) extends to – such that s and u in Equation (7.1) become complex affine coordinates – by using the transformations provided in Section 7.1. The only statement that remains to be proved is the statement about the torsion part of the Mordell-Weil group. All torsion sections for the families in Equation (7.4) are listed in Table 17. □

[6] A similar analysis can be carried out if the base surface is $\mathbb{P}^2$

μ	torsion	sections
$\frac{1}{2}$	[2]	$(X,Y) \;=\; \left(\frac{2}{3}(stu-2)\left(u^2-1\right)\left(s^2-1\right),0\right)$
$\frac{1}{3}$	[1]	–
$\frac{1}{4}$	[2]	$(X,Y) \;=\; \left(\frac{2}{3}\,u\,(u-1)\;s\,(s-1)\,,0\right)$
$\frac{1}{6}$	[1]	–

Table 17. Torsions sections of Equation (7.4)

Similarly, for the families of Jacobian elliptic K3 surfaces defined in Lemma 6.10 we define families of Gorenstein threefolds as Weierstrass models by mapping $t \mapsto t\,s$ and carrying out a different quadratic twist. We obtain new Weierstrass coefficients $\mathcal{G}_2$ and $\mathcal{G}_3$ by setting

$$
\begin{aligned}
t &\mapsto t\,s\,,\\
G_2(t,u) &\mapsto \mathcal{G}_2(s,t,u) = G_2(t\,s,u)\left[s^2-1\right]^2\,,\\
G_3(t,u) &\mapsto \mathcal{G}_3(s,t,u) = G_3(t\,s,u)\left[s^2-1\right]^3
\end{aligned}
\tag{7.5}
$$

in Equation (7.1) where G_2 and G_3 were defined in Equation (6.34).

Lemma 7.2. *For $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ the Weierstrass equations*

$$
Y^2 = 4\,X^3 - \underbrace{G_2(t\,s,u)\left[s^2-1\right]^2}_{=:\,\mathcal{G}_2(s,t,u)} X - \underbrace{G_3(t\,s,u)\left[s^2-1\right]^3}_{=:\,\mathcal{G}_3(s,t,u)}
\tag{7.6}
$$

with $G_2(t,u)$ and $G_3(t,u)$ given in Lemma 6.10 define families of elliptic Calabi-Yau threefolds over projective complex surfaces. The possible base surfaces for each family of Weierstrass models are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_k$ with $1 \le k < 1/\mu$. The torsion part of Mordell-Weil group is two-torsion for $n = 2,4$ and trivial for $n = 1,3$.

Proof. One can easily check what complex projective surfaces the Weierstrass model (7.6) extends to – such that s and u in Equation (7.1) become complex affine coordinates – by using the transformations from Section 7.1. The only statement that remains to be proved is the statement about the torsion part of the Mordell-Weil group. All torsion sections for the families in Equation (7.6) are listed in Table 18. □

μ	torsion	sections
$\frac{1}{2}$	[2]	$(X,Y) \;=\; \left(\frac{2}{3}\,(stu-2)\,u\,(u-1)\,s\,(s-1)\,,0\right)$
$\frac{1}{3}$	[1]	–
$\frac{1}{4}$	[2]	$(X,Y) \;=\; \left(\frac{2}{3}\left(u^2-1\right)\left(s^2-1\right),0\right)$
$\frac{1}{6}$	[1]	–

Table 18. Torsions sections of Equation (7.6)

7.2.2. *The extra case.* In the extra case, the construction of families of threefolds as Weierstrass models is based on the pure-twist construction applied to the families of Jacobian elliptic K3 surfaces with M_n-polarization for $n = 1, \dots, 4$ and additional rational transformation of the family parameter from Section 6.2.3. We define families of Gorenstein threefolds as Weierstrass models by mapping $t \mapsto s\,t$ and carrying out a quadratic twist. We obtain new Weierstrass coefficients $\mathcal{G}_2^{(\tilde{m})}$ and $\mathcal{G}_3^{(\tilde{m})}$ by setting

$$
\begin{aligned}
t &\mapsto s\,t\,, \\
G_2^{(\tilde{m})}(t,u) \mapsto \mathcal{G}_2^{(\tilde{m})}(s,t,u) &= G_2^{(\tilde{m})}(s\,t,u)\ \Big(s(s-1)\Big)^2\,, \\
G_3^{(\tilde{m})}(t,u) \mapsto \mathcal{G}_3^{(\tilde{m})}(s,t,u) &= G_3^{(\tilde{m})}(s\,t,u)\ \Big(s(s-1)\Big)^3
\end{aligned}
\tag{7.7}
$$

in Equation (7.10) where $G_2^{(\tilde{m})}$ and $G_3^{(\tilde{m})}$ with $\tilde{m} = 1, \dots, 5$ were defined in Lemma 6.7. We then have the following lemma:

Lemma 7.3. *For* $(n, \mu) \in \Big\{(1, \frac{1}{6}), (2, \frac{1}{4}), (3, \frac{1}{3}), (4, \frac{1}{2})\Big\}$ *the Weierstrass equations*

$$
Y^2 = 4\,X^3 - \underbrace{G_2^{(\tilde{m})}(s\,t,u)\ \Big(s\,(s-1)\Big)^2}_{=:\,\mathcal{G}_2^{(\tilde{m})}(s,t,u)}\ X - \underbrace{G_3^{(\tilde{m})}(s\,t,u)\ \Big(s\,(s-1)\Big)^3}_{=:\,\mathcal{G}_3^{(\tilde{m})}(s,t,u)}
\tag{7.8}
$$

with $G_2^{(\tilde{m})}(t,u)$ *and* $G_3^{(\tilde{m})}(t,u)$ *for* $1 \le \tilde{m} \le 5$ *taken from Equations (6.23)–(6.27) define families of elliptic Calabi-Yau threefolds over* $\mathbb{P}^1 \times \mathbb{P}^1$*, in the case* $\tilde{m} = 3$ *also over* $\mathbb{P}^2$ *and* $\mathbb{F}_1$*. The torsion part of the Mordell-Weil group is two-torsion for* $n = 2, 4$ *and trivial for* $n = 1, 3$*.*

Proof. One can easily check what complex projective surfaces the Weierstrass model (7.6) extends to – such that s and u in Equation (7.1) become complex affine coordinates – by using the transformation formulas from Section 7.1. The only statement that needs to be proved is the statement about the torsion part of the Mordell-Weil group. The transformations in (7.9) applied to the sections in Table 13 give explicit expressions for the various torsion sections.

(7.9)

$\tilde{m}$	transformation		
1	(t, X, Y)	$\mapsto$	$\Big(s\,t, X\,(1-s\,t)\,s\,(s-1), Y\,((1-s\,t)\,s\,(s-1))^{\frac{3}{2}}\Big)$
2	(t, X, Y)	$\mapsto$	$\Big(\frac{s\,t}{s\,t-1}, X\,(1-s\,t)\,s\,(s-1), Y\,((1-s\,t)\,s\,(s-1))^{\frac{3}{2}}\Big)$
3	(t, X, Y)	$\mapsto$	$\Big(4\,s\,t\,(1-s\,t), X\,s\,(s-1), Y\,(s\,(s-1))^{\frac{3}{2}}\Big)$
4	(t, X, Y)	$\mapsto$	$\Big(\frac{s^2\,t^2}{4\,(s\,t-1)}, X\,(1-s\,t)\,s\,(s-1), Y\,((1-s\,t)\,s\,(s-1))^{\frac{3}{2}}\Big)$
5	(t, X, Y)	$\mapsto$	$\Big(-\frac{4\,s\,t}{(1-s\,t)^2}, X\,(1-s\,t)^2\,s\,(s-1), Y\,(1-s\,t)^3(s\,(s-1))^{\frac{3}{2}}\Big)$

It follows that the transformations in Equation (7.9) can yield polynomial expressions in the affine coordinates only for two-torsion, i.e., if $Y = 0$. □

7.3. **Elliptic threefolds by the mixed-twist construction.** In this section, we will describe the *mixed-twist construction* for families of elliptic Calabi-Yau threefolds. We will only consider the *hypergeometric case*.

7.3.1. *The hypergeometric case.* In the previous section, we already constructed several families of elliptic Calabi-Yau threefolds that fall into the so-called hypergeometric case. However, a complete construction of all families of elliptic Calabi-Yau threefolds in the hypergeometric case can be only achieved by the mixed-twist construction with generalized functional invariant (k, l, β) applied to the families of Jacobian elliptic K3 surfaces with M_n-polarization of Picard rank 19. For two positive integers (k, l), we define a ramified map $s \mapsto c_{kl}/[s^k(s+1)^l]$ of degree $\deg = k + l$ from $\mathbb{P}^1 \to \mathbb{P}^1$ where $c_{kl} = (-1)^k\, k^k\, l^l/(k+l)^{k+l}$. The map is totally ramified over 0, has a simple ramification point over 1, and is ramified to degrees k and l over ∞.

For each family of Jacobian elliptic K3 surfaces with M_n-polarization defined in Section 6.1 we can then define families of Gorenstein threefolds as Weierstrass models by mapping $t \mapsto \frac{c_{kl}\, t}{s^k(s+1)^l}$ and carrying out a quadratic twist. In this way, we obtain new Weierstrass coefficients $\mathcal{G}_2$ and $\mathcal{G}_3$ by setting

$$
\begin{aligned}
t &\mapsto \frac{c_{kl}\, t}{s^k(s+1)^l}\;, \\
G_2(t,u) \mapsto \mathcal{G}_2(s,t,u) &= G_2\left(\frac{c_{kl}\, t}{s^k(s+1)^l}, u\right)\left(s(s+1)^\beta\right)^4, \\
G_3(t,u) \mapsto \mathcal{G}_3(s,t,u) &= G_3\left(\frac{c_{kl}\, t}{s^k(s+1)^l}, u\right)\left(s(s+1)^\beta\right)^6,
\end{aligned}
\tag{7.10}
$$

in Equation (7.1) where G_2 and G_3 were defined in Lemma 6.2. To ensure clearing of the denominators in Equation (7.10) as well as minimal Weierstrass fibrations we impose the constraints $1 \le k \le 1/\mu$ and $1 \le l \le \beta/\mu$ with $\beta \in \{\frac{1}{2}, 1\}$ and $(n,\mu) \in \{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\}$. We have the following lemma:

Lemma 7.4. *For* $(n,\mu) \in \left\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\right\}$ *the Weierstrass equations*

$$
Y^2 = 4\,X^3 - \underbrace{G_2\left(\frac{c_{kl}\, t}{s^k(s+1)^l}, u\right)\left(s(s+1)^\beta\right)^4}_{=:\,\mathcal{G}_2(s,t,u)} X - \underbrace{G_3\left(\frac{c_{kl}\, t}{s^k(s+1)^l}, u\right)\left(s(s+1)^\beta\right)^6}_{=:\,\mathcal{G}_3(s,t,u)}
\tag{7.11}
$$

with $G_2(t,u)$ *and* $G_3(t,u)$ *given in Lemma 6.2 and* $1 \le k \le 1/\mu$ *and* $1 \le l \le \beta/\mu$ *with* $\beta \in \{\frac{1}{2}, 1\}$ *define families of elliptic Calabi-Yau threefolds over* $\mathbb{P}^1 \times \mathbb{P}^1$*. For* $\beta = 1$*, the torsion part of the Mordell-Weil group is* n*-torsion* $\mathbb{Z}/n\mathbb{Z}$*. For* $\beta = 1/2$*, the torsion part of the Mordell-Weil group is two-torsion for* $n = 2, 4$ *and trivial for* $n = 1, 3$*.*

Proof. The only statement that needs to be proved is the statement about the torsion part of the Mordell-Weil group. The transformation

$$
\left(t, X, Y\right) \mapsto \left(\frac{c_{kl}\, t}{s^k(s+1)^l}, X\left(s(s+1)^\beta\right)^2, Y\left(s(s+1)^\beta\right)^3\right)
\tag{7.12}
$$

applied to the sections in Table 8 gives the explicit expressions for the various torsion sections. For $\beta = 1/2$, transformation (7.12) is only well-defined for two-torsion. □

The construction of five additional families of elliptic Calabi-Yau threefolds in the hypergeometric case is based on the mixed-twist construction applied to the families of Jacobian elliptic K3 surfaces of Picard rank 18. For a positive integer m, we define a ramified map $s \mapsto (1+s^2)^m/(2s)^m$ of degree deg $= 2m$ from $X_g = \mathbb{P}^1 \to X'_{g'} = \mathbb{P}^1$. The maps has a branching point at 0 with corresponding ramification points $s = \pm i$ in the pre-image with branch number $m-1$ each, a branching point at ∞ with corresponding ramification points $s = 0$ and $s = \infty$ in the pre-image with branch number $m-1$ each. For m even, we have only one more branching point at 1 with corresponding ramification points $s = \pm 1$ in the pre-image with branch number 1 each. For m odd, we have a branching point at 1 with corresponding ramification point $s = 1$ in the pre-image with branch number 1, and a branching point at -1 with corresponding ramification point $s = -1$ in the pre-image with branch number 1. It follows that the Riemann-Hurwitz formula $g = B/2 + \deg \cdot g' - \deg + 1$ is satisfied for $g = g' = 0$, $B = 4(m-1)+2$, $\deg = 2m$. Therefore, the map is indeed a ramified map from $\mathbb{P}^1$ to $\mathbb{P}^1$.

For each family of Jacobian elliptic K3 surfaces with lattice polarization $\tilde{M}'$, M', $\tilde{M}$, M defined in Lemma 6.11 we define families of threefolds as Weierstrass models by mapping $t \mapsto \frac{t\,(1+s^2)^m}{(2s)^m}$ and carrying out a quadratic twist. We obtain new Weierstrass coefficients $\mathcal{G}_2$ and $\mathcal{G}_3$ by setting

$$
\begin{aligned}
t &\mapsto \frac{t\,(1+s^2)^m}{(2s)^m}\,, \\
G_2(t,u) &\mapsto \mathcal{G}_2(s,t,u) = G_2\left(\frac{t\,(1+s^2)^m}{(2s)^m}, u\right)\, s^4\,, \qquad (7.13)\\
G_3(t,u) &\mapsto \mathcal{G}_3(s,t,u) = G_3\left(\frac{t\,(1+s^2)^m}{(2s)^m}, u\right)\, s^6\,,
\end{aligned}
$$

in Equation (7.1) where G_2 and G_3 were defined in Lemma 6.11. To ensure clearing of the denominators in Equation (7.10) as well as minimal Weierstrass fibrations we impose the constraint $1 \le m \le 1/\mu$ for $(\Lambda,\mu) \in \{(M,\frac{1}{6}),(\tilde{M},\frac{1}{4}),(M',\frac{1}{3}),(\tilde{M}',\frac{1}{2})\}$. We also impose the condition that m is odd. We then have the following lemma:

Lemma 7.5. *For* $(\Lambda,\mu) \in \left\{(M,\frac{1}{6}),(\tilde{M},\frac{1}{4}),(M',\frac{1}{3}),(\tilde{M}',\frac{1}{2})\right\}$ *the Weierstrass equations*

$$
Y^2 = 4\,X^3 - \underbrace{G_2\left(\frac{t\,(1+s^2)^m}{(2s)^m}, u\right)\, s^4}_{=:\,\mathcal{G}_2(s,t,u)}\, X - \underbrace{G_3\left(\frac{t\,(1+s^2)^m}{(2s)^m}, u\right)\, s^6}_{=:\,\mathcal{G}_3(s,t,u)} \qquad (7.14)
$$

with $G_2(t,u)$ *and* $G_3(t,u)$ *given in Lemma 6.11 and* $1 \le m \le 1/\mu$, *m odd, define families of elliptic Calabi-Yau threefolds over* $\mathbb{P}^1 \times \mathbb{P}^1$. *The torsion part of the Mordell-Weil group is n-torsion* $\mathbb{Z}/n\mathbb{Z}$.

Proof. The only statement that needs to be proved is the statement about the torsion part of the Mordell-Weil group. The transformation

$$
\Big(t, X, Y\Big) \mapsto \left(\frac{t\,(1+s^2)^m}{(2s)^m}, X\,s^2, Y\,s^3\right) \qquad (7.15)
$$

applied to elements in Table 15 gives the explicit expressions for the various torsion sections. □

Remark. The origin of the constraint that m must be odd can be understood as follows: the transformation in Equation (7.13) is in fact a mixed-twist construction with generalized functional invariant $(m/2, m/2, 1/2)$. Concretely, if we set $k = l = m/2, \beta = 1$ and $s = -\frac{S^{1\pm1}}{1+S^2}$ in Equation (7.10) we obtain

(7.16)
$$
\begin{aligned}
t &\mapsto \frac{c_{\frac{m}{2},\frac{m}{2}}\, t}{s^{\frac{m}{2}}(s+1)^{\frac{m}{2}}} = \frac{t\,(1+S^2)^m}{(2S)^m} ,\\
G_2(t,u) \mapsto \mathcal{G}_2(s,t,u) &= G_2\left(\frac{c_{\frac{m}{2},\frac{m}{2}}\, t}{s^{\frac{m}{2}}(s+1)^{\frac{m}{2}}}, u\right) \Big(s(s+1)\Big)^4 = G_2\left(\frac{t\,(1+S^2)^m}{(2S)^m}, u\right) \frac{S^8}{(S+1)^8} ,\\
G_3(t,u) \mapsto \mathcal{G}_3(s,t,u) &= G_3\left(\frac{c_{\frac{m}{2},\frac{m}{2}}\, t}{s^{\frac{m}{2}}(s+1)^{\frac{m}{2}}}, u\right) \Big(s(s+1)\Big)^6 = G_3\left(\frac{t\,(1+S^2)^m}{(2S)^m}, u\right) \frac{S^{12}}{(S+1)^{12}} ,
\end{aligned}
$$

which is transformation in Equation (7.13) up to an additional quadratic twist.

7.4. **Elliptic threefolds by the product-twist construction.** The *product-twist* construction is achieved by building a Weierstrass model of a Gorenstein threefold directly from two Jacobian extremal rational elliptic surfaces that each have a star-fiber. In this section, we will describe the product-twist construction for families of elliptic Calabi-Yau threefolds. There will be two cases to be considered, the so-called *hypergeometric case* and the *even case*.

7.4.1. *The hypergeometric case.* In the hypergeometric case, we obtain new Weierstrass coefficients in Equation (7.10) by setting

(7.17)
$$
\begin{aligned}
\mathcal{G}_2(s,t,u) &= s^4\, g_2^{(\mu')}\left(\frac{t}{s}\right) F^4 ,\\
\mathcal{G}_3(s,t,u) &= s^6\, g_3^{(\mu')}\left(\frac{t}{s}\right) F^6 ,
\end{aligned}
$$

where we have set

(7.18)
$$F^2 = 4\,u^3 - g_2^{(\mu)}(s)\; u - g_3^{(\mu)}(s) \;,$$

and $(g_2^{(\mu)}, g_3^{(\mu)})$ and $(g_2^{(\mu')}, g_3^{(\mu')})$ are two pairs (g_2, g_3) taken from Table 5. Notice that only F^2 in Equation (7.18) is a polynomial in s and u. F itself is only well-defined on a branched double cover with (u, F) being elliptic coordinates of a family of elliptic curves with parameter s.
We then have the following lemma:

Lemma 7.6. *For* $\mu' \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$ *and* $(n,\mu) \in \left\{(1,\frac{1}{6}), (2,\frac{1}{4}), (3,\frac{1}{3}), (4,\frac{1}{2})\right\}$*, the Weierstrass equations*

(7.19)
$$Y^2 = 4\,X^3 - \underbrace{s^4\, g_2^{(\mu')}\left(\frac{t}{s}\right) F^4}_{=:\,\mathcal{G}_2(s,t,u)} X - \underbrace{s^6\, g_3^{(\mu')}\left(\frac{t}{s}\right) F^6}_{=:\,\mathcal{G}_3(s,t,u)}$$

with $F^2 = 4\,u^3 - g_2^{(\mu)}(s)\; u - g_3^{(\mu)}(s)$ *and* $(g_2^{(\mu)}, g_3^{(\mu)}), (g_2^{(\mu')}, g_3^{(\mu')})$ *taken from Table 5, define families of elliptic Calabi-Yau threefolds. The possible base surfaces for each family of Weierstrass models are* $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, *and* $\mathbb{F}_k$ *with* $1 \leq k \leq 5 - n$ *for* $n \leq 3$ *and* $\mathbb{P}^2$ *for* $n = 4$.

Proof. One can easily check what complex projective surfaces the Weierstrass model (7.6) extends to – such that s and u in Equation (7.1) become complex affine coordinates – by using the transformations from Section 7.1. □

7.4.2. *The even case.* In the hypergeometric case, we obtain new Weierstrass coefficients in Equation (7.10) by setting

$$\begin{aligned} \mathcal{G}_2(s,t,u) &= s^4\,\tilde{g}_2^{(\tilde{\mu})}\left(\frac{t}{s}\right)\,F^4\;, \\ \mathcal{G}_3(s,t,u) &= s^6\,\tilde{g}_3^{(\tilde{\mu})}\left(\frac{t}{s}\right)\,F^6\;, \end{aligned} \tag{7.20}$$

where we have set

$$F^2 = 4\,u^3 - g_2^{(\mu)}(s)\;u - g_3^{(\mu)}(s)\;\;, \tag{7.21}$$

where $g_2, \tilde{g}_2$ and $g_3, \tilde{g}_3$ are taken from Table 5. We have the following lemma:

Lemma 7.7. *For $\tilde{\mu} \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$ and $(n,\mu) \in \left\{(1,\frac{1}{6}), (2,\frac{1}{4}), (3,\frac{1}{3}), (4,\frac{1}{2})\right\}$, the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{s^4\,\tilde{g}_2^{(\tilde{\mu})}\left(\frac{t}{s}\right)\,F^4}_{=:\,\mathcal{G}_2(s,t,u)}\,X - \underbrace{s^6\,\tilde{g}_3^{(\tilde{\mu})}\left(\frac{t}{s}\right)\,F^6}_{=:\,\mathcal{G}_3(s,t,u)} \tag{7.22}$$

with $F^2 = 4\,u^3 - g_2^{(\mu)}(s)\;u - g_3^{(\mu)}(s)$ and $(g_2^{(\mu)}, g_3^{\mu)}), (\tilde{g}_2^{(\tilde{\mu})}, \tilde{g}_3^{(\tilde{\mu})})$ taken from Table 5, define families of elliptic Calami-Yau threefolds. The possible base surfaces for each family of Weierstrass models are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_k$ with $1 \le k \le 5-n$ for $n \le 3$ and $\mathbb{P}^2$ for $n = 4$.

Proof. One can easily derive what complex projective surfaces the Weierstrass model (7.6) extends to – such that s and u in Equation (7.1) become complex affine coordinates – by using the transformations from Section 7.1. □

7.5. **Period computation for families of Calabi-Yau threefolds.** In this section we will compute the Picard-Fuchs operators and holomorphic periods for all families of Calabi-Yau threefolds constructed in the previous section.

7.5.1. *Pure-twist construction.* In this section we will compute the Picard-Fuchs operators and holomorphic periods for all families of elliptic Calabi-Yau threefolds obtained by the mixed-twist construction.

The hypergeometric case. In Section 6.2.1 we constructed families of closed two-cycles $\Sigma_2 = \Sigma_2(t)$ for the families of K3 surfaces of Picard rank 19 from Lemma 6.1. We now iterate this construction to obtain families of closed three-cycles Σ_3 on the total space of Equation (7.1) where $\mathcal{G}_2$ and $\mathcal{G}_3$ were obtained in Equation (7.3): for each $t \in D_1^*(0)$, we follow a Pochhammer contour $\Sigma_1 = \Sigma_1(t)$ around $s = 0$ and $s = 1$ in the affine s-plane such that $0 < |st| < 1$. For each $s \in \Sigma_1(t)$, we trace out the cycle in the (u, X, Y)-coordinates of the K3 fiber that is obtained from $\Sigma_2(st)$ defined in Section 6.2.1 after rescaling $(X, Y) \to (s\,(s-1)\,X, (s\,(s-1))^{3/2}\,Y)$ to account for the quadratic twist. For each $t \in D_1^*(0)$, we obtain a continuously varying family of closed three-cycles that we denote as warped product $\Sigma_3 = \Sigma_1 \times_s \Sigma_2$.

We can evaluate the period integral for the holomorphic two $ds/2 \wedge du/2 \wedge dX/Y$ over the three-cycle Σ_3, i.e.,

$$\omega_3 = \frac{1}{4} \oiiint_{\Sigma_3} ds \wedge du \wedge \frac{dX}{Y} \,. \tag{7.23}$$

We then have the following lemma:

Lemma 7.8. *For $t \in D_1^*(0)$ the period of the holomorphic three-form $ds/2 \wedge du/2 \wedge \frac{dX}{Y}$ over the three-cycle Σ_3 for the families of Weierstrass models from Lemma 7.1 is given by*

$$\omega_3^{(\mu)} = (2\pi i)^3 \; {}_4F_3\left(\begin{matrix} \mu, \frac{1}{2}, \frac{1}{2}, 1-\mu \\ 1, 1, 1 \end{matrix} \middle| \, t \right) . \tag{7.24}$$

Proof. To integrate the holomorphic three-form over a three-cycle Σ_3 we use Fubini's theorem. Integrating over the A-cycle first, we obtain

$$\frac{1}{2\pi i} \oiiint_{\Sigma_3} ds \wedge du \wedge \frac{dX}{Y} = \oint_{\Sigma_1(t)} \frac{ds}{\sqrt{s\,(s-1)}} \oint_{\Sigma_1(st)} \frac{du}{\sqrt{u\,(u-1)}} \; {}_2F_1\left(\begin{matrix} \mu, 1-\mu \\ 1 \end{matrix} \middle| \, s\,t\,u \right) .$$

Recalling that $|stu| < 1$ for all $u \in \Sigma_1(st)$, the series expansion of the hypergeometric function converges absolutely and uniformly. Using Equation (6.8), we obtain

$$\frac{1}{2\pi i} \oiiint_{\Sigma_3} ds \wedge du \wedge \frac{dX}{Y} = (4\pi i) \oint_{\Sigma_1(t)} \frac{ds}{\sqrt{s\,(s-1)}} \; {}_3F_2\left(\begin{matrix} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{matrix} \middle| \, s\,t \right) . \tag{7.25}$$

Using $|st| < 1$ for all $s \in \Sigma_1(t)$, one repeats the integration step and obtains for the period integral

$$(4\pi i)^2 \; {}_1F_0\left(\frac{1}{2} \middle| \, t \right) \star \; {}_3F_2\left(\begin{matrix} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{matrix} \middle| \, t \right) = (4\pi i)^2 \; {}_4F_3\left(\begin{matrix} \mu, \frac{1}{2}, \frac{1}{2}, 1-\mu \\ 1, 1, 1 \end{matrix} \middle| \, t \right) .$$

□

A well-known result for hypergeometric functions then yields the following corollary.

Corollary 7.9. *The period $\omega_3^{(\mu)}$ in Equation (7.24) is the solution of the differential equation $L_4^{(\mu)} \, \omega_3^{(\mu)} = 0$ that is holomorphic at $t = 0$ with*

$$L_4^{(\mu)} = \theta^4 - t \left(\theta + \frac{1}{2}\right)^2 \left(\theta + \mu\right) \left(\theta + 1 - \mu\right) . \tag{7.26}$$

□

In Section 6.4.1 we constructed families of closed two-cycles $\hat{\Sigma}_2 = \hat{\Sigma}_2(t)$ for the families of K3 surfaces of Picard rank 18 from Lemma 6.10. We now iterate this construction to obtain families of closed three-cycles $\hat{\Sigma}_3$ on the total space of Equation (7.1) where $\mathcal{G}_2$ and $\mathcal{G}_3$ were obtained in Equation (7.5): for each $t \in D_1^*(0)$, we follow a Pochhammer contour $\hat{\Sigma}_1$ invariant under $s \mapsto -s$ around $s = -1$ and $s = 1$ in the affine s-plane such that $0 < |st| < 1$. For each $s \in \hat{\Sigma}_1(t)$, we trace out the cycle in the (u, X, Y)-coordinates of the K3-fiber that is obtained from the cycle $\hat{\Sigma}_2(st)$ defined Section 6.4.1 after rescaling $(X, Y) \to \big((s^2-1)\,X, (s^2-1)^{3/2}\,Y \big)$ to account for the quadratic twist. For $t \in D_1^*(0)$ we obtain a continuously varying family of closed three-cycles that we denote as warped product $\hat{\Sigma}_3 = \hat{\Sigma}_1 \times_s \hat{\Sigma}_2$.

We can now evaluate the period integral for the holomorphic two $ds/2 \wedge du/2 \wedge dX/Y$ over the three-cycle $\hat{\Sigma}_3$, i.e.,

$$\hat{\omega}_3 = \frac{1}{4} \oiiint_{\hat{\Sigma}_3} ds \wedge du \wedge \frac{dX}{Y} \,. \tag{7.27}$$

We then have the following lemma:

Lemma 7.10. *For $t \in D_1^*(0)$ the period of the holomorphic three-form $ds/2 \wedge du/2 \wedge \frac{dX}{Y}$ over the three-cycle $\hat{\Sigma}_3$ for the families of Weierstrass models from Lemma 7.2 is given by*

$$\hat{\omega}_3^{(\mu)} = (2\pi i)^3 \; {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, 1 \end{array} \right| t^2 \right) . \tag{7.28}$$

Proof. To integrate the holomorphic three-form over a three-cycle $\hat{\Sigma}_3$ we use Fubini's theorem. Integrating over the A-cycle first, we obtain

$$\frac{1}{2\pi i} \oiiint_{\hat{\Sigma}_3} ds \wedge du \wedge \frac{dX}{Y} = i^2 \oint_{\hat{\Sigma}_1(t)} \frac{ds}{\sqrt{1-s^2}} \oint_{\hat{\Sigma}_1(st)} \frac{du}{\sqrt{1-u^2}} \; {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| s\,t\,u \right) .$$

Recalling that $|stu| < 1$ for all $u \in \hat{\Sigma}_1(st)$, the series expansion of the hypergeometric function converges absolutely and uniformly. Using Equation (6.42), we obtain

$$\frac{1}{2\pi i} \oiiint_{\hat{\Sigma}_3} ds \wedge du \wedge \frac{dX}{Y} = (4\pi i) \oint_{\hat{\Sigma}_1(t)} \frac{ds}{\sqrt{s^2-1}} \; {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| (st)^2 \right) .$$

We can break up the Pochhammer contour $\hat{\Sigma}_1$ into two parts with $\mathrm{Re}(s) \geq 0$ and $\mathrm{Re}(s) \leq 0$. Using the two branches of the square-root function, the coordinate transformations $s = \sqrt{\tilde{s}}$ and $s = -\sqrt{\tilde{s}}$ transform the right half and the left half into the same cycle $\tilde{\Sigma}_1$ which is itself a Pochhammer contour around $\tilde{s} = 0$ and $\tilde{s} = 1$ with $0 < |\tilde{s}|^{1/2}\,|t| < 1$ for all $\tilde{s} \in \tilde{\Sigma}_1$. But in this case the integrand is the same for both contours. Therefore, we obtain

$$\frac{1}{2\pi i} \oiiint_{\hat{\Sigma}_3} ds \wedge du \wedge \frac{dX}{Y} = (4\pi i) \oint_{\tilde{\Sigma}_1(t)} \frac{d\tilde{s}}{\sqrt{\tilde{s}(\tilde{s}-1)}} \; {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| \tilde{s}\,t^2 \right) .$$

Using $|\tilde{s}t^2| < |\tilde{s}|^{1/2}\,|t| < 1$ for all $\tilde{s} \in \tilde{\Sigma}_1(t)$, one then obtains for the period integral

$${}_1F_0\left(\left. \frac{1}{2} \right| t^2 \right) \star {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| \tilde{s}\,t^2 \right) = {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1, 1, 1 \end{array} \right| t^2 \right) .$$

□

A well-known result for hypergeometric functions then yields the following corollary.

Corollary 7.11. *The period $\hat{\omega}_3^{(\mu)}$ in Equation (7.28) is the solution of the differential equation $\hat{L}_4^{(\mu)}\,\hat{\omega}_4^{(\mu)} = 0$ that is holomorphic at $t = 0$ with*

$$\hat{L}_4^{(\mu)} = \theta^4 - t^2\,(\theta+\mu)\,(\theta+1-\mu)\,(\theta+1+\mu)\,(\theta+2-\mu) \,. \tag{7.29}$$

□

The extra case. In Section 6.2.3 we constructed families of closed two-cycles $\tilde{\Sigma}_2^{\prime\prime\,(\tilde{m})} = \tilde{\Sigma}_2^{\prime\prime\,(\tilde{m})}(t)$ for the families of K3 surfaces of Picard rank 19 from Lemma 6.7. We now define families of closed three-cycles $\tilde{\Sigma}_3^{(\tilde{m})}$ on the total space of Equation (7.1) where $\mathcal{G}_2$ and $\mathcal{G}_3$ were obtained in Equation (7.7): for each $t \in D_1^*(0)$, we follow a Pochhammer contour $\Sigma_1 = \Sigma_1(t)$ around $s = 0$ and $s = 1$ in the affine s-plane such that $0 < |st| < 1$. For each $s \in \Sigma_1(t)$, we trace out the cycle in the (u, X, Y)-coordinates of the K3 fiber that is obtained from $\tilde{\Sigma}_2^{\prime\prime\,(\tilde{m})}(st)$ defined in Section 6.2.1 after rescaling $(X, Y) \to (s\,(s-1)\,X, (s\,(s-1))^{3/2}\,Y)$ to account for the quadratic twist. For each $t \in D_1^*(0)$, we obtain a continuously varying family of closed three-cycles that we denote as warped product $\tilde{\Sigma}_3^{(\tilde{m})} = \Sigma_1 \times_s \tilde{\Sigma}_2^{\prime\prime\,(\tilde{m})}$. We have the following lemma:

Lemma 7.12. *For $t \in D_1^*(0)$ the period of the holomorphic three-form $ds/2 \wedge du/2 \wedge dX/Y$ over the three-cycle $\tilde{\Sigma}_3^{(\tilde{m})}$ for the families of Weierstrass models in Lemma 7.3 is given by the Hadamard product*

$$\tilde{\omega}_3^{(\tilde{p},\tilde{q})} = (2\pi i)^3 \;{}_1F_0\left(\frac{1}{2}\middle|\, t\right) \star \left(\frac{1}{(1-t)^{\frac{1-\tilde{p}-\tilde{q}}{2}}}\,{}_2F_1\left(\begin{matrix}\tilde{p}, \tilde{q}\\ 1\end{matrix}\middle|\, t\right)\right)^2 \tag{7.30}$$

with all possible values $(\tilde{p}, \tilde{q})$ given in Table 19.

Proof. The proof follows exactly the proof of Lemma 7.8 but with ${}_3F_2(\mu, 1-\mu, \frac{1}{2}; 1, 1 \,|\, t)$ replaced by ${}_2F_1(\tilde{p}, \tilde{q}; 1 \,|\, t)^2/(1-t)^{1-\tilde{p}-\tilde{q}}$. □

The following corollary produces the complete list of all 16 Calabi-Yau operators of the so-called *extra case*:

Corollary 7.13. *The period $\tilde{\omega}_3^{(\tilde{p},\tilde{q})}$ in Equation (7.30) is the solution of the differential equation $\tilde{L}_4^{(\tilde{p},\tilde{q})}\,\omega_3^{(\tilde{p},\tilde{q})} = 0$ that is holomorphic at $t = 0$ with*

$$\begin{aligned}\tilde{L}_4^{(\tilde{p},\tilde{q})} = {} & \theta^4 - 2\,t\left(\theta + \frac{1}{2}\right)^2\left(\theta^2 + \theta + 2\,\tilde{p}\,\tilde{q} - \tilde{p} - \tilde{q} + 1\right) \\ & + t^2\left(\theta + \frac{1}{2}\right)\left(\theta + \frac{3}{2}\right)\left(\theta + 1 + \tilde{p} - \tilde{q}\right)\left(\theta + 1 - \tilde{p} + \tilde{q}\right).\end{aligned} \tag{7.31}$$

The operator (7.31) is invariant under the transformation $(\tilde{p}, \tilde{q}) \mapsto (1-\tilde{p}, 1-\tilde{q})$.

□

Remark. The fourth-order and degree-two Calabi-Yau operators obtained in this way together with their classification number (and any alternative name used) in the AESZ database [4] are summarized in Table 19.

#	AESZ	Name	$(\tilde{p},\tilde{q})$	$(M_n,\tilde{m})$
1	(17)	$35, 3^*$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(M_4,3),(M_4,4),(M_4,5)$
2	–	–	$\left(\frac{1}{3},\frac{1}{2}\right)$	$(M_3,4)$
3	(66)	6^*	$\left(\frac{1}{4},\frac{1}{2}\right)$	$(M_2,4)$
4	–	14^*	$\left(\frac{1}{6},\frac{1}{2}\right)$	$(M_1,4)$
5	(39)	4^*	$\left(\frac{1}{3},\frac{1}{3}\right)$	$(M_3,5)$
6	(20)	$46, 4^{**}$	$\left(\frac{1}{3},\frac{2}{3}\right)$	$(M_3,3)$
7	(45)	8^*	$\left(\frac{1}{6},\frac{1}{3}\right)$	$(M_3,1)$
8	(34)	8^{**}	$\left(\frac{1}{6},\frac{2}{3}\right)$	$(M_3,2)$
9	(38)	10^*	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(M_2,5),(M_4,1)$
10	(32)	$111, 10^{**}$	$\left(\frac{1}{4},\frac{3}{4}\right)$	$(M_2,3),(M_4,2)$
11	(40)	13^*	$\left(\frac{1}{6},\frac{1}{6}\right)$	$(M_1,5)$
12	(21)	$47, 13^{**}$	$\left(\frac{1}{6},\frac{5}{6}\right)$	$(M_1,3)$
13	(44)	7^*	$\left(\frac{1}{8},\frac{3}{8}\right)$	$(M_2,1)$
14	(41)	7^{**}	$\left(\frac{1}{8},\frac{5}{8}\right)$	$(M_2,2)$
15	(43)	9^*	$\left(\frac{1}{12},\frac{5}{12}\right)$	$(M_1,1)$
16	(42)	9^{**}	$\left(\frac{1}{12},\frac{7}{12}\right)$	$(M_1,2)$

TABLE 19. The extra case by the pure-twist construction

7.5.2. *Mixed-twist construction.* In this section we will compute the Picard-Fuchs operators and holomorphic periods for all families of elliptic Calabi-Yau threefolds obtained by the mixed-twist construction.

We define a family of closed three-cycles Σ_3'' for the families of Calabi-Yau threefolds in Equation (7.11) as follows: for each $t \in D_1^*(0)$ and generalized functional invariant (k,l,β), we follow the positively oriented circle $C_r(0) : |s| = r = ((|t|/(1+k/l)^l)^{1/k}+1)/2$ in the affine s-plane. For each s, we trace out the cycle in the (u,X,Y)-coordinates of the K3 fiber that is obtained from the cycle $\Sigma_2''(t\, c_{kl}/(s^k\,(s+1)^l))$ defined in Section 6.2.2 after rescaling $(X,Y) \to (s^2\,(s+1)^{2\beta}\,X, s^3\,(s+1)^{3\beta}\,Y)$ to account for the twist. For $t \in D_1^*(0)$ we obtain a continuously varying family of closed three-cycles that we denote as warped product $\Sigma_3'' = \Sigma_1' \times_s \Sigma_2''$.
We have the following lemma:

Lemma 7.14. *For $t \in D_1^*(0)$ the period of the holomorphic three-form $ds \wedge du/2 \wedge \frac{dX}{Y}$ over the three-cycle Σ_3'' for the families of Weierstrass models in Lemma 7.4 is given by the Hadamard product*

$$\omega_3^{(k,l,\beta),(\mu)} = (2\pi i)^3 \; {}_{k+l}F_{k+l-1}\left(\begin{array}{c} \frac{\beta}{k+l}, \quad \ldots \quad \frac{\beta+k+l-1}{k+l} \\ \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{\beta}{l}, \ldots, \frac{\beta+l-1}{l} \end{array}\middle|\, t\right) \star {}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{array}\middle|\, t\right) . \tag{7.32}$$

Proof. The proof follows closely the proofs of Lemmas 6.6 and 5.5. We assume that the holomorphic period of $du/2 \wedge dX/Y$ for the family of K3 surfaces is a complex analytic function of the form $\omega(t) = \sum_{n\geq 0} \omega_n\, t^n$ for $|t| < 1$.

We evaluate the period of the holomorphic three-from $ds \wedge du/2 \wedge dX/Y$ over the three-cycle Σ_3''. To evaluate the holomorphic period by a residue computation we recall that

$$\left| -\frac{t\, c_{kl}}{4\,u\,(u+1)\,s^k\,(s+1)^k} \right|, \; \left| \frac{t\, c_{kl}}{s^k\,(s+1)^l} \right|, \; |t| < 1 .$$

We obtain for the period of the three-form:

$$\begin{aligned}
&\frac{1}{(2\pi i)^3}\oiiint_{\Sigma_3''} ds\wedge du\wedge\frac{dX}{2Y} = \frac{1}{(2\pi i)}\oint_{C_r(0)}\frac{ds}{s(s+1)^\beta}\,\omega\left(\frac{t\,c_{kl}}{s^k\,(s+1)^l}\right)\\
&= \frac{1}{(2\pi i)}\sum_{n=0}^\infty (t\,c_{kl})^n \oint_{C_r(0)}\frac{ds}{s^{1+kn}(1+s)^{ln+\beta}}\,\omega_n\\
&= \sum_{n=0}^\infty \left((-1)^k\,t\,c_{kl}\right)^n \frac{\Gamma((k+l)n+\beta)}{\Gamma(kn+1)\,\Gamma(ln+\beta)}\,\omega_n \qquad (7.33)\\
&= \sum_{n=0}^\infty \left((-1)^k\,t\,c_{kl}\,\frac{(k+l)^{k+l}}{k^k\,l^l}\right)^n \frac{\prod_{p=0}^{k+l-1}\left(\frac{\beta}{k+l}+\frac{p}{k+l}\right)_n}{\prod_{p'=1}^{k-1}\left(\frac{p'}{k}\right)_n\,\prod_{p''=0}^{l-1}\left(\frac{\beta+p''}{l}\right)_n\;n!}\,\omega_n\\
&= {}_{k+l}F_{k+l-1}\left(\left.\begin{array}{c}\frac{\beta}{k+l},\;\ldots\;\frac{\beta+k+l-1}{k+l}\\ \frac{1}{k},\ldots,\frac{k-1}{k},\frac{\beta}{l},\ldots,\frac{\beta+l-1}{l}\end{array}\right|t\right)\star\,\omega(t)\;.
\end{aligned}$$

In the case $\beta=1$, we obtain for the holomorphic period of the family of Calabi-Yau threefolds in a neighborhood of $t=0$ the expression

$$ {}_{2+k+l}F_{1+k+l}\left(\left.\begin{array}{c}\mu,1-\mu,\frac{1}{2},\frac{1}{k+l},\;\ldots\;\frac{k+l-1}{k+l}\\ 1,\frac{1}{k},\ldots,1,\frac{1}{l},\ldots,1\end{array}\right|t\right)\;. \qquad (7.34)$$

In the case $\beta=\frac{1}{2}$, we obtain for the holomorphic period

$$ {}_{3+k+l}F_{2+k+l}\left(\left.\begin{array}{c}\mu,1-\mu,\frac{1}{2},\frac{1}{2(k+l)},\;\ldots\;\frac{2(k+l)-1}{2(k+l)}\\ 1,1,\frac{1}{k},\ldots,\frac{k}{k},\frac{1}{2l},\ldots,\frac{2l-1}{2l}\end{array}\right|t\right)\;. \qquad (7.35)$$

□

We define a family of closed three-cycles Σ_3'' for the families of Calabi-Yau threefolds in Equation (7.14) as follows: for each $t\in D_1^*(0)$ and generalized functional invariant $(m/2,m/2,\beta)$, we follow the positively oriented circle $C_r(0): |s|=r=(1+|t|^{1/m})/2$ in the affine s-plane. For each s, we trace out the cycle in the (u,X,Y)-coordinates of the K3 fiber that is obtained from the cycle $\hat{\Sigma}_2''(t\,(1+s^2)^m/(2s)^m)$ defined in Section 6.4.2 after rescaling $(u,X,Y)\to(s^2\,X,s^3\,Y)$ to account for the twist. For $t\in D_1^*(0)$ we obtain a continuously varying family of closed three-cycles that we denote as warped product $\hat{\Sigma}_3''=\hat{\Sigma}_1'\times_s\hat{\Sigma}_2''$. We have the following lemma:

Lemma 7.15. *For $t\in D_1^*(0)$ the period of the holomorphic three-form $ds\wedge du/2\wedge\frac{dX}{Y}$ over the three-cycle $\hat{\Sigma}_3''$ for the families of Weierstrass models from Lemma 7.5 is given by*

$$\hat{\omega}_3^{(m/2,m/2,1/2),(\mu)} = (2\pi i)^3\;{}_{4+m}F_{3+m}\left(\left.\begin{array}{c}\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2},1-\frac{\mu}{2},\frac{1}{2m},\frac{3}{2m},\ldots,\frac{2m-1}{2m}\\ \frac{1}{2},\frac{1}{m},\frac{2}{m},\ldots,\frac{m}{m},1,1\end{array}\right|t^2\right)\;. \qquad (7.36)$$

Proof. The proof follows closely the proofs of Lemmas 6.16 and 5.5. We evaluate the period of the holomorphic three-from $ds\wedge du/2\wedge dX/Y$ over the three-cycle $\hat{\Sigma}_3''$. To evaluate the holomorphic period by a residue computation we recall that

$$\left|\frac{t\,(1+s^2)^m}{(2s)^m}\left(1+\frac{1}{2\,u\,(u+1)}\right)\right|,\ \left|\frac{t\,(1+s^2)^m}{(2s)^m}\right|,\ |t|<1\ .$$

We obtain the period of the holomorphic three-form from the following computation

$$\begin{aligned}(7.37)\qquad &\frac{1}{(2\pi i)^3}\oiiint_{\hat{\Sigma}''_3} ds\wedge du\wedge\frac{dX}{2Y}\\ &=\frac{1}{2\pi i}\oint_{C_r(0)}\frac{ds}{s}\;{}_4F_3\left(\left.\begin{array}{c}\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2},1-\frac{\mu}{2}\\ 1,1,\frac{1}{2}\end{array}\right|\frac{t^2\,(1+s^2)^{2m}}{(2s)^{2m}}\right)\\ &={}_{4+m}F_{3+m}\left(\left.\begin{array}{c}\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2},1-\frac{\mu}{2},\frac{1}{2m},\frac{3}{2m},\dots,\frac{2m-1}{2m}\\ \frac{1}{2},\frac{1}{m},\frac{2}{m},\dots,\frac{m}{m},1,1\end{array}\right|t^2\right)\ .\end{aligned}$$

□

The following corollary produces all 14 Calabi-Yau operators of the so-called *hypergeometric case*:

Corollary 7.16. *All generalized functional invariants (k,l,β) for which the period in Equation (7.32) reduces to the hypergeometric function*

$$(7.38)\qquad \omega_4^{(p,q)}=(2\pi i)^3\;{}_4F_3\left(\left.\begin{array}{c}p,q,1-q,1-p\\ 1,1,1\end{array}\right|t\right)$$

with $p,q\in\mathbb{Q}$ are given in Table 20. The period is annihilated by the fourth-order and degree-one Picard-Fuchs operator

$$(7.39)\qquad L_4^{(p,q)}=\theta^4-t\,(\theta+p)\,(\theta+q)\,(\theta+1-q)\,(\theta+1-p)\ .$$

All possible generalized functional invariants $(m/2,m/2,0)$ for which Equation (7.36) reduces to the hypergeometric function

$$(7.40)\qquad \hat{\omega}_4^{(p,q)}=(2\pi i)^3\;{}_4F_3\left(\left.\begin{array}{c}p,q,1-q,1-p\\ 1,1,1\end{array}\right|t^2\right)$$

are given in Table 20. The period is annihilated by the fourth-order and degree-two Picard-Fuchs operator

$$(7.41)\qquad \hat{L}_4^{(p,q)}=\theta^4-t^2\,(\theta+2p)\,(\theta+2q)\,(\theta+2-q)\,(\theta+2-p)\ .$$

Proof. For the proof we have to check for cancellations among the defining fractions in the hypergeometric functions from Lemmas 7.14 and 7.15 to obtain a hypergeometric function of type ${}_4F_3$. □

Remark. In Table 20 we have listed all possible 14 rational combinations

$$(p,q,q'=1-q,p'=1-p)$$

and the corresponding classification numbers in the AESZ database [4] for which Corollary 7.16 now provides a geometric construction of the corresponding Calabi-Yau operator. Moreover,

#	AESZ	lattice pol.	M_4	M_3	M_2	M_1	M_4	M_3	M_2	M_1	$\tilde{M}'$	M'	$\tilde{M}$	M
		(p,q,q',p')	$(k,l,\beta=1)$				$(k,l,\beta=\frac{1}{2})$				$(\frac{m}{2},\frac{m}{2},\frac{1}{2})$			
1	(3)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(1,1)$											
2	(5)	$(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3})$	$(1,2)$	$(1,1)$										
3	(4)	$(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$		$(1,2)$										
4	(6)	$(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4})$	$(2,2)$	$(1,3)$	$(1,1)$		$(1,1)$							
5	(11)	$(\frac{1}{4},\frac{1}{3},\frac{2}{3},\frac{3}{4})$		$(2,2)$	$(1,2)$			$(1,1)$						
6	(10)	$(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$			$(2,2)$				$(1,1)$		(1)			
7	(14)	$(\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{5}{6})$		$(3,3)$		$(1,1)$	$(2,1)$		$(1,2)$					
8	(8)	$(\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6})$			$(2,4)$	$(1,2)$		$(2,1)$				(1)		
9	(12)	$(\frac{1}{6},\frac{1}{4},\frac{3}{4},\frac{5}{6})$				$(2,2)$			$(2,1)$	$(1,1)$				
10	(13)	$(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6})$								$(2,1)$		(3)		
11	(1)	$(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$		$(2,3)$	$(1,4)$									
12	(7)	$(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$			$(4,4)$			$(3,1)$	$(2,2)$	$(1,3)$			(1)	
13	(2)	$(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$							$(4,1)$	$(2,3)$				
14	(9)	$(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12})$							$(4,2)$					(1)

Table 20. The hypergeometric case by the mixed-twist construction

Table 20 also reproduces all generalized functional invariants that were used in [29] to construct threefolds fibered by M_n-polarized K3 surfaces using toric geometry.

Remark. It was shown in [31] that the regular singular points $t = 0, 1, \infty$ of the Picard-Fuchs operator in Equation (7.39) correspond to the conifold limit, large complex structure limit, and the orbifold point, respectively. In particular, the monodromy around $t = \infty$ is maximally unipotent.

7.5.3. *Product-twist construction.* In this section we will compute the Picard-Fuchs operators and holomorphic periods for all families of elliptic Calabi-Yau threefolds obtained by the product-twist construction.

We define families of closed three-cycles Σ'_3 on the total space of Equation (7.6) as follows: for each $t \in D^*_1(0)$, we follow the positively oriented circle in the affine s-plane such that $0 < |st| < |s| < 1$. For each s, we trace out the product cycle in the (u, X, Y)-coordinates of the fiber that is obtained from $\Sigma'_1(s) \times \Sigma'_1(t/s)$ – where Σ'_1 was defined in Section 6.2.1 – after rescaling $(X, Y) \to ((sF)^2 X, (sF)^3 Y)$ to account for the twist. For each $t \in D^*_1(0)$, we obtain a continuously varying family of closed three-cycles that we denote as warped product $\Sigma'_3 = \Sigma'_1 \times_s (\Sigma'_1 \times \Sigma'_1)$. We have the following lemma:

Lemma 7.17. *For $t \in D^*_1(0)$ the period of the holomorphic three-form $ds \wedge du \wedge \frac{dX}{Y}$ over the three-cycle Σ'_3 for the families of Weierstrass models in Lemma 7.6 is given by*

$$\omega_3^{(\mu,\mu')} = (2\pi i)^3 \; {}_4F_3\left(\begin{matrix} \mu, \mu', 1-\mu', 1-\mu \\ 1, 1, 1 \end{matrix} \middle| \, t \right) . \tag{7.42}$$

Proof. We assume that the holomorphic period of the one-form $dX/(2\pi i Y)$ over the cycle $\Sigma'_1(t)$ for the family of elliptic curves defined by $g_2^{(\mu')}(t)$ and $g_3^{(\mu')}(t)$ is a complex analytic function of the form $\tilde{\omega}^{(\mu')}(t) = \sum_{n\ge 0} \tilde{\omega}^{(\mu')}_n \, t^n$ for $|t|<1$ We evaluate the period of the holomorphic three-from $du \wedge ds \wedge dX/Y$ over the three-cycle $\Sigma'_3 = \Sigma'_1 \times_s (\Sigma'_1 \times \Sigma'_1)$ using Fubini's theorem. We first observe that for $0<|st|<|s|=r<1$ we have

$$\begin{aligned} \frac{1}{2\pi i} \oiiint_{\Sigma'_3} du \wedge ds \wedge \frac{dX}{Y} &= \frac{1}{2\pi i} \oint_{|s|=r} \frac{ds}{s} \oint_{\Sigma'_1(s)} \frac{du}{F} \oint_{\Sigma'_1(t/s)} \frac{dX}{Y} \\ &= \oint_{|s|=r} \frac{ds}{s} \oint_{\Sigma'_1(s)} \frac{du}{F} \; \tilde{\omega}^{(\mu')}\Big(\frac{t}{s}\Big) \; . \end{aligned} \tag{7.43}$$

We denote the holomorphic period of the one-form $du/(2\pi i F)$ over the cycle $\Sigma'_1(s)$ for the family of elliptic curves defined by Equation (7.18) as a complex analytic function of the form $\omega^{(\mu)}(s) = \sum_{n\ge 0} \omega^{(\mu)}_n \, s^n$ for $|s|<1$. We can then express the holomorphic period using a Hadamard product as follows

$$\begin{aligned} \frac{1}{(2\pi i)^2} \oiiint_{\Sigma'_3} du \wedge ds \wedge \frac{dX}{Y} &= \oint_{|s|=r<1} \frac{ds}{s} \quad \omega^{(\mu)}(s) \quad \tilde{\omega}^{(\mu')}\Big(\frac{t}{s}\Big) \\ = \sum_{m,n\ge 0} \omega^{(\mu)}_m \, \tilde{\omega}^{(\mu')}_n \, t^n \oint_{|s|=r<1} \frac{ds}{s^{1-m+n}} &= (2\pi i) \;\; \omega^{(\mu)}(t) \star \tilde{\omega}^{(\mu')}(t) \; . \end{aligned} \tag{7.44}$$

Substituting the holomorphic periods from Lemma 5.6, we obtain the desired result from the identity

$$ {}_2F_1\left(\left. \begin{array}{c} \mu', 1-\mu' \\ 1 \end{array} \right| t \right) \star {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| t \right) = {}_4F_3\left(\left. \begin{array}{c} \mu, \mu', 1-\mu', 1-\mu \\ 1, 1, 1 \end{array} \right| t \right) \; . \tag{7.45}$$

□

It follows immediately:

Corollary 7.18. *The period $\omega_3^{(\mu,\mu')}$ in Equation (7.42) is the solution of the differential equation $L_4^{(\mu,\mu')} \, \omega_3^{(\mu,\mu')} = 0$ that is holomorphic at $t=0$ with*

$$L_4^{(\mu,\mu')} = \theta^4 - t \, (\theta+\mu) \, (\theta + \mu') \, (\theta + 1 - \mu) \, (\theta + 1 - \mu') \; . \tag{7.46}$$

□

We define families of closed three-cycles $\tilde{\Sigma}'_3$ on the total space of Equation (7.7) as follows: for each $t \in D^*_1(0)$ we follow the positively oriented circle in the affine s-plane such that $0<|st|<|s|<1$. For each s, we trace out the product cycle in the (u, X, Y)-coordinates of the fiber that is obtained from $\Sigma'_1(s) \times \tilde{\Sigma}'_1(t/s)$ after rescaling $(X,Y) \to \big((sF)^2 \, X, (sF)^3 \, Y\big)$ to account for the twist. For each $t \in D^*_1(0)$ we obtain a continuously varying family of closed three-cycles that we denote as warped product $\tilde{\Sigma}'_3 = \Sigma'_1 \times_s (\Sigma'_1 \times \tilde{\Sigma}'_1)$. We have the following lemma:

Lemma 7.19. *For $t \in D^*_1(0)$ the period of the holomorphic three-form $ds \wedge du \wedge \frac{dX}{Y}$ over the three-cycle $\tilde{\Sigma}'_3$ for the families of Weierstrass models in Lemma 7.7 is given by*

$$\tilde{\omega}_3^{(\mu,\tilde{\mu})} = {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| t \right) \star \left[\frac{1}{1-t} \, {}_2F_1\left(\left. \begin{array}{c} \tilde{\mu}, 1-\tilde{\mu} \\ 1 \end{array} \right| \frac{t}{t-1} \right) \right] \tag{7.47}$$

Proof. The proof is completely analogous to the proof of Lemma 7.17 with $\tilde{\omega}(t)$ replace by the period in Equation (5.19). □

The following corollary produces all 16 Calabi-Yau operators of the so-called *even case*:

Corollary 7.20. *The period $\tilde{\omega}_3^{(\mu,\tilde{\mu})}$ in Equation (7.47) is the solution of the differential equation $\tilde{L}_4^{(\mu,\tilde{\mu})}\,\tilde{\omega}_3^{(\mu,\tilde{\mu})} = 0$ that is holomorphic at $t = 0$ with*

$$\begin{aligned}\tilde{L}_4^{(\mu,\tilde{\mu})} &= \theta^4 - t\left(2\,\theta^2 + 2\,\theta + \tilde{\mu}^2 - \tilde{\mu} + 1\right)(\theta+\mu)\,(\theta-\mu+1)\\ &\quad + t^2\,(\theta+2-\mu)\,(\theta+1+\mu)\,(\theta+\mu)\,(\theta+1-\mu)\;.\end{aligned} \tag{7.48}$$

□

#	AESZ	Name	$(\mu,\tilde{\mu})$
1	(32)	111	$\left(\frac{1}{2},\frac{1}{2}\right)$
2	(31)	110	$\left(\frac{1}{3},\frac{1}{2}\right)$
3	(15)	30	$\left(\frac{1}{4},\frac{1}{2}\right)$
4	(33)	112	$\left(\frac{1}{6},\frac{1}{2}\right)$
5	(34)	141, 8**	$\left(\frac{1}{2},\frac{1}{3}\right)$
6	(35)	142	$\left(\frac{1}{3},\frac{1}{3}\right)$
7	-	196	$\left(\frac{1}{4},\frac{1}{3}\right)$
8	(36)	143	$\left(\frac{1}{6},\frac{1}{3}\right)$
9	(41)	189, 7**	$\left(\frac{1}{2},\frac{1}{4}\right)$
10	(46)	194	$\left(\frac{1}{3},\frac{1}{4}\right)$
11	(48)	197	$\left(\frac{1}{4},\frac{1}{4}\right)$
12	(50)	199	$\left(\frac{1}{6},\frac{1}{4}\right)$
13	(42)	190, 9**	$\left(\frac{1}{2},\frac{1}{6}\right)$
14	(47)	195	$\left(\frac{1}{3},\frac{1}{6}\right)$
15	(49)	198	$\left(\frac{1}{4},\frac{1}{6}\right)$
16	(23)	61	$\left(\frac{1}{6},\frac{1}{6}\right)$

Table 21. The even case by the product-twist construction

8. Step 4: From families of Calabi-Yau threefolds to families of Calabi-Yau fourfolds

In this section we will describe the fourth step in our iterative construction that produces families of elliptic Calabi-Yau fourfolds with section over $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The families are constructed from the families of elliptic Calabi-Yau threefolds obtained in the so-called hypergeometric case of Section 7. In our exposition of Step 4, we will restrict ourselves to the pure-twist construction and will construct families only over $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. In particular, the iterative construction at Step 4 will produce families of (singular) Calabi-Yau fourfolds as Weierstrass models that realize all 14 one-parameter variations of Hodge structure of weight four and type $(1,1,1,1,1)$ over a one-dimensional rational deformation space of the so-called odd case.

We will also compute the Picard-Fuchs operators and holomorphic periods for all families. Using facts about the relation of fourth-order and fifth-order Calabi-Yau operators proved in [2, 3], we will show that the Yifan-Yang pullbacks of the geometrically realized fifth-order

Picard-Fuchs operators produce all remaining symplectically rigid fourth-order Calabi-Yau operators of the *odd case* that were classified by Bogner and Reiter [15].

First, we will discuss how families of Weierstrass models in affine charts can be extended to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We will describe families of fourfolds by exhibiting their Weierstrass equations

$$Y^2 = 4\,X^3 - \mathsf{G}_2(s,t,u,v)\,X - \mathsf{G}_3(s,t,u,v) \tag{8.1}$$

where G_2 and G_3 are polynomials in $s,t,u,v \in \mathbb{C}$. t is a complex family parameter taking values in $\mathbb{P}^1\backslash\{0,1,\infty\}$ and s,u,v are affine coordinates.

The Weierstrass fibration of Equation (8.1) extends onto $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, if we produce a minimal Weierstrass equation when introducing projective variables $[s_0 : s_1] \in \mathbb{P}^1$, $[u_0 : u_1] \in \mathbb{P}^1$, $[v_0 : v_1] \in \mathbb{P}^1$, and $[x : y : z] \in \mathbb{WP}_{(2,3,1)}$ and writing each fiber as the hypersurface

$$y^2 z \; = 4\,x^3 - \mathsf{G}_2\left(\frac{s_0}{s_1}, t, \frac{u_0}{u_1}, \frac{v_0}{v_1}\right) s_1^8\, u_1^8\, v_1^8\, x\, z^2 - \mathsf{G}_3\left(\frac{s_0}{s_1}, t, \frac{u_0}{u_1}, \frac{v_0}{v_1}\right) s_1^{12}\, u_1^{12}\, v_1^{12}\, z^3. \tag{8.2}$$

Four $\mathbb{C}^*$-group actions on the defining variables in Equation (8.2) are given by the weights listed in Table 22 where *deg* denotes the total weight of Equation (8.2) and *sum* denotes the sum of weights of the defining variables. Since the condition is satisfied that the total weight equals the sum of weights for each acting $\mathbb{C}^*$, a Calabi-Yau fourfold is obtained by removing the loci $\{s_0 = s_1 = 0\}$, $\{u_0 = u_1 = 0\}$, $\{v_0 = v_1 = 0\}$, $\{x = y = z = 0\}$ from the solution set of Equation (8.2), and taking the quotient $(\mathbb{C}^*)^4$.

$\mathbb{C}^*$	deg	x	y	z	s_0	s_1	u_0	u_1	v_0	v_1	Σ
λ_1	3	1	1	1	0	0	0	0	0	0	3
λ_2	12	4	6	0	1	1	0	0	0	0	12
λ_3	12	4	6	0	0	0	1	1	0	0	12
λ_4	12	4	6	0	0	0	0	0	1	1	12

Table 22. Weights of variables in Weierstrass equation

8.1. **Elliptic fourfolds by the pure-twist construction.** In this section we will describe the *pure-twist construction* for families of elliptic Calabi-Yau fourfolds. To obtain Weierstrass coefficients G_2 and G_3 of a fourfold, we set

$$\begin{aligned} t &\mapsto t\,v\,, \\ \mathcal{G}_2(s,t,u) &\mapsto \mathsf{G}_2(s,t,u,v) = \mathcal{G}_2(s,t\,v,u)\Big[v\,(v-1)\Big]^2\,, \\ \mathcal{G}_3(s,t,u) &\mapsto \mathsf{G}_3(s,t,u,v) = \mathcal{G}_3(s,t\,v,u)\Big[v\,(v-1)\Big]^3\,, \end{aligned} \tag{8.3}$$

in Equation (8.1) where $\mathcal{G}_2$ and $\mathcal{G}_3$ are taken from Lemma 7.4. We then have the following lemma:

Lemma 8.1. *For* $(n,\mu) \in \Big\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\Big\}$ *and* $1 \le k \le 1/\mu$ *and* $1 \le l \le \beta/\mu$ *with* $\beta \in \{\frac{1}{2},1\}$*, the Weierstrass equations*

$$Y^2 = 4\,X^3 - \underbrace{\mathcal{G}_2(s,t\,v,u)\Big[v\,(v-1)\Big]^2}_{=:\,\mathsf{G}_2(s,t,u,v)} X - \underbrace{\mathcal{G}_3(s,t\,v,u)\Big[v\,(v-1)\Big]^3}_{=:\,\mathsf{G}_3(s,t,u,v)} \tag{8.4}$$

with $\mathcal{G}_2(s,t,u)$ and $\mathcal{G}_3(s,t,u)$ given in Lemmas 7.4 and 7.5 define families of elliptic Calabi-Yau fourfolds over $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The torsion part of the Mordell-Weil group is two-torsion for $n=2,4$ and trivial for $n=1,3$.

Proof. The only statement that needs to be proved is the statement about the torsion part of the Mordell-Weil group. The transformation

$$\Big(t,X,Y\Big)\mapsto\left(\frac{c_{kl}\,v\,t}{s^k(s+1)^l},X\,\Big(s(s+1)^\beta\Big)^2\,v\,(v-1),Y\,\Big(s(s+1)^\beta\Big)^3\,\big(v\,(v-1)\big)^{3/2}\right)\tag{8.5}$$

applied to elements in Table 15 gives the explicit expressions for the various torsion sections. It follows that the transformations can yield polynomial expressions in the affine coordinate only for two-torsion. □

Similarly, for the families of elliptic threefolds from Lemma 7.5, we define families of elliptic fourfolds as Weierstrass models by mapping $t\mapsto t\,v$ and carrying out a different quadratic twist. We obtain new Weierstrass coefficients G_2 and G_3 by setting

$$\begin{aligned}t&\mapsto t\,v\,,\\ \mathcal{G}_2(s,t,u)&\mapsto \mathsf{G}_2(s,t,u,v)=\mathcal{G}_2(s,t\,v,u)\Big[v^2-1\Big]^2\,,\\ \mathcal{G}_3(s,t,u)&\mapsto \mathsf{G}_3(s,t,u,v)=\mathcal{G}_3(s,t\,v,u)\Big[v^2-1\Big]^3\,,\end{aligned}\tag{8.6}$$

in Equation (8.1) where G_2 and G_3 are defined in Lemma 7.5.

Lemma 8.2. *For $(n,\mu)\in\Big\{(1,\frac{1}{6}),(2,\frac{1}{4}),(3,\frac{1}{3}),(4,\frac{1}{2})\Big\}$ and $1\le m\le 1/\mu$ with m odd, the Weierstrass equations*

$$Y^2=4\,X^3-\underbrace{\mathcal{G}_2(s,t\,v,u)\Big[v^2-1\Big]^2}_{=:\,\mathsf{G}_2(s,t,u,v)}X-\underbrace{\mathcal{G}_3(s,t\,v,u)\Big[v^2-1\Big]^3}_{=:\,\mathsf{G}_3(s,t,u,v)}\tag{8.7}$$

with $\mathcal{G}_2(s,t\,v,u)$ and $\mathcal{G}_3(s,t\,v,u)$ given in Lemma 7.5 define families of elliptic Calabi-Yau fourfolds over $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The torsion part of the Mordell-Weil group is two-torsion for $n=2,4$ and trivial for $n=1,3$.

Proof. The only statement that needs to be proved is the statement about the torsion part of the Mordell-Weil group. The transformation

$$\Big(t,X,Y\Big)\mapsto\left(\frac{t\,v\,(1+s^2)^m}{(2s)^m},X\,s^2\,(v^2-1),Y\,s^3\,(v^2-1)^{3/2}\right)\tag{8.8}$$

applied to elements in Table 15 gives the explicit expressions for the various torsion sections. It follows that the transformations can yield polynomial expressions in the affine coordinate only for two-torsion. □

8.2. **Period computation.** In this section we will compute the Picard-Fuchs operators and holomorphic periods for all families of Calabi-Yau fourfolds constructed in the previous section.

In Section 7.5.1 we constructed families of closed three-cycles $\Sigma''_3 = \Sigma''_3(t)$ for the families of elliptic threefolds in Lemma 7.4. We now define a family of closed four-cycles Σ_4 as follows: for each $t \in D^*_1(0)$, we follow the Pochhammer contour Σ_1 around $v = 0$ and $v = 1$ in the affine v-plane such that $0 < |t\,v| < 1$. For each $v \in \Sigma_1(t)$, we trace out the cycle in the (s, u, X, Y)-coordinates of the threefold-fiber that is obtained from $\Sigma''_3(vt)$ defined in Section 7.5.2 after rescaling $(X, Y) \to \big(v\,(v-1)\,X, (v\,(v-1))^{3/2}\,Y\big)$ to account for the quadratic twist. For each $t \in D^*_1(0)$, we obtain a continuously varying family of closed four-cycles that we denote as warped product $\Sigma_4 = \Sigma_1 \times_v \Sigma''_3$.

We now evaluate the period integral for the holomorphic four-form $ds \wedge du/2 \wedge dv/2 \wedge dX/Y$ over the four-cycle Σ_4, i.e.,

$$\omega_4 = \frac{1}{4} \iiiint_{\Sigma_4} ds \wedge du \wedge dv \wedge \frac{dX}{Y} \,. \tag{8.9}$$

We then have the following lemma:

Lemma 8.3. *For $t \in D^*_1(0)$ the period of the holomorphic four-form $ds \wedge du/2 \wedge dv/2 \wedge \frac{dX}{Y}$ over the four-cycle Σ_4 for the families given in Lemma 8.4 is given by*

$$\omega_4^{(p,q)} = (2\pi i)^4 \; {}_5F_4\left(\left. \begin{array}{c} p, q, \frac{1}{2}, 1-q, 1-p \\ 1, 1, 1, 1 \end{array} \right| t \right) \tag{8.10}$$

with $(p, q, 1-q, 1-p)$ given in Table 20.

Proof. The proof follows closely the proof of Lemma 7.8. One repeats the integration step and obtains the period integral using

$${}_1F_0\left(\left. \frac{1}{2} \right| t \right) \star {}_4F_3\left(\left. \begin{array}{c} p, q, 1-q, 1-p \\ 1, 1, 1 \end{array} \right| t \right) = {}_5F_4\left(\left. \begin{array}{c} p, q, \frac{1}{2}, 1-q, 1-p \\ 1, 1, 1, 1 \end{array} \right| t \right) \,.$$

□

A well-known result for hypergeometric functions then yields the following corollary.

Corollary 8.4. *The period $\omega_4^{(p,q)}$ in Equation (8.10) is the solution of the differential equation $L_5^{(p,q)}\,\omega_4^{(p,q)} = 0$ that is holomorphic at $t = 0$ with*

$$L_5^{(p,q)} = \theta^5 - t\Big(\theta + \frac{1}{2}\Big)\Big(\theta + p\Big)\Big(\theta + q\Big)\Big(\theta + 1 - p\Big)\Big(\theta + 1 - q\Big) \,. \tag{8.11}$$

□

In Section 7.5.1, we also constructed families of closed three-cycles $\hat{\Sigma}''_3 = \hat{\Sigma}''_3(t)$ for the families of elliptic threefolds in Lemma 7.4. We now define a family of closed four-cycles $\hat{\Sigma}_4$ as follows: for each $t \in D^*_1(0)$, we follow a Pochhammer contour $\hat{\Sigma}_1$ invariant under $v \mapsto -v$ around $v = -1$ and $v = 1$ in the affine v-plane such that $0 < |t\,v| < 1$. For each $v \in \hat{\Sigma}_1(t)$, we trace out the cycle in the (s, u, X, Y)-coordinates of the threefold-fiber that is obtained from the cycle $\hat{\Sigma}''_3(vt)$ defined Section 7.5.2 after rescaling $(X, Y) \to \big((v^2-1)\,X, (v^2-1)^{3/2}\,Y\big)$ to account for the quadratic twist. For $t \in D^*_1(0)$ we obtain a continuously varying family of closed four-cycles that we denote as warped product $\hat{\Sigma}_4 = \hat{\Sigma}_1 \times_v \hat{\Sigma}''_3$. We can evaluate the period for the holomorphic two $ds \wedge du/2 \wedge dv/2 \wedge dX/Y$ over the four-cycle $\hat{\Sigma}_4$. We have the following lemma:

Lemma 8.5. *For $t \in D^*_1(0)$ the period of the holomorphic four-form $ds \wedge du/2 \wedge dv/2 \wedge \frac{dX}{Y}$ over the four-cycle $\hat{\Sigma}_4$ for the families of Weierstrass models in Lemma 8.2 is given by*

$$\hat{\omega}_4^{(p,q)} = (2\pi i)^4 \; {}_5F_4\left(\begin{matrix} p, q, \frac{1}{2}, 1-q, 1-p \\ 1, 1, 1, 1 \end{matrix} \middle| \, t^2 \right) \tag{8.12}$$

with $(p, q, 1-q, 1-p)$ given in Table 20.

Proof. The proof follows the proof of Lemma 7.10. One repeats the integration procedure and obtains the period integral using

$${}_1F_0\left(\frac{1}{2} \middle| \, t^2 \right) \star {}_4F_3\left(\begin{matrix} p, q, 1-q, 1-p \\ 1, 1, 1 \end{matrix} \middle| \, t^2 \right) = {}_5F_4\left(\begin{matrix} p, q, \frac{1}{2}, 1-q, 1-p \\ 1, 1, 1, 1 \end{matrix} \middle| \, t^2 \right) .$$

□

A well-known result for hypergeometric functions then yields the following corollary.

Corollary 8.6. *The period $\hat{\omega}_4^{(p,q)}$ in Equation (8.12) is the solution of the differential equation $\hat{L}_5^{(p,q)} \, \hat{\omega}_4^{(p,q)} = 0$ that is holomorphic at $t = 0$ with*

$$\hat{L}_5^{(p,q)} = \theta^5 - t^2 \left(\theta + 1\right)\left(\theta + 2\,p\right)\left(\theta + 2\,q\right)\left(\theta + 2 - 2\,p\right)\left(\theta + 2 - 2\,q\right) . \tag{8.13}$$

□

8.3. **The Yifan-Yang pullback.** In this section we will show that the so-called Yifan-Yang pullback of the now geometrically realized fifth-order Picard-Fuchs operators from the previous section produces all remaining symplectically rigid fourth-order Calabi-Yau operators of the so-called *odd case*.

Let L_n be a general nth-order linear differential operator with at most regular singular points in the variable t, i.e.,

$$L_n = \partial^n + \sum_{i=0}^{n-1} a_i(t)\, \partial^i \ ,$$

with $\partial = \frac{d}{dt}$ and suitable coefficient functions $a_i(t)$ for $1 \leq i \leq n-1$. Its dual operator is defined to be

$$L_n^* = \partial^n + \sum_{i=0}^{n-1} (-1)^{n+i}\, \partial^i a_i(t) \ .$$

We call the operator L_n self-dual if there exists a non-trivial function $f(t)$ such that $L_n f(t) = f(t) L_n^*$. The self-duality condition implies as a necessary condition that the function $f(t)$ satisfies the differential equation

$$f'(t) = -\frac{2}{n}\, a_{n-1}(t) \cdot f(t) \ .$$

Let L_4 be a general fourth-order linear differential operator acting on a function $y(t)$, i.e.,

$$L_4\, y(t) = \frac{d^4}{dt^4} y(t) + a_3(t)\, \frac{d^3}{dt^3} y(t) + a_2(t)\, \frac{d^2}{dt^2} y(t) + a_1(t)\, \frac{d}{dt} y(t) + a_0(t)\, y(t)\ .$$

We have the following lemma:

Lemma 8.7. *The following statements are equivalent:*

(1) *The operator L_4 is self-dual.*
(2) *The exterior square of L_4 is a fifth-order operator.*
(3) *The following condition holds*

$$0 = 8\, a_1(t) - 8 \frac{da_2(t)}{dt} + 4 \frac{d^2 a_3(t)}{dt^2} - 4\, a_2(t)\, a_3(t) + 6\, a_3(t)\, \frac{da_3(t)}{dt} + a_3(t)^3\ . \tag{8.14}$$

Proof. The proof follows by an explicit computation. □

Let L_5 be a general fifth-order linear differential operator acting on a function $Y(t)$, i.e.,

$$\begin{aligned} L_5\, Y(t) = & \frac{d^5}{dt^5} Y(t) + b_4(t)\, \frac{d^4}{dt^4} Y(t) + b_3(t)\, \frac{d^3}{dt^3} Y(t) \\ & + b_2(t)\, \frac{d^2}{dt^2} Y(t) + b_1(t)\, \frac{d}{dt} Y(t) + b_0(t)\, Y(t)\ . \end{aligned}$$

We have the following lemma:

Lemma 8.8. *The following statements are equivalent:*

(1) *The operator L_5 is self-dual.*
(2) *The operator L_5 is the exterior square of a fourth-order self-dual operator L_4.*
(3) *The projective monodromy group of the operator L_5 is a discrete subgroup of* $\mathrm{Sp}_4(\mathbb{R})$.
(4) *The following two conditions hold*

$$b_2(t) = \frac{3}{2}\, \frac{db_3(t)}{dt} + \frac{3}{5}\, b_4(t)\, b_3(t) - \frac{d^2 b_4(t)}{dt^2} - \frac{6}{5} b_4(t)\, \frac{db_4(t)}{dt} - \frac{4}{25} b_4(t)^3 \tag{8.15}$$

and

$$\begin{aligned} b_0(t) = & \frac{1}{5} \frac{d^4 b_4(t)}{dt^4} - \frac{1}{4} \frac{d^3 b_3(t)}{dt^3} + \frac{2}{5} b_4(t)\, \frac{d^3 b_4(t)}{dt^3} - \frac{3}{10} b_4(t)\, \frac{d^2 b_3(t)}{dt^2} \\ & + \left(\frac{8}{25} b_4(t)^2 + \frac{4}{5} \frac{db_4(t)}{dt} - \frac{1}{10} b_3(t) \right) \frac{d^2 b_4(t)}{dt^2} + \frac{1}{2} \frac{db_1(t)}{dt} \\ & + \left(-\frac{3}{25} b_4(t)^2 - \frac{3}{10} \frac{db_4(t)}{dt} \right) \frac{db_3(t)}{dt} + \frac{12}{25} b_4(t) \left(\frac{db_4(t)}{dt} \right)^2 \\ & + \left(-\frac{3}{25}\, b_3(t)\, b_4(t) + \frac{16}{125}\, b_4(t)^3 \right) \frac{db_4(t)}{dt} - \frac{2}{125} b_3(t)\, b_4(t)^3 \\ & + \frac{1}{5}\, b_1(t)\, b_4(t) + \frac{16}{3125}\, b_4(t)^5\ . \end{aligned} \tag{8.16}$$

Proof. The equivalence of (1), (2), (4) follows by an explicit computation. Yang and Zudilin [73] proved that L_5 has a projective monodromy group that is a discrete subgroup of $\mathrm{Sp}_4(\mathbb{R})$, if and only if L_5 satisfies conditions (8.15) and (8.16). In fact, we have the following identification with the polynomials p_1, p_2, and p_3 in [73, Theorem 4]:

$$\begin{aligned}
\frac{p_1(t)}{t} &= \frac{1}{10}b_4(t) - \frac{1}{t}\ ,\\
\frac{p_2(t)}{t^2} &= \frac{1}{5}b_3(t) - \frac{1}{5}\frac{db_4(t)}{dt} - \frac{7}{100}b_4(t)^2\ ,\\
\frac{p_3(t)}{t^4} &= \frac{1}{250}b_3(t)\,b_4(t)^2 - \frac{3}{10}b_4(t)\,\frac{db_3(t)}{dt} + \frac{1}{50}b_3(t)\,\frac{db_4(t)}{dt} + \frac{17}{50}\left(\frac{db_4(t)}{dt}\right)^2\\
&\quad + \frac{2}{5}\frac{d^3b_4(t)}{dt^3} + \frac{29}{125}b_4(t)^2\frac{db_4(t)}{dt} + \frac{14}{25}b_4(t)\,\frac{d^2b_4(t)}{dt^2} + \frac{9}{1250}b_4(t)^4\\
&\quad - \frac{2}{25}b_3(t)^2 + \frac{1}{2}b_1(t) - \frac{9}{20}\frac{d^2b_3(t)}{dt^2}\ .
\end{aligned} \tag{8.17}$$

□

For a fifth-order linear differential operator L_5 satisfying conditions (8.15) and (8.16), the fourth-order linear differential operator L_4 whose exterior square is L_5 is unique. Thus, we will write $L_5 = \wedge^2 L_4$ and $L_4 = \vee_2 L_5$.

Lemma 8.9. *For a self-dual fifth-order linear differential operator L_5, the unique fourth-order self-dual linear differential operator $L_4 = \vee_2 L_5$ is determined by setting*

$$\begin{aligned}
a_3(t) &= \frac{2}{5}\,b_4(t)\ ,\\
a_2(t) &= -\frac{7\,b_4(t)^2}{50} - \frac{2}{5}\frac{\mathrm{d}}{\mathrm{d}t}b_4(t) + \frac{1}{2}\,b_3(t)\ ,\\
a_1(t) &= -\frac{9\,b_4(t)^3}{250} - \frac{12\,b_4(t)\,\frac{\mathrm{d}}{\mathrm{d}t}b_4(t)}{25} + \frac{1}{10}\,b_4(t)\,b_3(t) - \frac{3}{5}\frac{\mathrm{d}^2}{\mathrm{d}t^2}b_4(t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}b_3(t)\,,\\
a_0(t) &= -\frac{2}{5}\frac{\mathrm{d}^3}{\mathrm{d}t^3}b_4(t) + \frac{3}{8}\frac{\mathrm{d}^2}{\mathrm{d}t^2}b_3(t) - \frac{23\,b_4(t)\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}b_4(t)}{50} + \frac{1}{5}\,b_4(t)\,\frac{\mathrm{d}}{\mathrm{d}t}b_3(t)\\
&\quad - \frac{27\left(\frac{\mathrm{d}}{\mathrm{d}t}b_4(t)\right)^2}{100} + \left(-\frac{18\,b_4(t)^2}{125} - \frac{1}{20}\,b_3(t)\right)\frac{\mathrm{d}}{\mathrm{d}t}b_4(t) - \frac{19\,b_4(t)^4}{10000}\\
&\quad - \frac{3\,b_4(t)^2\,b_3(t)}{200} + \frac{1}{16}\,b_3(t)^2 - \frac{1}{4}\,b_1(t)
\end{aligned} \tag{8.18}$$

in Equation (8.3).

Proof. The proof follows by an explicit computation. □

Remark. It is easy to check that a fifth-order linear differential operator L_5 satisfies the conditions of Equations (8.15) and (8.16) if and only for any function $g(t)$ the operator

$$L_5^{\langle 2g(t)\rangle} = e^{2\,g(t)}\,L_5\,e^{-2\,g(t)}$$

does. It then follows that $\vee_2 L_5^{\langle 2g(t)\rangle}$ coincides with

$$L_4^{\langle g(t)\rangle} = e^{\,g(t)}\, L_4\, e^{-g(t)}\ .$$

For a fifth-order self-dual linear differential operator L_5 we call the fourth-order self-dual linear differential operator $L_4^{\langle g(t)\rangle} = \vee_2 L_5^{\langle 2g(t)\rangle}$ following [2] the *Yifan-Yang pullback* of L_5.

We then have the following Lemma:

Lemma 8.10. *The differential operators $L_5^{(p,q)}$ in Corollary 8.4 with $(p, q, 1-q, 1-p)$ given in Table 20 are self-dual fifth-order linear differential operators whose Yifan-Yang pullbacks are given by*

$$\begin{aligned} L_4^{(p,q),\langle g(t)\rangle} &= \theta^4 - \frac{1}{4}\, t\, \Big(8\,\theta^4 + 16\,\theta^3 - 2\,(p^2+q^2-p-q-9)\,\theta^2 \\ &\quad -2\,(p^2+q^2-p-q-5)\,\theta + 2 + p + q - pq - p^2 - q^2 + p^2q + p\,q^2 + p^2q^2\Big) \\ &\quad + \frac{1}{16}\, t^2\, (2\,\theta+2-q+p)\,(2\,\theta+1+q+p)\,(-2\,\theta-2-q+p)\,(-2\,\theta-3+p+q) \end{aligned} \tag{8.19}$$

for $\exp(-g(t)) = \sqrt[4]{t^2\,(t-1)^3}$.

Proof. The proof follows from an explicit computation. □

Similarly, we have the following lemma:

Lemma 8.11. *The differential operators $\hat{L}_5^{(p,q)}$ in Corollary 8.6 with $(p, q, 1-q, 1-p)$ given in Table 20 are self-dual rank-five linear differential operators whose Yifan-Yang pullbacks are given by*

(8.20)

$$\begin{aligned} \hat{L}_4^{(p,q),\langle \hat{g}(t)\rangle} &= \theta^4 - 2\,t^2\, \Big(\theta^4 + 4\,\theta^3 - (p^2+q^2-p-q-9)\,\theta^2 \\ &\quad -2\,(p^2+q^2-p-q-5)\theta + 4 + 2\,p + 2\,q - 2\,p\,q - 2\,p^2 - 2\,q^2 + 2\,p^2\,q + 2\,p\,q^2 + 2\,p^2\,q^2\Big) \\ &\quad + t^4\, (\theta+2-q+p)\,(\theta+1+q+p)\,(-\theta-2-q+p)\,(-\theta-3+p+q) \end{aligned}$$

for $\exp(-\hat{g}(t)) = \sqrt[4]{t^2\,(t^2-1)^3}$.

Proof. The proof follows from an explicit computation. □

Remark. Replacing t by t^2 maps θ to $\theta/2$ and $L_5^{(p,q)}$ in Corollary 8.4 to $\hat{L}_5^{(p,q)}/32$ in Corollary 8.6. Similarly, the operator $L_4^{(p,q),\langle g(t)\rangle}$ in Lemma 8.10 is mapped to $\hat{L}_4^{(p,q),\langle \hat{g}(t)\rangle}/16$ in Lemma 8.11.

Remark. In Table 23 we have listed the fourth-order and degree-two Calabi-Yau operators obtained as Yifan-Yang pullback of all 14 Picard-Fuchs operators for the Calabi-Yau fourfolds constructed in Lemma 8.1 and Lemma 8.7. They realize all operators of the so-called *odd case*. In Table 23 we list classification numbers in the AESZ database [4] (and any alternative name used).

#	AESZ	Name	(p, q)
1	(51)	$\tilde{3}, 204$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
2	(92)	$\tilde{5}$	$\left(\frac{1}{3}, \frac{1}{2}\right)$
3	(91)	$\tilde{4}$	$\left(\frac{1}{3}, \frac{1}{3}\right)$
4	(93)	$\tilde{6}$	$\left(\frac{1}{4}, \frac{1}{2}\right)$
5	(98)	$\tilde{11}$	$\left(\frac{1}{4}, \frac{1}{3}\right)$
6	(97)	$\tilde{10}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$
7	(101)	$\tilde{14}$	$\left(\frac{1}{6}, \frac{1}{2}\right)$
8	(95)	$\tilde{8}$	$\left(\frac{1}{6}, \frac{1}{3}\right)$
9	(99)	$\tilde{12}$	$\left(\frac{1}{6}, \frac{1}{4}\right)$
10	(100)	$\tilde{13}$	$\left(\frac{1}{6}, \frac{1}{6}\right)$
11	(89)	$\tilde{1}$	$\left(\frac{1}{5}, \frac{2}{5}\right)$
12	(94)	$\tilde{7}$	$\left(\frac{1}{8}, \frac{3}{8}\right)$
13	(90)	$\tilde{2}$	$\left(\frac{1}{10}, \frac{3}{10}\right)$
14	(96)	$\tilde{9}$	$\left(\frac{1}{12}, \frac{5}{12}\right)$

TABLE 23. The odd case by Yifan-Yang pullback

9. PROOF OF THEOREM 2.1

In Sections 5-8 we have defined an iterative construction that produces families of elliptically fibered Calabi-Yau n-folds with section from families of elliptic Calabi-Yau varieties of one dimension lower by a combination of a quadratic twist and a rational base transformation encoded in the generalized functional invariant. Moreover, all Weierstrass models are obtained through a sequence of constructions that start with the quadric pencil in Equation (2.1). Each step $n = 1, 2, 3, 4$ of our iterative construction has also provided a family of a closed transcendental n-cycle for each family of n-folds as the warped product of the corresponding transcendental cycle in dimension $n - 1$ with a Pochhammer contour. Upon integration of this cycle with the holomorphic n-form we obtain a period for the family of elliptically fibered Calabi-Yau n-folds with section. By construction, the period is holomorphic on the unit disk about the point $t = 0$ of maximally unipotent monodromy. Each holomorphic period is then annihilated by a Picard-Fuchs operator which is a Calabi-Yau type operator in the sense of [5].

n	n-form	construction	Eqns	n-cycles	period	Eqns	operator	Eqns
1	$\frac{dx}{2y}$	pure-twist	(5.2)	Σ_1	ω_1	(5.5)	L_2	(4.6)
1	$\frac{dX}{Y}$	mixed-twist	(5.7)	Σ'_1	$\omega_1^{(\mu)}$	(5.12)	$L_2^{(\mu)}$	(5.13)
1	$\frac{dX}{Y}$	rational transfo	(5.18)	$\tilde{\Sigma}'_1$	$\tilde{\omega}_1^{(\mu)}$	(5.19)	$\tilde{L}_2^{(\mu)}$	(5.20)
2	$\frac{du}{2} \wedge \frac{dX}{Y}$	pure-twist	(6.2) (6.34)	$\Sigma_2 = \Sigma_1 \times_u \Sigma'_1$ $\hat{\Sigma}_2 = \hat{\Sigma}_1 \times_u \Sigma'_1$	$\omega_2^{(\mu)}$ $\hat{\omega}_2^{(\mu)}$	(6.8) (6.42)	$L_3^{(\mu)}$ $\hat{L}_3^{(\mu)}$	(6.11) (6.48)
2	$\frac{du}{2} \wedge \frac{dX}{Y}$	mixed-twist	(6.4) (6.36)	$\Sigma''_2 = \Sigma''_1 \times_u \Sigma'_1$ $\hat{\Sigma}''_2 = \hat{\Sigma}''_1 \times_u \Sigma'_1$	$\omega_2^{(\mu)}$ $\hat{\omega}_2^{(\mu)}$	(6.15) (6.55)	$L_3^{(\mu)}$ $\hat{L}_3^{(\mu)}$	(6.11) (6.48)
2	$\frac{du}{2} \wedge \frac{dX}{Y}$	rational transfo	(6.28)	$\tilde{\Sigma}''^{(\tilde{m})}_2 = \Sigma''^{(\tilde{m})}_1 \times_u \Sigma'_1$	$\tilde{\omega}_2^{(\tilde{p},\tilde{q})}$	(6.29)	$\tilde{L}_3^{(\tilde{p},\tilde{q})}$	(6.30)
3	$\frac{ds}{2} \wedge \frac{du}{2} \wedge \frac{dX}{Y}$	pure-twist	(7.4) (7.6)	$\Sigma_3 = \Sigma_1 \times_s \Sigma_2$ $\hat{\Sigma}_3 = \hat{\Sigma}_1 \times_s \hat{\Sigma}_2$	$\omega_3^{(\mu)}$ $\hat{\omega}_3^{(\mu)}$	(7.24) (7.28)	$L_4^{(\mu)}$ $\hat{L}_4^{(\mu)}$	(7.26) (7.29)
3	$\frac{ds}{2} \wedge \frac{du}{2} \wedge \frac{dX}{Y}$	pure-twist	(7.8)	$\tilde{\Sigma}_3^{(\tilde{m})} = \Sigma_1 \times_s \tilde{\Sigma}''^{(\tilde{m})}_2$	$\tilde{\omega}_3^{(\tilde{p},\tilde{q})}$	(7.30)	$\tilde{L}_4^{(\tilde{p},\tilde{q})}$	(7.31)
3	$ds \wedge \frac{du}{2} \wedge \frac{dX}{Y}$	mixed-twist	(7.11) (7.14)	$\Sigma''_3 = \Sigma'_1 \times_s \Sigma''_2$ $\hat{\Sigma}''_3 = \hat{\Sigma}'_1 \times_s \hat{\Sigma}''_2$	$\omega_3^{(p,q)}$ $\hat{\omega}_3^{(p,q)}$	(7.24) (7.28)	$L_4^{(p,q)}$ $\hat{L}_4^{(p,q)}$	(7.39) (7.41)
3	$ds \wedge du \wedge \frac{dX}{Y}$	product-twist	(7.19) (7.22)	$\Sigma'_3 = \Sigma'_1 \times_s (\Sigma'_1 \times \Sigma'_1)$ $\tilde{\Sigma}'_3 = \Sigma'_1 \times_s (\tilde{\Sigma}'_1 \times \Sigma'_1)$	$\omega_3^{(\mu,\mu')}$ $\tilde{\omega}_3^{(\mu,\tilde{\mu})}$	(6.15) (6.55)	$L_4^{(\mu,\mu')}$ $\tilde{L}_4^{(\mu,\tilde{\mu})}$	(7.46) (7.48)
4	$ds \wedge \frac{du}{2} \wedge \frac{dv}{2} \wedge \frac{dX}{Y}$	pure-twist	(7.11) (7.14)	$\Sigma_4 = \Sigma_1 \times_v \Sigma''_3$ $\hat{\Sigma}_4 = \hat{\Sigma}_1 \times_v \hat{\Sigma}''_3$	$\omega_4^{(p,q)}$ $\hat{\omega}_4^{(p,q)}$	(7.38) (7.40)	$L_5^{(p,q)}$ $\hat{L}_5^{(p,q)}$	(8.11) (8.13)

TABLE 24. Iterative construction at Steps 1-4: cycles, periods, Calabi-Yau operators

The proof of Theorem 2.1 proceeds as follows: Bogner and Reiter classified all $Sp_4(\mathbb{C})$-rigid, quasi-unipotent local systems which have a maximal unipotent element and are induced by fourth-order Calabi-Yau operators. In particular, they obtained explicit expressions for all Calabi-Yau operators and closed formulas for special solutions of them. We prove that we have realized all of these operators and periods. There are four cases:

(1) The *hypergeometric case* consist of 14 operators called $P_1(4, 10, 4)$ [15, Theorem 6.1]. These operators precisely coincide with the 14 operators of Equations (7.39) and (7.40) obtained by the mixed-twist construction for the parameters listed in Table 20 in Section 7.5.2. Among these 14 case we found alternative geometric description for 8 and 10 by the pure-twist construction in Section 7.5.1 and the product-twist construction in Section 7.5.3, respectively.
(2) The *extra case* consist of 16 operators called $P_2(4, 6, 6)$ [15, Theorem 6.3]. These operators precisely coincide with the 16 operators of Equation (7.31) obtained by the pure-twist construction for the parameters listed in Table 19 in Section 7.5.2.
(3) The *even case* consist of 16 operators called $P_2(4, 6, 8)$ [15, Theorem 6.4]. These operators precisely coincide with the 16 operators of Equation (7.48) obtained by the product-twist construction for the parameters listed in Table 21 in Section 7.5.3.
(4) The *odd case* consist of 14 operators called $P_1(4, 8, 4)$ [15, Theorem 6.2]. These operators precisely coincide with the Yifan-Yang pullbacks in Equations (8.19) and (8.20) of the 14 operators of Equations (8.11) and (8.13) obtained by the pure-twist construction for the parameters listed in Table 23 in Section 8.2.

References to equations describing the constructed Weierstrass model for each family of elliptically fibered Calabi-Yau n-folds with section, the family of transcendental cycles and the corresponding period, as well as the Picard-Fuchs operator are listed in Table 24. □

10. Mirror families and the mixed-twist construction

In this section we will show that the generalized functional invariant of the mixed-twist construction captures all key features of the one-parameter mirror Calabi-Yau families of Fermat pencils necessary to compute holomorphic period integrals and monodromy matrices. In fact, the Euler integral transform for the holomorphic periods allows us to give a description of the Picard-Fuchs equations as non-resonant GKZ systems. Applying the results in [11] we deduce that the GKZ system has a basis of solutions as integrals of Mellin-Barnes type. We then use the analytic continuation of solutions and determine the transition matrix between a basis of solutions of the local monodromy around $t = 0$ and $t = \infty$. Since the monodromy group is generated by these two matrices, the group is determined. In the case of Calabi-Yau threefolds this reproduces the monodromy (up to conjugation) in [16]. In [19] the authors used a related procedure to construct the monodromy group of the Picard-Fuchs differential equations associated with the one-parameter families of Calabi-Yau threefolds in [31]. As mentioned in the introduction, there is the "14th case" that was not fully considered at that time, as it corresponds to a Calabi-Yau variety with a singular point. Special attention must then be exercised to guarantee that the ambient space is a Gorenstein toric Fano variety [29]. The generalized functional invariants also allow us to construct Weierstrass models for the mirror families and relations among the holomorphic periods given in the form of middle convolutions.

10.1. **GKZ description of the mirror families.** Let $\mathbb{P}^n(n+1)$ be the general family of hypersurfaces of degree $(n+1)$ in $\mathbb{P}^n$. Equation (1.1) from the introduction determines the one-parameter single-monomial deformation of a Fermat hypersurface $X_\lambda^{(n-1)}$ in $\mathbb{P}^n$. Cox and Katz determined [22] when general deformations of Calabi-Yau hypersurfaces remain Calabi-Yau. For example, for $n = 5$ there are 101 parameters associated with the complex structure of these manifolds, which can be thought of as the coefficients of additional terms in the quintic polynomials. Starting with a Fermat-type hypersurfaces V in $\mathbb{P}^n$, Yui [74] and Goto [36] classified all discrete symmetries G such that the quotients V/G are singular Calabi-Yau varieties with at worst Abelian quotient singularities. A theorem by Greene, Roan, Yau [41] guarantees that there is a crepant resolution of V/G. This is known as the Greene-Plesser orbifolding construction.

For the family (1.1) the discrete group of symmetries needed for the Greene-Plesser orbifolding is easily constructed: it is generated by the action $(X_0, X_j) \mapsto (\zeta_{n+1}^n X_0, \zeta_{n+1} X_j)$ for $1 \le j \le n$ with $\zeta_{n+1} = \exp\left(\frac{2\pi i}{n+1}\right)$. In virtue of the fact that the product of all generators multiplies the homogeneous coordinates by a common phase, the symmetry group is $G_{n-1} = (\mathbb{Z}/(n+1)\,\mathbb{Z})^{n-1}$. The affine variables

(10.1)
$$t = \frac{(-1)^{n+1}}{\lambda^{n+1}}\,, \quad x_1 = \frac{X_1^n}{(n+1)\, X_0 \cdot X_2 \cdots X_n\, \lambda}\,, \quad x_2 = \frac{X_2^n}{(n+1)\, X_0 \cdot X_1 \cdot X_3 \cdots X_n\, \lambda}, \quad \text{etc.}$$

are invariant under the action of G_{n-}, hence coordinates on $X_\lambda^{(n-1)}/G_{n-1}$. A family of special hypersurfaces $Y_t^{(n-1)}$ is then defined by the remaining relation between $x_1, \dots, x_n$, namely the equation

$$f_n(x_1, \dots, x_n, t) = x_1 \cdots x_n \Big(x_1 + \cdots + x_n + 1\Big) + \frac{(-1)^{n+1}\, t}{(n+1)^{n+1}} = 0\,. \tag{10.2}$$

Moreover, it was proved in [7] that the the family of special Calabi-Yau hypersurfaces $Y_t^{(n-1)}$ of degree $(n+1)$ in $\mathbb{P}^n$ given by Equation (10.2) is in fact the mirror family of a general hypersurface $\mathbb{P}^n(n+1)$ of degree $(n+1)$ and co-dimension one in $\mathbb{P}^n$. The subspace of the cohomology $H^{n-1}(X_\lambda^{(n-1)}, \mathbb{Q})$ invariant under the obvious action of G_{n-1} or, equivalently, the cohomology $H^{n-1}(Y_t^{(n-1)}, \mathbb{Q})$ as dimension n and has the Hodge numbers $(1, \dots, 1)$. We have the following:

Lemma 10.1. *For each $n \ge 2$, the family of hypersurfaces $Y_t^{(n-1)}$ given by Equation (10.2) is a fibration over $\mathbb{P}^1$ by hypersurfaces $Y_{\tilde t}^{(n-2)}$ where x_n is the affine base coordinate, and $\tilde t = -\frac{n^n\, t}{(n+1)^{n+1} x_n\, (x_n+1)^n}$ is given by the generalized functional invariant* $(1, n, 1)$.

Proof. For each $x_n \neq 0, -1$ substituting $\tilde x_i = x_i/(x_n+1)$ for $1 \le i \le n-1$ and $\tilde t = -n^n\, t/((n+1)^{n+1} x_n\, (x_n+1)^n)$ defines a fibration of the hypersurface (10.2) by $f_{n-1}(\tilde x_1, \dots, \tilde x_{n-1} \tilde t) = 0$ since

$$f_n(x_1, \dots, x_n, t) = x_n\, (x_n+1)^n\, f_{n-1}(\tilde x_1, \dots, \tilde x_{n-1}, \tilde t\,) = 0\,. \tag{10.3}$$

This is the mixed-twist construction with generalized functional invariant $(1, n, 1)$. □

In the GKZ formalism the construction of the family $Y_t^{(n-1)}$ is described as follows: from the homogeneous degrees of the defining Equation (1.1) and the coordinates of the ambient projective space for the family $X_\lambda^{(n-1)}$ we obtain the lattice $\mathbb{L}' = \mathbb{Z}(-(n+1), 1, 1, \dots, 1) \subset \mathbb{Z}^{n+2}$. We define a matrix $\mathsf{A}' \in \mathrm{Mat}(n+1, n+2; \mathbb{Z})$ by

$$\begin{pmatrix} 1 & 0 & 0 & \dots & (n+1) \\ 0 & 1 & 0 & \dots & -1 \\ 0 & \ddots & \ddots & \ddots & -1 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots\ 0 & 1 & -1 \end{pmatrix} \quad \Leftrightarrow \quad \mathsf{A}' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & \ddots & \ddots & \ddots & -1 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots\ 0 & 1 & -1 \end{pmatrix}, \tag{10.4}$$

and let $\mathcal{A}' = \{\vec{a}'_1, \dots, \vec{a}'_{n+2}\}$ denote the columns of the right-handed matrix obtained by a basis transformation in $\mathbb{Z}^{n+1}$ from the matrix on the left hand side. The finite subset $\mathcal{A}' \subset \mathbb{Z}^{n+1}$ generates $\mathbb{Z}^{n+1}$ as an abelian group and can be equipped with a group homomorphism $h' : \mathbb{Z}^{n+1} \to \mathbb{Z}$, in this case the projection onto the first coordinate, such that $h'(\mathcal{A}') = 1$. This means that $\mathcal{A}'$ lies in an affine hyperplane in $\mathbb{Z}^{n+1}$. The lattice of linear relations between the vectors in $\mathcal{A}'$ is easily checked to be precisely $\mathbb{L}' = \mathbb{Z}(-(n+1), 1, 1, \dots, 1) \subset \mathbb{Z}^{n+2}$. From A' we form the Laurent polynomial

$$P_{\mathsf{A}'}(z_1, \dots, z_{n+1}) = \sum_{\vec{a}' \in \mathcal{A}'} c_{\vec{a}}\, z_1^{a_1} \cdot z_2^{a_2} \cdots z_{n+1}^{a_{n+1}} = c_1\, z_1 + c_2\, z_1\, z_2 + c_3\, z_1\, z_3 + \dots + c_{n+2} z_1\, z_2^{-1} \cdots z_{n+1}^{-1}\ ,$$

and observe that the dehomogenized Laurent polynomial yields

$$\frac{x_1 \cdots x_n}{c_1}\ P_{\mathsf{A}'}\left(1, \frac{c_1 x_1}{c_2}, \frac{c_1 x_2}{c_3}, \dots, \frac{c_1 x_n}{c_{n+1}}\right) = f_n\left(x_1, \dots, x_n, t = (-1)^{n+1} \frac{(n+1)^{n+1}\, c_2 \cdots c_{n+2}}{c_1^{n+1}}\right)\ .$$

In the context of toric geometry this is interpreted as follows: a secondary fan can be constructed from the data $(\mathcal{A}', \mathbb{L}')$. This secondary fan is a complete fan of strongly convex polyhedral cones in $\mathbb{L}'^{\vee}_{\mathbb{R}} = \mathrm{Hom}(\mathbb{L}', \mathbb{R})$ which are generated by vectors in the lattice $\mathbb{L}'^{\vee}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{L}', \mathbb{Z})$. As the coefficients $c_1, \dots, c_{n+2}$ – or effectively t – vary, the zero locus of $P_{\mathcal{A}'}$ sweeps out the family of hypersurfaces $Y_t^{(n-1)}$ in $(\mathbb{C}^*)^{n+1}/\mathbb{C}^* = (\mathbb{C}^*)^n$. Both $(\mathbb{C}^*)^n$ and the hypersurfaces can then be suitably compactified. The members of the family $Y_t^{(n-1)}$ are Calabi-Yau varieties since the original Calabi-Yau varieties had codimension one in the ambient space.

10.2. **Recurrence relations between the period integrals.** We now describe the construction of the period integrals. The unique holomorphic $(n-1)$-form on $Y_t^{(n-1)}$ is given by

$$\eta_t^{(n-1)} = \frac{dx_2 \wedge dx_3 \wedge \dots \wedge dx_n}{\partial_{x_1} f_n(x_1, \dots, x_n, t)}\ . \tag{10.5}$$

One defines an $(n-1)$-cycle Σ on $Y_t^{(n-1)}$ by requiring that the period integral of $\eta_t^{(n-1)}$ over Σ corresponds by a residue computation in x_1 to the integral over the torus $T_n(r) = S^1_r \times \dots S^1_r$ with $S^1_r = \{|x| = r\} \subset \mathbb{C}$, i.e.,

(10.6)

$$\underbrace{\int \!\! \dots \!\! \int}_{\Sigma} \frac{dx_2 \wedge \dots \wedge dx_n}{\partial_{x_1} f_n(x_1, \dots, x_n, t)} = \frac{c_1}{2\pi i} \underbrace{\int \!\! \dots \!\! \int}_{T_n(r)} P_{\mathcal{A}}\left(1, \frac{c_1 x_1}{c_2}, \frac{c_1 x_2}{c_3}, \dots, \frac{c_1 x_n}{c_{n+1}}\right)^{-1} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

The right hand side of Equation (10.6) is the resonant $\mathcal{A}$-hypergeometric integral in the sense of [34, Thm. 2.7] derived from the data $(\mathcal{A}', \mathbb{L}')$ and

$$\vec{\alpha}' = \langle \alpha_1', -\beta_1' - 1, \ldots, -\beta_n' - 1\rangle^t = \langle -1, 0, \ldots, 0\rangle^t = \sum_{i=1}^{n+2} \gamma_i' \, \vec{a}_i'$$

with $\boldsymbol{\gamma}_0' = (\gamma_1', \ldots, \gamma_{n+2}') = (-1, 0, \ldots, 0)$. We will denote the period integral by $\omega_{n-1}(t) = \oint_K \eta_t^{(n-1)}$. We have the following:

Proposition 10.2. *For $n \geq 1$ and $|t| \leq 1$, there is a family of transcendental $(n-1)$-cycle Σ_{n-1} on $Y_t^{(n-1)}$ such that*

$$\omega_{n-1}(t) = \oint_{\Sigma_{n-1}} \eta_t^{(n-1)} = (2\pi i)^{n-1} \, {}_nF_{n-1}\left(\begin{matrix} \frac{1}{n+1} & \ldots & \frac{n}{n+1} \\ 1 & \ldots & 1 \end{matrix} \middle| \, t \right) . \tag{10.7}$$

The iterative structure (10.3) induces the iterative period relation

$$\omega_{n-1}(t) = (2\pi i) \, {}_nF_{n-1}\left(\begin{matrix} \frac{1}{n+1} & \ldots & \frac{n}{n+1} \\ \frac{1}{n} & \ldots & \frac{n-1}{n} \end{matrix} \middle| \, t \right) \star \omega_{n-2}(t) \quad \text{for } n \geq 2, \tag{10.8}$$

and the cycles Σ_n determined by $T_n(r_n) = \frac{n}{n+1} \cdot \left(T_{n-1}(r_{n-1}) \times S^1_{r_{n-1}}\right)$ with $r_n = 1 - \frac{n}{n+1}$ where $\frac{n}{n+1} \cdot \left(T_{n-1}(r_{n-1}) \times S^1_{r_{n-1}}\right)$ indicates that each coordinate in $\mathbb{C}^n$ is scaled by a factor of $\frac{n}{n+1}$.

Proof. We prove the statement by induction over n. For $n = 1$ we set $\Sigma_0 = \{t\}$ and obtain for $|t| < 1$

$$\omega_0(t) = \frac{1}{\partial_{x_1} f_1(x_1, t)} = (1-t)^{-\frac{1}{2}} = {}_1F_0\left(\frac{1}{2} \middle| \, t \right) .$$

We set $r_1 = \frac{1}{2}$ such that for $|t| < 1$ and $|x_1| = r_1$ it follows

$$\left| \frac{t}{4\, x_1(x_1+1)} \right| < \frac{1}{2(1-|x_1|)} = 1 .$$

Using the formula $(1+z)^{-r} = \sum_{l \geq 0} \frac{\Gamma(l+r)}{\Gamma(r)\Gamma(l+1)} (-z)^l$ for $|z| < 1$, we can carry out the following residue computation

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|x_1|=r_1} \frac{dx_1}{f_1(x_1,t)} &= \sum_{k \geq 0} \left(-\frac{t}{4}\right)^k \frac{1}{2\pi i} \oint_{|x_1|=r_1} \frac{dx_1}{x_1^{k+1}(x_1+1)^{k+1}} \\ &= \sum_{k \geq 0} \left(-\frac{t}{4}\right)^k \sum_{l \geq 0} \frac{\Gamma(k+l+1)}{\Gamma(k+1)\,\Gamma(l+1)} (-1)^l \frac{1}{2\pi i} \oint_{|x_1|=r_1} \frac{dx_1}{x_1^{k-l+1}} \\ &= \sum_{k \geq 0} \frac{(2k)!}{2^{2k} k!} \frac{t^k}{k!} = \sum_{k \geq 0} \frac{\left(\frac{1}{2}\right)_k t^k}{k!} = {}_1F_0\left(\frac{1}{2} \middle| \, t \right), \end{aligned} \tag{10.9}$$

where in the last step we used the product formula

$$\frac{n^{nk}\,((n+1)\,k)!}{(n+1)^{(n+1)k}\,(nk)!} = \frac{\prod_{m=1}^{n} \left(\frac{m}{n+1}\right)_k}{\prod_{m=1}^{n-1} \left(\frac{m}{n}\right)_k} . \tag{10.10}$$

Let $n \geq 2$: we set $\tilde{r} = r_{n-1} = 1 - \frac{n-1}{n}$ and assume that there is a transcendental $(n-2)$-cycle $\tilde{\Sigma}_{n-2}$ on $\tilde{Y}^{(n-2)}_{\tilde{t}}$ such that for $|\tilde{t}| < 1$ the period integral for $\tilde{\eta}^{(n-2)}_{\tilde{t}}$ is given by

$$\omega_{n-2}(\tilde{t}) = \oint_{\tilde{\Sigma}_{n-2}} \tilde{\eta}^{(n-2)}_{\tilde{t}} = \frac{1}{2\pi i} \underbrace{\int \dots \int}_{\tilde{T}(\tilde{r})_{n-1}} \frac{d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_{n-1}}{f_{n-1}(\tilde{x}_1, \dots, \tilde{x}_{n-1}, \tilde{t})}\,, \tag{10.11}$$

with an absolutely convergent series expansion for the period $\omega_{n-2}(\tilde{t}) = \sum_{k \geq 0} f_k\, \tilde{t}^k$. We set $r = r_n = 1 - \frac{n}{n+1}$ such that for $|t| < 1$ and $|x_n| = r$ we have

$$\left| -\frac{n^n\, t}{(n+1)^{n+1}\, x_n\, (x_n+1)^n} \right| < \frac{n^n}{(n+1)^{n+1}\, (1 - \frac{n}{n+1}) \left(\frac{n}{n+1}\right)^n} = 1\,.$$

We carry out the following residue computation

$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{|x_n|=r} \frac{dx_n}{x_n(x_n+1)}\, \omega_{n-2}\left(-\frac{n^n\, t}{(n+1)^{n+1}\, x_n\, (x_n+1)^n} \right) \\
&= \sum_{k \geq 0} f_k \left(-\frac{n^n t}{(n+1)^{n+1}} \right)^k \frac{1}{2\pi i} \oint_{|x_n|=r} \frac{dx_n}{x_n^{k+1} (x_n+1)^{nk+1}} \\
&= \sum_{k \geq 0} f_k \left(-\frac{n^n t}{(n+1)^{n+1}} \right)^k \sum_{l \geq 0} \frac{\Gamma(nk+l+1)}{\Gamma(nk+1)\,\Gamma(l+1)} (-1)^l \frac{1}{2\pi i} \oint_{|x_n|=r} \frac{dx_n}{x_n^{k-l+1}} \\
&= \sum_{k \geq 0} f_k\, \frac{n^{nk}\, ((n+1)\,k)!}{(n+1)^{(n+1)k}\, (nk)!\, k!}\, t^k = \sum_{k \geq 0} f_k\, \frac{\prod_{m=1}^{n} \left(\frac{m}{n+1}\right)_k}{k!\, \prod_{m=1}^{n-1} \left(\frac{m}{n}\right)_k}\, t^k \\
&= {}_nF_{n-1}\left(\left. \begin{array}{c} \frac{1}{n+1} \quad \dots \quad \frac{n}{n+1} \\ \frac{1}{n} \quad \dots \quad \frac{n-1}{n} \end{array} \right| t \right) \star \omega_{n-2}(t)\,.
\end{aligned} \tag{10.12}$$

For $x_n \neq 0, -1$ the coordinate transformation $\tilde{x}_i = x_i/(x_n+1)$ for $1 \leq i \leq n-1$ and $\tilde{t} = -n^n\, t/((n+1)^{n+1} x_n\, (x_n+1)^n)$ yields

$$\eta^{(n-1)}_t = \tilde{\eta}^{(n-2)}_{\tilde{t}} \wedge \frac{dx_n}{x_n(x_n+1)}\,. \tag{10.13}$$

For any fixed x_n on S^1_r write $1/(x_n+1) = R\, e^{i\varphi}$ with $\frac{n+1}{n+2} \leq R \leq \frac{n+1}{n}$, the transformation $\tilde{x}_i = x_i/(x_n+1)$ maps the circle $\tilde{x}_i = \tilde{r}\, e^{it}$ to the circle $x_i = R\, \tilde{r}\, e^{i(t+\varphi)}$ with $0 < R\, \tilde{r} \leq \frac{n+1}{n^2} < 1$ as $n \geq 2$. Since poles in the period integrands are located at $|x_i| = 0, 1$ we can deform this circle into the circle $0 < |x_i| = r < 1$ in a residue computation to obtain

$$\underbrace{\int \dots \int}_{T(r)_n} \frac{dx_1 \wedge \dots \wedge dx_n}{f_n(x_1, \dots, x_n, t)} = \oint_{|x_n|=C} \frac{dx_n}{x_n(x_n+1)} \underbrace{\int \dots \int}_{\tilde{T}(\tilde{r})_{n-1}} \frac{d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_{n-1}}{f_{n-1}(\tilde{x}_1, \dots, \tilde{x}_{n-1}, \tilde{t})}\,. \tag{10.14}$$

Equation (10.8) follows. □

10.3. **GKZ description of the period integrals.** After having established the recursive relation of the holomorphic period integrals for the mirror families in Proposition 10.2, we will show how the corresponding monodromy representations are derived. In fact, the so-called Euler-integral formula for the hypergeometric functions ${}_nF_{n-1}$ generates a second set of *non-resonant* GKZ data $(\mathcal{A}, \mathbb{L}, \boldsymbol{\gamma}_0)$ from the *resonant* GKZ data$(\mathcal{A}', \mathbb{L}', \boldsymbol{\gamma}'_0)$ by integration. The GKZ data $(\mathcal{A}, \mathbb{L}, \boldsymbol{\gamma}_0)$ determines local Frobenius bases of solutions around $t = 0$ and $t = \infty$. Their Mellin-Barnes integral representation determines the transition matrix between them by analytic continuation.

We will always assume that we have n rational parameters, namely $\rho_1, \dots, \rho_n \in (0,1) \cap \mathbb{Q}$, and consider the generalized hypergeometric function

$$ {}_nF_{n-1}\left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix} \middle| t \right) , $$

which include all periods from Proposition 10.2. The *Euler-integral formula* then specializes to the identity

(10.15)

$$ \left[\prod_{i=1}^{n-1} \Gamma(\rho_i)\, \Gamma(1-\rho_i) \right] {}_nF_{n-1}\left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix} \middle| t \right) = \left[\prod_{i=1}^{n-1} \int_0^1 \frac{dz_i}{z_i^{1-\rho_i} (1-z_i)^{\rho_i}} \right] (1 - t\, z_1 \cdots z_{n-1})^{-\rho_n} . $$

The rank-n hypergeometric differential equation satisfied by ${}_nF_{n-1}$ is given by

$$ \Big[\theta^n - t\,(\theta+\rho_1)\cdots(\theta+\rho_n) \Big] F(t) = 0 \tag{10.16} $$

with $\theta = t\frac{d}{dt}$, and it has the Riemann symbol

$$ \mathcal{P} \left(\begin{array}{ccc|c} 0 & 1 & \infty & \\ \hline 0 & 0 & \rho_1 & \\ 0 & 1 & \rho_2 & \\ \vdots & \vdots & \vdots & t \\ 0 & n-2 & \rho_{n-1} & \\ 0 & n-1-\sum_{j=1}^n \rho_j & \rho_n & \end{array} \right) . \tag{10.17} $$

From the Euler-integral (10.15) we immediately read off the A-matrix $\mathsf{A} \in \operatorname{Mat}(2n-1, 2n; \mathbb{Z})$ given by

$$ \left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 1 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 \end{array} \right) \quad \Leftrightarrow \quad \mathsf{A} = \left(\begin{array}{cccc|c|cccc|c} 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 1 & 0 \end{array} \right) , \tag{10.18} $$

and let $\mathcal{A} = \{\vec{a}_1, \dots, \vec{a}_{2n}\}$ denote the columns of the matrix. The entries for the matrix on the left hand side of (10.18) are determined as follows: the first n entries in each column label which of the n terms $(1-z_i)^{\rho_i}$ or $(1 - t\, z_1 \cdots z_{n-1})^{-\rho_n}$ in the integrand of the Euler-integral (10.15) is specified. For each term, two column vectors are needed and the entries in rows $n+1,\dots,2n-1$ label the exponents of variables z_i appearing. For example, the last two columns determine the term $(1 - t\, z_1 \cdots z_{n-1})^{-\rho_n}$. The finite subset $\mathcal{A} \subset \mathbb{Z}^{2n-1}$ generates $\mathbb{Z}^{2n-1}$ as an abelian group and is equipped with a group homomorphism $h : \mathbb{Z}^{2n-1} \to \mathbb{Z}$, in this case the sum of the first n coordinates, such that $h(\mathcal{A}) = 1$. The lattice of linear relations between the vectors in $\mathcal{A}$ is easily checked to be $\mathbb{L} = \mathbb{Z}(1,\dots,1,-1,\dots,-1) \subset \mathbb{Z}^{2n}$. The toric data $(\mathsf{A},\mathbb{L})$ has an associated GKZ system of differential equations which is equivalent to the differential equation (10.16). Equivalently, the right hand side of the Equation (10.15) is the $\mathcal{A}$-hypergeometric integral in the sense of [34, Thm. 2.7] derived from the data $(\mathcal{A},\mathbb{L})$ and the additional vector

$$\vec{\alpha} = \langle \alpha_1,\dots,\alpha_{n-1},-\beta_1-1,\dots,-\beta_n-1\rangle^t = \langle -\rho_1,\dots,-\rho_n,-\rho_1,\dots,-\rho_{n-1}\rangle^t = \sum_{i=1}^{2n} \gamma_i\, \vec{a}_i$$

where we have set $\boldsymbol{\gamma}_0 = (\gamma_1,\dots,\gamma_{2n}) = (0,\dots,0,-\rho_1,\dots,-\rho_n) \subset \mathbb{Z}^{2n}$. We always have the freedom to shift $\boldsymbol{\gamma}_0$ by elements in $\mathbb{L}\otimes\mathbb{R}$ while leaving $\vec{\alpha}$ and any $\mathcal{A}$-hypergeometric integral unchanged. We observe that $\alpha_i,\beta_j \notin \mathbb{Z}$ for $i = 1,\dots,n-1$ and $j = 1,\dots,n$ and $\sum_i \alpha_i + \sum_j \beta_j \equiv -\rho_n \mod (1) \notin \mathbb{Z}$. It was proved in [34, Ex. 2.17] that this is equivalent to the non-resonance of the GKZ system.

10.3.1. *Remarks on the toric data.* The B-matrix $\mathsf{B} = (1,\dots,1,-1,\dots,-1)$ contains an integer basis of $\mathbb{L}$ and satisfies $\mathsf{A}\cdot\mathsf{B}^t = 0$. We also write $\mathsf{B} = \sum b_i \hat{e}_i$ in terms of the standard basis $\{\hat{e}_i\}_{i=1}^{2n}$. The space $\mathbb{L}\otimes\mathbb{R} \subset \mathbb{R}^{2n}$ is parameterized by the tuple $(s,\dots,s,-s,\dots,-s) \in \mathbb{R}^{2n}$ with $s\in\mathbb{R}$. The polytope $\Delta_{\mathcal{A}}$ defined as convex hull of the vectors contained in $\mathcal{A}$ is the primary polytope associated with $\mathcal{A}$. A short exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{2n} \longrightarrow \mathbb{Z}^{2n-1} \to 0$$

is obtained by mapping each vector $\boldsymbol{\ell} = \sum l_i \hat{e}_i \in \mathbb{Z}^{2n}$ to the vector $\sum l_i\, \vec{a}_i \in \mathbb{Z}^{2n-1}$. As the linear relations between vectors in $\mathcal{A}$ are given by the lattice $\mathbb{L}$, this sequence is exact. The corresponding dual short exact sequence (over $\mathbb{R}$) is given by

$$0 \longrightarrow \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n} \overset{\pi}{\longrightarrow} \mathbb{L}^{\vee}_{\mathbb{R}} \cong \mathbb{R} \longrightarrow 0,$$

with $\pi(u_1,\dots,u_{2n}) = u_1 + \dots + u_n - u_{n+1} - \dots - u_{2n}$. Restricting π to the positive orthant in $\mathbb{R}^{2n}$ and calling it $\hat{\pi}$, we observe that for each $s\in\mathbb{R}$ the set $\hat{\pi}^{-1}(s)$ is a convex polyhedron. For $s \in \mathbb{L}^{\vee}_{\mathbb{R}}$, there are two maximal cones C_+ and C_- in the secondary fan of $\mathcal{A}$ for positive and negative real value s, respectively. The lists of vanishing components for vertex vectors in each $\hat{\pi}^{-1}(s)$ are given by

$$T_{C_+} = \bigcup_{k=1}^{n} \Big\{ \underbrace{\{1,\dots,\widehat{k},\dots,n,n+1,\dots 2n\}}_{=:I_k} \Big\}, \quad T_{C_-} = \bigcup_{k=1}^{n} \Big\{ \underbrace{\{1,\dots,n,n+1,\widehat{k+n},\dots\dots 2n\}}_{=:I_{k+n}} \Big\}.$$

The symbol $\widehat{k}$ indicates that the entry k has been suppressed. For each member I of $T_{C_\pm}$, we define $\boldsymbol{\gamma}^I = \boldsymbol{\gamma}_0 - \mu^I \mathsf{B}$ such that $\boldsymbol{\gamma}^I_i = 0$ for $i \notin I$. We have

$$\boldsymbol{\gamma}^I = \begin{cases} \boldsymbol{\gamma}_0 & \text{for } I \in T_{C_+}, \quad \mu^I = 0, \\ (-\rho_k,\dots,-\rho_k,\rho_k-\rho_1,\dots,0,\dots,\rho_k-\rho_n) & \text{for } I = I_{n+k} \in T_{C_-}, \quad \mu^{I_{n+k}} = \rho_k. \end{cases} \tag{10.19}$$

For $I_k \in T_{C_\pm}$ we denote the so-called convergence direction by $\boldsymbol{\nu}^{I_k} = (\nu_1, \dots, \nu_{2p}) = (\delta_i^k)_{i=1}^{2p} \in \mathbb{L} \otimes \mathbb{R}$ such that $\hat{\pi}(\boldsymbol{\nu}^{I_k}) = \pm 1$. Using the B-matrix one defines the so-called zonotope

$$\mathsf{Z}_\mathsf{B} = \left\{ \frac{1}{4} \sum_{i=1}^{2n} \mu_i\, b_i \,\middle|\, \mu_i \in (-1,1) \right\} = \left(-\frac{n}{2}, \frac{n}{2} \right) \subset \mathbb{L}_\mathbb{R}^\vee \cong \mathbb{R}.$$

Let $\boldsymbol{u} = (u_1, \dots, u_{2n})$ with $u_j = |u_j| \exp(2\pi i \tau_j)$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{2n})$. For any $\boldsymbol{u}$ with $\boldsymbol{\tau}$ such that $\sum b_i \tau_i \in \mathsf{Z}_\mathcal{B}$ and any $\boldsymbol{\gamma}$ equivalent to $\boldsymbol{\gamma}_0$ up to elements in $\mathbb{L} \otimes \mathbb{R}$ with $\gamma_{n+i} < \sigma < -\gamma_i$ for all $i = 1, \dots, n$, it follows from [11, Cor. 4.2] that the Mellin-Barnes integral given by

$$\mathsf{M}_{\boldsymbol{\tau}}(u_1, \dots, u_{2n}) = \int_{\sigma + i\mathbb{R}} \left[\prod_{i=1}^{2n} \Gamma(-\gamma_i - b_i s)\, u_i^{\gamma_i + b_i s} \right] ds\ , \tag{10.20}$$

is *absolutely convergent* and satisfies the GKZ differential system for $(\mathcal{A}, \mathbb{L})$ or, equivalently, as explained below the differential equation (10.16) after restricting to $u_2 = \dots = u_{2n} = 1$ and $u_1 = (-1)^n t$. In the case of the hypergeometric function it follows from [11, Prop. 4.6] that the differential equation (10.16) even allows for a *basis* of such Mellin-Barnes integrals since the zonotope $\mathsf{Z}_\mathcal{B}$ contains n distinct points $\{-\frac{n-1}{2} + k\}_{k=0}^{n-1}$ whose coordinates differ by integers.

A toric variety $\mathcal{V}_\mathcal{A}$ can be associated with the secondary fan by glueing together certain affine schemes, one scheme for every maximal cone in the secondary fan. Details can be found in [64]. Here, the secondary fan has two maximal cones C_+ and C_-, and one can easily see that the toric variety $\mathcal{V}_\mathcal{A}$ is the projective line $\mathcal{V}_\mathcal{A} = \mathbb{P}^1$ which is the the domain of definition for the variable t in Equation (10.15). Each member in the list for a maximal cone contains $2n - 1$ integers and define a subdivision of the primary polytope $\Delta_\mathcal{A}$ by polytopes generated by the subdivision, called regular triangulation. In our case these regular triangulations are unimodular, i.e.,

$$\forall I_k \in T_{C_\pm} : \quad \left| \det(\vec{a}_i)_{i \in I_k} \right| = \left| b_k \right| = 1\ .$$

Given $\mathcal{A}$ and its secondary fan, we define a ring $\mathcal{R}_\mathcal{A}$ by dividing the free polynomial ring in $2n$ variables by the ideal $\mathcal{I}_\mathcal{A}$ generated by the linear relations of $\mathcal{A}$ and the ideal $\mathcal{I}_{C_\pm}$ generated by the regular triangulations. In our situation, we obtain $\mathcal{R}_\mathcal{A}$ from the list of generators given by

$$\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{2n}) = \epsilon\,(1, \dots, 1, -1, \dots, -1) \in \mathcal{R}_\mathcal{A}$$

with relation $\epsilon^n = 0$, i.e., $\mathcal{R}_\mathcal{A} = \mathbb{Z}[\epsilon]/(\epsilon^n)$ is a free $\mathbb{Z}$-module of rank n.

10.3.2. *Basis of solutions around zero.* Using the toric data we derive a local basis of solutions of the differential equation (10.16) around the point $t = 0$ [64]. For the convergence direction $\boldsymbol{\nu}^{I_1}$ in T_{C_+}, the Γ-series is a series solutions of the GKZ system for $(\mathbb{L}, \boldsymbol{\gamma}_0)$ and given by

$$\Phi_{\mathbb{L}, \boldsymbol{\gamma}_0}(u_1, \dots, u_{2n}) = \sum_{\boldsymbol{\ell} \in \mathbb{L}} \frac{u_1^{\gamma_1 + \ell_1} \cdots u_{2n}^{\gamma_{2n} + \ell_{2n}}}{\Gamma(\gamma_1 + \ell_1 + 1) \cdots \Gamma(\gamma_{2n} + \ell_{2n} + 1)}\ . \tag{10.21}$$

We have the following:

Lemma 10.3. *For the convergence direction $\boldsymbol{\nu}^{I_1}$ in T_{C_+}, the Γ-series for $(\mathbb{L}, \boldsymbol{\gamma}_0)$ equals*

$$\Phi_{\mathbb{L}, \boldsymbol{\gamma}_0}(u_1, \dots, u_{2n}) = \left[\prod_{i=1}^{n} \frac{1}{\Gamma(1 - \rho_i)\, u_{n+i}^{\rho_i}} \right] {}_nF_{n-1}\!\left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix} \middle| t \right) \tag{10.22}$$

with $t = (-1)^n u_1 \cdots u_n/(u_{n+1} \cdots u_{2n})$ which we assume is positive. Moreover, convergence in convergence direction $\boldsymbol{\nu}^{I_1} = (\nu_1, \dots, \nu_{2p})$ is guaranteed for all $u_1, \dots, u_{2n}$ with $|u_i| = t^{\nu_i}$ and $0 \le t < 1$.

Proof. We observe that

$$\begin{aligned} &\Phi_{\mathbb{L},\boldsymbol{\gamma}_0}(u_1, \dots, u_{2n}) \sum_{k \ge 0} \frac{u_1^k \cdots u_n^k \cdot u_{n+1}^{-\rho_1 - k} \cdots u_{2n}^{-\rho_n - k}}{(k!)^n \, \Gamma(-\rho_1 - k + 1) \cdots \Gamma(-\rho_n - k + 1)} \\ &= \left[\prod_{i=1}^n \frac{1}{\Gamma(1 - \rho_i) \, u_{n+i}^{\rho_i}} \right] \sum_{k \ge 0} \frac{(\rho_1)_k \cdots (\rho_n)_k}{(k!)^n} \, t^k \, . \end{aligned} \tag{10.23}$$

The summation over $\mathbb{L}$ reduces to non-negative integers as the other terms vanish when $1/\Gamma(k + 1) = 0$ for $k < 0$. Using the identities

$$(\rho)_k = (-1)^k \frac{\Gamma(1 - \rho)}{\Gamma(1 - k - \rho)}, \quad \Gamma(z) \, \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \, , \tag{10.24}$$

we obtain Equation (10.22). Equation (10.21) shows that restricting the variables $u_2 = \cdots = u_{2n} = 1$ to a base point, the convergence of the Γ-series $\Phi_{\mathbb{L},\boldsymbol{\gamma}_0}((-1)^n t, 1 \dots, 1)$ is guaranteed for $|u_1| = t$ with t sufficiently small. □

Remark. Notice that we obtain the same Γ-series for all convergence directions $\boldsymbol{\nu}^{I_r}$ with $1 \le r \le n$ in T_{C_+}. This is due to the fact that in the Riemann symbol (10.17) at $t = 0$ the critical exponent 0 has multiplicity n.

However, from the maximal cone C_+ of the secondary fan of $\mathcal{A}$, we can still construct a local basis of solutions of the GKZ system around $t = 0$ by expanding the twisted power series $\Phi_{\mathbb{L},\boldsymbol{\gamma}_0 + \boldsymbol{\epsilon}}(u_1, \dots, u_{2n})$ over $\mathcal{R}_{\mathcal{A}}$ [64]. We have the following:

Lemma 10.4. *For $|t| < 1$, choosing the branch $t^\epsilon = \exp(\epsilon \ln t)$ we obtain for the twisted power series over $\mathcal{R}_{\mathcal{A}}$*

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}_0 + \boldsymbol{\epsilon}}(u_1, \dots, u_{2n}) = \frac{e^{2\pi i \epsilon}}{\Gamma(1 + \epsilon)^n} \left[\prod_{i=1}^n \frac{1}{\Gamma(1 - \rho_i - \epsilon) \, u_{n+i}^{\rho_i}} \right] \; t^\epsilon \, {}_nF_{n-1}^{(\epsilon)} \left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix} \middle| \, t \right). \tag{10.25}$$

Proof. The proof uses $1/(1 + \epsilon)_k^n = O(\epsilon^n) = 0$ for $k < 0$ because for $k \in \mathbb{Z}$ we have

$$\frac{1}{(1 + \epsilon)_k} = \frac{\Gamma(1 + \epsilon)}{\Gamma(k + 1 + \epsilon)} = \begin{cases} \epsilon(\epsilon - 1) \cdots (\epsilon + k + 1) & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ \dfrac{1}{(1 + \epsilon)(2 + \epsilon) \cdots (m + \epsilon)} & \text{if } k > 0. \end{cases}$$

□

We renormalize the generating function for the local solutions by introducing

$$f(\epsilon, t) = t^\epsilon \, {}_nF_{n-1}^{(\epsilon)} \left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix} \middle| \, t \right) = \sum_{k \ge 0} \frac{(\rho_1 + \epsilon)_k \cdots (\rho_n + \epsilon)_k}{(1 + \epsilon)_k^n} \, t^{k + \epsilon}. \tag{10.26}$$

For $r = 0, \ldots, n-1$, we also introduce the functions

$$y_r(t) = \frac{1}{r!}\,\frac{\partial^r}{\partial\epsilon^r}\bigg|_{\epsilon=0} {}_nF^{(\epsilon)}_{n-1}\left(\begin{matrix}\rho_1 & \ldots & \rho_n\\ 1 & \ldots & 1\end{matrix}\,\bigg|\, t\right), \qquad y_0(t) = f(0,t) = {}_nF_{n-1}\left(\begin{matrix}\rho_1 & \ldots & \rho_n\\ 1 & \ldots & 1\end{matrix}\,\bigg|\, t\right). \tag{10.27}$$

We have the following:

Lemma 10.5. *For $|t| < 1$, we obtain*

$$f(\epsilon,t) = \sum_{m=0}^{n-1}\left(2\pi i\epsilon\right)^m f_m(t) = \sum_{m=0}^{n-1}\left(2\pi i\epsilon\right)^m \sum_{r=0}^{m}\frac{1}{r!}\left(\frac{\ln t}{2\pi i}\right)^r \frac{y_{m-r}(t)}{(2\pi i)^{m-r}}\,, \tag{10.28}$$

such that $f_m(t) = \frac{1}{(2\pi i)^m m!}\frac{\partial^m}{\partial\epsilon^m}|_{\epsilon=0} f(\epsilon,t)$ for $m = 0, \ldots, n-1$.

As proved in [64], the functions $\{f_r\}_{r=0}^{n-1}$ form a local basis of solutions around $t = 0$, and the functions $y_r(t)$ with $r = 0, \ldots n-1$ are holomorphic in a neighborhood of $t = 0$. The local monodromy group is generated by the cycle $(u_1, \ldots, u_{2n}) = (R_1 \exp(i\varphi), R_2, \ldots, R_{2n})$ based at the point $(R_1, \ldots, R_{2n})$ for $\varphi \in [0,1]$ such that $|u_2| = \cdots = |u_{2n}| = 1$ and $|u_1| = t$. Equivalently, we consider the local monodromy of the hypergeometric differential equation generated by $t = t_0 \exp(i\varphi)$ for $0 < t_0 < 1$ and $\varphi \in [0, 2\pi]$. We have the following:

Lemma 10.6. *The local monodromy of the basis $\boldsymbol{f}^t = \langle f_{n-1}, \ldots, f_0\rangle^t$ of solutions to the differential equation (10.16) at $t = 0$ is given by*

$$\mathsf{m}_0 = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \ldots & \frac{1}{(n-2)!}\\ 0 & 1 & 1 & \ldots & \frac{1}{(n-3)!}\\ \vdots & \ddots & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & 1\\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}. \tag{10.29}$$

10.3.3. *Basis of solutions around infinity.* We will assume that $0 < \rho_1 < \cdots < \rho_n < 1$. Using the toric data we derive a local basis of solutions of the differential equation (10.16) around the point $t = \infty$. For the convergence direction $\boldsymbol{\nu}^{I_{n+r}}$ in T_{C_-}, the Γ-series is a series solutions of the GKZ system for $(\mathbb{L}, \boldsymbol{\gamma}^{I_{n+r}})$ and given by

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1, \ldots, u_{2n}) = \sum_{\boldsymbol{\ell}\in\mathbb{L}} \frac{u_1^{\gamma_1-\mu^{I_{r+n}}+\ell_1}\cdots u_{2n}^{\gamma_{2n}+\mu^{I_{r+n}}+\ell_{2n}}}{\Gamma(\gamma_1-\mu^{I_{r+n}}+\ell_1+1)\cdots\Gamma(\gamma_{2n}+\mu^{I_{r+n}}+\ell_{2n}+1)}\,. \tag{10.30}$$

We have the following:

Lemma 10.7. *For the convergence direction $\boldsymbol{\nu}^{I_{n+r}}$ in T_{C_-}, the Γ-series is a series solutions for $(\mathbb{L}, \boldsymbol{\gamma}^{I_{n+r}})$ equals*

$$\begin{aligned}\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1, \ldots, u_{2n}) &= \frac{e^{\pi i n\rho_r}}{\Gamma(1-\rho_r)^n}\left[\prod_{i=1}^{n}\frac{1}{\Gamma(1+\rho_r-\rho_i)\,u_{n+i}^{\rho_i}}\right]\\ &\times t^{-\rho_r}\,{}_nF_{n-1}\left(\begin{matrix}\rho_r & \ldots & \ldots & \rho_r\\ 1+\rho_r-\rho_1 & \ldots\widehat{1}\ldots & & 1+\rho_r-\rho_n\end{matrix}\,\bigg|\,\frac{1}{t}\right)\end{aligned} \tag{10.31}$$

with $t = (-1)^n u_1 \cdots u_n/(u_{n+1} \cdots u_{2n})$ which we assumed is positive. Moreover, restricting the variables $u_1 = \cdots = \widehat{u_{n+r}} = \cdots = u_{2n} = 1$ to a base point, the convergence of the Γ-series $\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(1, \dots, (-1)^n/t, \dots, 1)$ is guaranteed for $t > 1$. The symbol $\widehat{1}$ indicates that the entry $1 + \rho_r - \rho_i$ for $i = r$ has been suppressed.

Proof. We obtain for the Γ-series

$$\begin{aligned}\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1, \dots, u_{2n}) &= \sum_{k\le 0} \frac{u_1^{-\rho_r+k} \cdots u_n^{-\rho_r+k} \cdot u_{n+1}^{\rho_r-\rho_1-k} \cdots u_{2n}^{\rho_r-\rho_n-k}}{\Gamma(1+k-\rho_r)^n\, \Gamma(\rho_r-\rho_1-k+1) \cdots \Gamma(1-k) \cdots \Gamma(\rho_r-\rho_n-k+1)} \\ &= \frac{e^{\pi i n \rho_r}}{\Gamma(1-\rho_r)^n} \left[\prod_{i=1}^n \frac{1}{\Gamma(1+\rho_r-\rho_i)\, u_{n+i}^{\rho_i}}\right] \left(\frac{u_{n+1}\cdots u_{2n}}{(-1)^n u_1 \cdots u_n}\right)^{\rho_r} \\ &\times \sum_{k\ge 0} \frac{(\rho_r)_k^n}{(1+\rho_r-\rho_1)_k \cdots (1+\rho_r-\rho_n+1)_k} \left(\frac{u_{n+1}\cdots u_{2n}}{(-1)^n u_1 \cdots u_n}\right)^k .\end{aligned}$$

The result follows. □

Remark. Based on the assumption that $0 < \rho_1 < \cdots < \rho_n < 1$, we have obtained *n different* Γ-series for the different convergence directions $\boldsymbol{\nu}^{I_{n+r}}$ with $1 \le r \le n$ in T_{C_-}.

The local monodromy group is generated by the cycle based at the point $(R_1, \dots, R_{2n})$ given by $(u_1, \dots, u_{2n}) = (R_1, \dots, R_{n+r} \exp(-i\varphi), \dots, R_{2n})$ for $\varphi \in [0, 2\pi]$ such that $|u_1| = \cdots = |u_{2n}| = 1$ and $|u_{n+r}| = 1/t$. Equivalently, we consider the local monodromy generated by $t = t_0 \exp(i\varphi)$ for $t_0 \gg 1$ and $\varphi \in [0, 2\pi]$. From the Riemann symbol (10.17) we observe that the functions

$$F_r(t) = A_r\, t^{-\rho_r}\, {}_nF_{n-1}\left(\begin{array}{cccc} \rho_r & \dots & \dots & \rho_r \\ 1+\rho_r-\rho_1 & \dots \widehat{1} \dots & & 1+\rho_r-\rho_n \end{array} \middle| \frac{1}{t}\right) \tag{10.32}$$

for $r = 1, \dots, n$ and any non-zero constants A_r, form a Frobenius basis of solutions to the differential equation (10.16) at $t = \infty$. We have the following:

Lemma 10.8. *The local monodromy of the basis $\boldsymbol{F}^t = \langle F_n, \dots, F_1 \rangle^t$ of solutions to the differential equation (10.16) at $t = \infty$ is given by*

$$\mathsf{M}_\infty = \begin{pmatrix} e^{-2\pi i \rho_n} & & \\ & \ddots & \\ & & e^{-2\pi i \rho_1} \end{pmatrix} . \tag{10.33}$$

10.3.4. *The transition matrix.* We will again assume that $0 < \rho_1 < \cdots < \rho_n < 1$. The solution (10.26) has an integral representation of Mellin-Barnes type [19] given by

$$f(\epsilon, t) = \frac{t^\epsilon}{2\pi i} \frac{\Gamma(1+\epsilon)^n}{\Gamma(\rho_1+\epsilon) \cdots \Gamma(\rho_n+\epsilon)} \int_{\sigma+i\mathbb{R}} ds\, \frac{\Gamma(s+\rho_1+\epsilon) \cdots \Gamma(s+\rho_n+\epsilon)}{\Gamma(s+1+\epsilon)^n} \cdot \frac{\pi\,(-t)^s}{\sin(\pi s)} , \tag{10.34}$$

where $\sigma \in (-\rho_1, 0)$. For $|t| < 1$ the contour integral can be closed to the right. We have the following:

Lemma 10.9. *For $|t| < 1$, Equation (10.34) coincides with Equation (10.26).*

Proof. For $|t| < 1$ the contour integral can be closed to the right, and the Γ-series in Equation (10.26) is recovered as a sum over the enclosed residua at $r \in \mathbb{N}_0$ where we have used

$$\forall r \in \mathbb{N}_0 : \ \mathrm{Res}_{s=r}\left(\frac{\pi\,(-t)^s}{\sin(\pi s)}\right) = t^r.$$

□

For $|t| > 1$ the contour integral must be closed to the left. The relation to the local basis of solutions at $t = \infty$ can be explicitly computed. We obtain:

Proposition 10.10. *For $|t| > 1$, we obtain for $f(\epsilon, t)$ in Equation (10.34)*

$$f(\epsilon, t) = \sum_{r=1}^{n} B_r(\epsilon)\, F_r(t) \tag{10.35}$$

where $F_r(t)$ is given by

$$F_r(t) = A_r\, t^{-\rho_r}\, {}_nF_{n-1}\left(\begin{matrix} \rho_r & \ldots & \ldots & \rho_r \\ 1+\rho_r-\rho_1 & \ldots \widehat{1} \ldots & & 1+\rho_r-\rho_n \end{matrix}\,\middle|\, \frac{1}{t}\right) \tag{10.36}$$

and

$$A_r = -e^{-\pi i \rho_r} \prod_{\substack{i=1\\ i\neq r}}^{n} \frac{\Gamma(\rho_r-\rho_i)}{\Gamma(\rho_i)\,\Gamma(1-\rho_i)},\ \ B_r(\epsilon) = e^{-\pi i \epsilon}\left[\prod_{i=1}^{n} \frac{\Gamma(\rho_i)\,\Gamma(1+\epsilon)}{\Gamma(\rho_i+\epsilon)}\right] \frac{\sin(\pi\rho_r)}{\sin(\pi\rho_r+\pi\epsilon)}, \tag{10.37}$$

such that $B_r(0) = 1$ for $r = 1, \ldots, n$.

Proof. For $|t| > 1$ the contour integral in Equation (10.34) must be closed to the left. Using $1/(1+\epsilon)^n_k = O(\epsilon^n) = 0$ for $k < 0$, we observe that the poles are located at $s = -\epsilon - \rho_i - k$ for $i = 1, \ldots, n$ and $k \in \mathbb{N}_0$. Using

$$\forall r \in \mathbb{N}_0 : \ \mathrm{Res}_{s=-r}\Big(\Gamma(s)\,(-t)^s\Big) = \frac{t^{-r}}{r!}.$$

and Equations (10.24) the result follows. □

Equation (10.35) allows us to compute the transition matrix between the Frobenius basis $\langle f_{n-1}, \ldots, f_0\rangle^t$ of solutions to the differential equation (10.16) at $t = 0$ with local monodromy given by the matrix (10.29) and the Frobenius basis $\langle F_n, \ldots, F_1\rangle^t$ of solutions at $t = \infty$ with local monodromy given by the matrix (10.33). We obtain:

Corollary 10.11. *The transition matrix P between the analytic continuation of the Frobenius basis $\boldsymbol{f}^t = \langle f_{n-1}, \ldots, f_0\rangle^t$ at $t = 0$ with local monodromy given by the matrix m_0 in (10.29) and the analytic continuation of the Frobenius basis $\boldsymbol{F}^t = \langle F_n, \ldots, F_1\rangle^t$ at $t = \infty$ with local monodromy given by the matrix M_∞ in (10.33) such that $\boldsymbol{f} = \mathsf{P}\cdot\boldsymbol{F}$ is given by*

$$\begin{pmatrix} f_{n-1} \\ \vdots \\ f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} \frac{B_n^{(n-1)}(0)}{(2\pi i)^{n-1}(n-1)!} & \cdots & \frac{B_1^{(n-1)}(0)}{(2\pi i)^{n-1}(n-1)!} \\ \vdots & \ddots & \vdots \\ \frac{B_n'(0)}{(2\pi i)} & \cdots & \frac{B_1'(0)}{2\pi i} \\ 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} F_n \\ \vdots \\ F_2 \\ F_1 \end{pmatrix} \tag{10.38}$$

with $B_r(\epsilon)$ given in Equation (10.37). The monodromy of the analytic continuation of $\boldsymbol{f}$ around $t=\infty$ and $t=1$ is given by $\mathsf{m}_\infty = \mathsf{P}\cdot\mathsf{M}_\infty\cdot\mathsf{P}^{-1}$ and $\mathsf{m}_1 = \mathsf{m}_\infty\cdot\mathsf{m}_0^{-1}$, respectively.

10.3.5. *Monodromy group after rescaling.* For $C>0$ the rescaled rank-n hypergeometric differential equation satisfied by ${}_nF_{n-1}(Ct)$ is given by

$$\Big[\theta^n - C\,t\,(\theta+\rho_1)\cdots(\theta+\rho_n)\Big]\tilde{F}(t) = 0\,. \tag{10.39}$$

For $|t|<1/C$ we introduce $\tilde{f}(\epsilon,t) = C^{-\epsilon} f(\epsilon,Ct)$ such that

$$\tilde{f}(\epsilon,t) = \sum_{m=0}^{n-1}\Big(2\pi i\epsilon\Big)^m \tilde{f}_m(t) \quad \text{with} \quad \tilde{f}_m(t) = \frac{1}{(2\pi i)^m m!}\,\frac{\partial^m}{\partial\epsilon^m}\bigg|_{\epsilon=0} f(\epsilon,Ct) \tag{10.40}$$

for $j=0,\dots,n-1$. The local monodromy with respect to the Frobenius basis $\langle\tilde{f}_{n-1},\dots,\tilde{f}_0\rangle^t$ of solutions to the differential equation (10.39) at $t=0$ is given by the matrix m_0 in (10.29). Similarly, for $|t|>1/C$ we introduce $\tilde{F}_k(t) = F_k(Ct)$ for $k=1,\dots,n$. The local monodromy with respect to the Frobenius basis $\langle\tilde{F}_n,\dots,\tilde{F}_1\rangle^t$ of solutions to the differential equation (10.39) at $t=\infty$ is given by the matrix M_∞ in (10.33). We obtain:

Corollary 10.12. *The transition matrix $\tilde{\mathsf{P}}$ between the analytic continuation of $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{F}}$ such that $\tilde{\boldsymbol{f}} = \tilde{\mathsf{P}}\cdot\tilde{\boldsymbol{F}}$ is given by*

$$\tilde{\mathsf{P}} = \Big(\tilde{\mathsf{P}}_{n-j,n+1-k}\Big)_{j=0,k=1}^{n-1,n} \quad \textit{with} \quad \tilde{\mathsf{P}}_{n-j,n+1-k} = \frac{1}{(2\pi i)^j j!}\,\frac{\partial^j}{\partial\epsilon^j}\bigg|_{\epsilon=0}\Big[C^{-\epsilon}B_k(\epsilon)\Big]\,. \tag{10.41}$$

The monodromy of the analytic continuation of $\tilde{\boldsymbol{f}}$ around $t=\infty$ and $t=1/C$ is given by $\mathsf{m}_\infty = \tilde{\mathsf{P}}\cdot\mathsf{M}_\infty\cdot\tilde{\mathsf{P}}^{-1}$ and $\mathsf{m}_{1/C} = \mathsf{m}_\infty\cdot\mathsf{m}_0^{-1}$, respectively.

Remark. If we set $\rho_r = \frac{r}{n+1}$ for $r=1,\dots n$ and $C=(n+1)^{n+1}$, we obtain from the multiplication formula for the Γ-function, i.e.,

$$\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)}\,m^{\frac{1}{2}-mz}\,\Gamma(mz), \tag{10.42}$$

the identity

$$C^{-\epsilon}B_k(\epsilon) = \frac{\Gamma(1+\epsilon)^{n+1}}{\Gamma(1+(n+1)\epsilon)}\,\frac{\sin(\pi\rho_k)}{\sin(\pi\rho_k+\pi\epsilon)}e^{-\pi i\epsilon}\,.$$

We then compute the monodromy of the analytic continuation of $\tilde{\boldsymbol{f}}$ around $t=0,1/C,\infty$ where we have set $\kappa_4 = -200\frac{\zeta(3)}{(2\pi i)^3}$ and $\kappa_5 = 420\frac{\zeta(3)}{(2\pi i)^3}$. We obtain the results listed in Table 25. The case $n=4$, reproduces up to conjugation the monodromy matrices for the quintic threefold case by Candelas et al. [17].

n	Y	m_0	$\mathsf{m}_{1/C}$	m_∞
2	EC	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$
3	K3	$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 \\ -4 & -4 & -2 \end{pmatrix}$
4	CY3	$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1+\kappa_4 & 0 & \frac{5\kappa_4}{12} & \frac{\kappa_4^2}{5} \\ -\frac{25}{12} & 1 & -\frac{125}{144} & -\frac{5\kappa_4}{12} \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -\frac{25}{12} & 1-\kappa_4 \end{pmatrix}$	$\begin{pmatrix} 1+\kappa_4 & 1+\kappa_4 & \frac{1}{2}+\frac{11\kappa_4}{12} & \frac{1}{6}+\frac{7\kappa_4}{12}+\frac{\kappa_4^2}{5} \\ -\frac{25}{12} & -\frac{13}{12} & -\frac{131}{144} & -\frac{103}{144}-\frac{5\kappa_4}{12} \\ 0 & 0 & 1 & 1 \\ -5 & -5 & -\frac{55}{12} & -\frac{23}{12}-\kappa_4 \end{pmatrix}$
5	CY4	$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{75}{64} & 0 & \frac{55}{512} & -\frac{11\kappa_5}{384} & -\frac{121}{24576} \\ -\kappa_5 & 1 & -\frac{5\kappa_5}{8} & \frac{\kappa_5^2}{6} & \frac{11\kappa_5}{384} \\ -\frac{15}{4} & 0 & -\frac{43}{32} & \frac{5\kappa_5}{8} & \frac{55}{512} \\ 0 & 0 & 0 & 1 & 0 \\ -6 & 0 & -\frac{15}{4} & \kappa_5 & \frac{75}{64} \end{pmatrix}$	$\begin{pmatrix} \frac{75}{64} & \frac{75}{64} & \frac{355}{512} & -\frac{11\kappa_5}{384}+\frac{155}{512} & -\frac{11\kappa_5}{384}+\frac{2399}{24576} \\ -\kappa_5 & -\kappa_5+1 & -\frac{9\kappa_5}{8}+1 & \frac{(4\kappa_5-3)(\kappa_5-4)}{24} & \frac{\kappa_5^2}{6}-\frac{125\kappa_5}{384}+\frac{1}{6} \\ -\frac{15}{4} & -\frac{15}{4} & -\frac{103}{32} & \frac{5\kappa_5}{8}-\frac{63}{32} & \frac{5\kappa_5}{8}-\frac{369}{512} \\ 0 & 0 & 0 & 1 & 1 \\ -6 & -6 & -\frac{27}{4} & \kappa_5-\frac{19}{4} & \kappa_5-\frac{61}{64} \end{pmatrix}$

TABLE 25. Monodromy matrices

Remark. All 14 one-parameter families of Calabi-Yau threefolds in [31] are complete intersection of hypersurfaces in weighted projective space. Their basic invariants are the degree, the second Chern number, and the Euler number. In [19] a procedure closely related to the one used in Corollary 10.12 was used to construct the monodromy group of the Picard-Fuchs differential equations in the remaining cases where P and $\mathsf{m}_{1/C}$ are modified according to the basic geometric invariants.

10.4. **Weierstrass models for mirror families.** In the previous section we established that generalized functional invariants allow us to build up iteratively mirror families by fibrations of mirror families of lower dimension. In this section, we will show that these generalized functional invariants allow us to construct Weierstrass models for the mirror families and relations among the holomorphic periods given in the form of middle convolutions.

10.4.1. *Cubic pencil.* For $n = 2$ the family $Y_t^{(1)}$ is equivalent to the elliptic modular surface over the rational modular curve $X_0(3)$. By using the birational transformation

$$x_1 = \frac{4\,t}{3\,(2\,X+3)}\,, \qquad x_2 = \frac{i\,\sqrt{2}\,Y - 4\,t}{6\,(2\,X+3)} - \frac{1}{2} \tag{10.43}$$

in Equation (10.2), we recover the Weierstrass normal form for X_{431} in Table 3. Moreover, the transformation (10.43) maps the holomorphic one-form $3\sqrt{2}idX/Y$ to $dx_2/f_{2,x_1}$. The period integral of dX/Y over the one-cycle $\Sigma_1' \cong K_1$ was shown to equal $\omega_1^{(\mu)}$ for $\mu = \frac{1}{3}$ and annihilated by the Picard-Fuchs operator $L_2^{(\mu)}$ in Lemma 5.6.

Remark. Lemma 10.1 shows that the mirror cubic is fibered by quadrics $f_1(\tilde{x}_1, \tilde{t}) = 0$. This corresponds to the fact that the family of curves X_{431} in Lemma 5.7 is obtained by the mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 2, 1)$ applied to the quadric family in Equation (4.1). In fact, combining $\tilde{x}_1 = x_1/(x_2 + 1)$ and $\tilde{t} = -4\,t/(27\,x_2\,(x_2 + 1)^2)$ with the transformation from Table 2 yields Equation (10.43). On the level of periods, this is reflected by the factorization of the holomorphic period in terms of the Hadamard product as

$$ {}_2F_1\left(\left. \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \end{array} \right| t \right) = {}_2F_1\left(\left. \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ \frac{1}{2} \end{array} \right| t \right) \star {}_1F_0\left(\left. \frac{1}{2} \right| t \right) . \tag{10.44} $$

Remark. The three other extremal families of elliptic curves $X_{141}, X_{321}, X_{211}$ constructed in Lemma 5.7 are also realized as deformation of hypersurfaces in weighted projective space. Their defining equations as hypersurfaces were discussed from the view point of the Greene-Plesser orbifolding construction in [74, Thm. 2.2 and Ex. 2.7], and from the view point of mirror symmetry in [1, Eqns. (4.43)-(4.46)]. The holomorphic period integral for each family was shown to be annihilated by the second-order and degree-one Picard-Fuchs operator $L_2^{(\mu)}$ given in Equation (5.13) in Section 5.

Remark. The families of elliptic curves $X_{141}, X_{431}, X_{321}, X_{211}$ constructed in Lemma 5.7 for $(n, \mu) \in \left\{(1, \frac{1}{6}), (2, \frac{1}{4}), (3, \frac{1}{3}), (4, \frac{1}{2})\right\}$ are also called mirror curves of E_N type (with $N = 9 - n$)[51]. In fact, they are the mirror curves in the B-model of the A-model for Calabi-Yau threefolds that are the canonical bundle of the del Pezzo surfaces dP_N, for $N = 5, 6, 7, 8$. The latter are non-compact Calabi-Yau threefolds whose Picard-Fuchs operators factor as products $L_2^{(\mu)}\,\theta$ of the operators given in Equation (5.13) and θ. The three regular singularities for $L_2^{(\mu)}$ are located at $t = 0, 1, \infty$ and known as the conifold limit, the large complex structure limit, and the orbifold point, respectively.

10.4.2. *Narumiya-Shiga model for the quartic pencil.* For $n = 3$ the family $Y_t^{(2)}$ is equivalent to a family of minimal Weierstrass models. Following Narumiya and Shiga [60], setting

$$ \begin{aligned} x_1 &= -\frac{\left(4\,Z^2\lambda^2 + 3\,X\lambda^2 - Z^3 - Z\right)\left(4\,Z^2\lambda^2 + 3\,X\lambda^2 - Z^3 + 2\,Z\right)}{6\lambda^2 Z\,(16\,Z^3\lambda^2 + 3\,iY\lambda^2 + 12\,XZ\lambda^2 - 4\,Z^4 - 4\,Z^2)}, \\ x_2 &= -\frac{16\,Z^3\lambda^2 + 3\,iY\lambda^2 + 12\,XZ\lambda^2 - 4\,Z^4 - 4\,Z^2}{8Z\,(4\,Z^2\lambda^2 + 3\,X\lambda^2 - Z^3 + 2\,Z)}, \\ x_3 &= -\frac{Z^2\left(4\,Z^2\lambda^2 + 3\,X\lambda^2 - Z^3 + 2\,Z\right)}{2\lambda^2\,(16\,Z^3\lambda^2 + 3\,iY\lambda^2 + 12\,XZ\lambda^2 - 4\,Z^4 - 4\,Z^2)} \end{aligned} \tag{10.45} $$

in Equation (10.2) with $t = 1/\lambda^4$ yields a one-parameter family of Jacobian elliptic K3 surfaces of Picard rank 19 in Weierstrass normal form given by $Y^2 = 4\,X^2 - G_2\,X - G_3$ with

$$ \begin{aligned} G_2 &= \frac{4}{3\,\lambda^4}\,Z^2\,\left(16\,Z^2\lambda^4 - 8\,Z^3\lambda^2 + Z^4 - 8\,Z\lambda^2 - Z^2 + 1\right)\,, \\ G_3 &= \frac{4}{27\,\lambda^6}\,Z^3\,\left(4\,Z\lambda^2 - Z^2 - 1\right)\left(32\,Z^2\lambda^4 - 16\,Z^3\lambda^2 + 2\,Z^4 - 16\,Z\lambda^2 - 5\,Z^2 + 2\right)\,. \end{aligned} \tag{10.46} $$

This Weierstrass model is closely connected to our mixed-twist construction: there is a fifth extremal rational elliptic surface with three singular fibers, often denoted by X_{411}, that is obtained from X_{141} in Table 3 by transferring a star and a base transformation $t \mapsto 1/t$. Its

Weierstrass model is given by the equation

$$Y^2 = 4X^3 - \underbrace{\frac{4}{3}\Big(16t^2 - 16t + 1\Big)}_{=:\,g_2(t)} X - \underbrace{\frac{8}{27}\Big(2t-1\Big)\Big(32t^2 - 32t - 1\Big)}_{=:\,g_3(t)}, \tag{10.47}$$

and has singular fibers of Kodaira-type I_1, I_1, and I_4^* over $t = 0$, 1, and ∞, respectively. For the family of elliptic curves in Equation (10.47) it follows easily that the holomorphic period is the same as the one we already computed for X_{141}, i.e.,

$$\oint_{\Sigma_1'} \frac{dX}{Y} = (2\pi i)\ {}_2F_1\left(\left.\begin{array}{c}\frac{1}{2}, \frac{1}{2}\\ 1\end{array}\right| t\right),$$

where Σ_1' is the A-cycle in the general fiber that is transformed into itself as one closely encircles the singular fiber at $t = 0$. If we carry out the coordinate transformation $t \mapsto \lambda^2 + \frac{1}{2} - \frac{Z}{4} - \frac{1}{4Z}$ and a twist in Equation (10.47), we obtain the Weierstrass equation

$$Y^2 = 4X^3 - \underbrace{g_2\left(\lambda^2 + \frac{1}{2} - \frac{Z}{4} - \frac{1}{4Z}\right)\left(\frac{Z}{\lambda}\right)^4}_{=\,G_2} X - \underbrace{g_3\left(\lambda^2 + \frac{1}{2} - \frac{Z}{4} - \frac{1}{4Z}\right)\left(\frac{Z}{\lambda}\right)^6}_{=\,G_3}, \tag{10.48}$$

which coincides with the Weierstrass equation (10.46). The elliptic fibration has four singular fibers of Kodaira-type I_1, two singular fibers of Kodaira-type I_4^*, and a Mordell-Weil group that contains the two-torsion section $(X, Y) = (-Z\,(4Z - Z^2 - 1)/(3\lambda^2), 0)$ generating its entire torsion. There is also an additional section of infinite order with height pairing one. Thus, the family of Jacobian elliptic K3 surfaces has Picard rank 19. It follows that the determinant of the discriminant group equals $4^2/2^2 = 2^2$. The family of Jacobian elliptic K3 surfaces in Equation (10.48) realizes a family of M_2-polarized K3 surfaces with $M_2 = H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$ and transcendental lattice $T = H \oplus \langle 4 \rangle$.

Our mixed-twist construction applied to the rational elliptic surface X_{411} given in Equation (10.47) produces Equation (10.48) (up to a coordinate transformation). In fact, we can apply the mixed-twist construction to the rational elliptic surface X_{411} given by Equation (10.47) and obtain the mixed-twist model (6.39) with $a = 1 + \lambda^2$, $b = \lambda^2$. It is the same Weierstrass model as the model obtained by setting

$$(X, Y, Z) \mapsto \left(\frac{X}{u^4\,\lambda^2}, -\frac{Y}{u^6\,\lambda^3}, 1 + \frac{1}{u}\right) \tag{10.49}$$

in Equation (10.48). In turn, under the transformation (10.49) the holomorphic two-form $dZ \wedge dX/Y$ coincides with holomorphic two-form $\lambda\, du \wedge dX/Y$ in Equation (6.39). This means that the holomorphic period of $dZ \wedge dX/Y$ is computed by the result in [21, Theorem 2.5] for $\mu = 1/2$, $a = 1 + \lambda^2$ and $b = \lambda^2$, yielding the period

$$\lambda\, \frac{(2\pi i)^2}{\lambda} \left({}_2F_1\left(\left.\begin{array}{c}\frac{1}{4}, \frac{3}{4}\\ 1\end{array}\right| A\right)\right)^2$$

with $A = \frac{1}{2}\left(1 - \sqrt{1-1/\lambda^4}\right)$. The well-known quadratic relation for the hypergeometric function

$$ {}_2F_1\left(\begin{array}{c} p, q \\ p+q+\frac{1}{2} \end{array} \middle| \frac{1}{\lambda^4}\right) = {}_2F_1\left(\begin{array}{c} 2p, 2q \\ p+q+\frac{1}{2} \end{array} \middle| \frac{1}{2}\left(1 - \sqrt{1-\frac{1}{\lambda^4}}\right)\right) $$

then implies that the period of $dZ \wedge dX/Y$ over $\underline{\Sigma}_2^{\#}$ equals

$$ (2\pi i)^2 \left({}_2F_1\left(\begin{array}{c} \frac{1}{8}, \frac{3}{8} \\ 1 \end{array} \middle| \frac{1}{\lambda^4}\right)\right)^2 = (2\pi i)^2 \left({}_2F_1\left(\begin{array}{c} \frac{1}{8}, \frac{3}{8} \\ 1 \end{array} \middle| t\right)\right)^2 , \tag{10.50} $$

which agrees with the holomorphic period computed by Narumiya and Shiga in [60].

10.4.3. *A second elliptic fibration on the quartic pencil.* For $n = 3$ the family $Y_t^{(2)}$ is equivalent to a family of minimal Weierstrass models given by the equation

$$ Y^2 = 4X^3 - \underbrace{g_2\left(-\frac{3^3\, t}{4^4\, u^3(u+1)}\right)\Big(u\,(u+1)\Big)^4}_{=: G_2(t,u)} X - \underbrace{g_3\left(-\frac{3^3 t}{4^4\, u^3(u+1)}\right)\Big(u\,(u+1)\Big)^6}_{=: G_3(t,u)} \tag{10.51} $$

where we have choose for $g_2(t)$ and $g_3(t)$ the Weierstrass coefficients for X_{431} given in Table 3. Equation (10.51) defines a family of Jacobian elliptic K3 surfaces of Picard rank 19 with a singular fiber of Kodaira-type IV^* over $u = 0$, a singular fiber of Kodaira-type I_{12} over $u = \infty$, and four fibers of Kodaira-type I_1. The Mordell-Weil group is pure three-torsion generated by the sections $(X, Y) = (-3/2\, u^2\,(u+1)^2\,, \pm\frac{27\,i}{128}\sqrt{2}tu^2)$. It follows that the determinant of the discriminant equals $3 \cdot 12/3^2 = 2^2$. In fact, the family of Jacobian elliptic K3 surfaces has an M_2-polarization. Applying the birational transformation

$$ \begin{aligned} x_1 &= -\frac{9\,(u+1)\,t}{64\,\left(3\,u^4 + 6\,u^3 + 3\,u^2 + 2\,X\right)}, \\ x_2 &= \frac{-64\,i\,\sqrt{2}\,Y + 9\left(64\,u^5 + 128\,u^4 + 64\,u^3 + \left(3t + \frac{128}{3}\right)\,u + t\right)(u+1)}{1152\left(u^4 + 2\,u^3 + u^4 + \frac{2}{3}\,u\right)(u+1)}, \\ x_3 &= -\,(u+1) \end{aligned} \tag{10.52} $$

in Equation (10.2) recovers the Weierstrass normal form in Equation (10.51).

Moreover, the transformation (10.52) maps the holomorphic two-form $3\sqrt{2}i du \wedge dX/Y$ to $dx_2 \wedge dx_3/f_{3,x_1}$. Our construction of two-cycles Σ_2'' in Section 6.2.2 is easily generalized to include mixed-twists for all generalized functional invariants, such as $(3, 1, 1)$. The holomorphic period for the two-form $du \wedge dX/Y$ over such a two-cycle Σ_2' is then given by the hypergeometric function $\omega_2^{(\mu)}$ for $\mu = \frac{1}{4}$ in Equation (6.8), agrees with Equation (10.50) by means of Clausen's identity (6.20), and is annihilated by the Picard-Fuchs operator $L_3^{(\mu)}$ in Equation (6.11).

Remark. Lemma 10.1 shows that the mirror quartic is fibered by cubics $f_2(\tilde{x}_1, \tilde{x}_2, \tilde{t}) = 0$. This structure is captured by the mixed-twist construction: the Weierstrass model in Equation (10.51) was obtained by applying the mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 3, 1)$ to the Jacobian rational elliptic surface X_{431}. In fact, combining $\tilde{x} = x/(z+1)$ and $\tilde{y} = y/(z+1)$ and $\tilde{t} = -27\,t/(256\,z\,(z+1)^3)$ with the transformation (10.43) yields the birational transformation (10.52). On the level of periods, this is reflected by the factorization of the holomorphic period in terms of the Hadamard product as

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{array} \middle| \, t \right) = {}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ \frac{1}{3}, \frac{2}{3} \end{array} \middle| \, t \right) \star {}_2F_1\left(\begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \end{array} \middle| \, t \right) . \tag{10.53} $$

Remark. The family in Equation (10.51) is a family of M_2-polarized K3 surfaces Y with transcendental lattice $T(Y) = H \oplus \langle 4 \rangle$. Following Dolgachev [27] its mirror partner $Y^\vee \cong X$ is the family of generic quartic surfaces in $\mathbb{P}^3$ with $\mathrm{NS}(X) = \langle 4 \rangle$ since $T(Y) = H \oplus \mathrm{NS}(X)$. Equivalently, it was proved in [60] that the mirror quartic is the family of the Calabi-Yau varieties arising from the polytope P_0^* which is easily constructed from the finite subset in Equation (10.18) for $n = 3$ and dual to the simplest reflexive polytope P_0 in dimension 3. The family X is the family of the Calabi-Yau varieties arising from the reflexive polytope P_0 and is the family of generic quartic surfaces in $\mathbb{P}^3$.

10.4.4. *Quintic Pencil.* For $n = 4$ the family $Y_t^{(3)}$ is equivalent to a family of minimal Weierstrass models given by the equation

$$ \begin{aligned} Y^2 = 4\,X^3 &- \underbrace{g_2\left(\frac{3^3\,t}{5^5\,u(u+1)\,s^3\,(s+1)^2} \right) \Big(u\,(u+1)\,s\,(s+1) \Big)^4}_{=: \mathcal{G}_2(s,t,u)} X \\ &- \underbrace{g_3\left(\frac{3^3\,t}{5^5\,u\,(u+1)\,s^3\,(s+1)^2} \right) \Big(u\,(u+1)\,s\,(s+1) \Big)^6}_{=: \mathcal{G}_3(s,t,u)}, \end{aligned} \tag{10.54} $$

where we have choose for $g_2(t)$ and $g_3(t)$ the Weierstrass coefficients for X_{431} given in Table 3. Equation (10.54) defines a minimal Weierstrass model over a two-dimensional base with affine coordinates u and s such that restricted to a generic s-slice we obtain a family of Jacobian elliptic K3 surfaces with M_3-polarization. Applying the birational transformation

$$ \begin{aligned} x_1 &= \frac{36\,t\,u(u+1)}{9375\,s^2(s+1)^2u^4+18750\,s^2(s+1)^2u^3+9375\,s^2(v+1)^2u^2+6250\,X} , \\ x_2 &= \frac{-3125\,i\sqrt{2}Y+28125\left(s^3(s+1)^2u^4+2\,s^3(s+1)^2u^3+\left(s^5+2\,s^4+s^3-\frac{12\,t}{3125}\right)u^2-\frac{12\,ut}{3125}+\frac{2}{3}\,s\,X\right)(u+1)u(s+1)}{\left(56250\,s^2(s+1)^2u^4+112500\,s^2(s+1)^2u^3+56250\,s^2(s+1)^2u^2+37500\,X\right)(u+1)u(s+1)} , \\ x_3 &= u\,(s+1)\,, \qquad x_4 = -(u+1)\,(s+1) \end{aligned} \tag{10.55} $$

in Equation (10.2) recovers the Weierstrass normal form in Equation (10.54).

Moreover, the transformation (10.55) maps the holomorphic two-form $3\sqrt{2}i ds \wedge du \wedge dX/Y$ to $dx_2 \wedge dx_3 \wedge dx_4/f_{4,x_1}$. The construction of a three-cycle for the mixed-twist construction that upon integration with the three-forms yields the holomorphic period and its fourth-order Picard-Fuchs operator is discussed in Section 7. In Lemma 7.16 we showed that the holomorphic period is given by $\omega_4^{(p,q)}$ with $p = \frac{1}{5}$ and $q = \frac{2}{5}$ which agrees with [16, Eq. (3.11)] and is annihilated by the fourth-order Picard-Fuchs operator $L_4^{(p,q)}$. The Picard-Fuchs operator is another of the 14 original Calabi-Yau operators mentioned in the introduction and labelled "(11)" in the AESZ database [4].

Remark. Lemma 10.1 shows that the mirror quintic is fibered by quartics $f_3(\tilde{x}_1, \dots, x_3, \tilde{t}) = 0$. This structure is captured by the iterated mixed-twist construction: the Weierstrass model in Equation (10.54) was obtained by applying the sequence of mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ and $(i, j, \alpha) = (2, 3, 1)$ to the Jacobian rational elliptic surface X_{431}. In fact, combining $\tilde{x} = x/(z+w+1)$, $\tilde{y} = y/(z+w+1)$, and $\tilde{t} = 3^3\, t/(5^5\, z\, w\, (z+w+1)^3)$ with the transformation (10.43) yields the birational transformation (10.55). On the level of periods, this is reflected by the factorization of the holomorphic period in terms of the Hadamard product as

$$(10.56)\qquad {}_4F_3\left(\left.\begin{array}{c}\frac{1}{5},\, \frac{2}{5},\, \frac{3}{5},\, \frac{4}{5}\\ 1,\, 1,\, 1\end{array}\right| t\right) = {}_4F_3\left(\left.\begin{array}{c}\frac{1}{5},\, \frac{2}{5},\, \frac{3}{5},\, \frac{4}{5}\\ \frac{1}{3},\, \frac{2}{3},\, \frac{1}{2}\end{array}\right| t\right) \star {}_1F_0\left(\left.\frac{1}{2}\right| t\right) \star {}_2F_1\left(\left.\begin{array}{c}\frac{1}{3},\, \frac{2}{3}\\ 1\end{array}\right| t\right) .$$

Remark. As shown in [16] the family of Calabi-Yau threefolds $Y_t^{(3)}$ has a general fiber with Hodge numbers $h^{2,1}(Y_t^{(3)}) = 1$ and $h^{1,1}(Y_t^{(3)}) = 101$. Following [16] its mirror partner $X_\lambda^{(3)}$ is the family of generic quintic surfaces in $\mathbb{P}^4$ with Hodge numbers $h^{1,1}(X_\lambda^{(3)}) = 1$ and $h^{2,1}(X_\lambda^{(3)}) = 101$.

11. Discussion and Outlook

We have successfully demonstrated that our iterative construction produces Weierstrass models whose Picard-Fuchs operators realize all symplectically rigid Calabi-Yau operators. Our iterative construction provides a unifying construction for many examples of elliptic curves, K3 surfaces and Calabi-Yau threefolds considered in the context of mirror symmetry, e.g., the families considered in [60, 72] and also the examples in [74]. Moreover, by specializing the family parameter to special values one easily obtains elliptic curves, K3 surfaces, and Calabi-Yau threefolds with properties such as CM, isogeny to an abelian surface with quaternionic multiplication, or rigidity. Some of these arithmetic properties were investigated in [70], and their fiberwise Picard-Fuchs equations computed in [71]. All of these results are contained in our iterative construction.

This raises the obvious question whether our method generalizes to include more examples from the list of Calabi-Yau operators in [4]. We were able to answer this question affirmatively. Using the Maple software package we have produced over 100 Weierstrass models whose Picard-Fuchs operators realize various Calabi-Yau operators from the database in [4]. A straightforward generalization works as follows: if – at Step 1 of our iterative construction – we allow all 10 non-isotrivial extremal Jacobian rational elliptic surfaces from the Miranda-Persson list [57], we obtain a substantially bigger number of families of Jacobian elliptic K3 surfaces at the next iteration step, including models for families of Jacobian elliptic M_n-polarized K3 surfaces for $n \leq 9$ with $n \neq 7$. It comes as no surprise that when used in our iterative construction at Step 3, these families produce an even larger number of families of Calabi-Yau threefolds with $h^{2,1} = 1$. On the level of periods, the role of the Gauss hypergeometric function is replaced by the Heun function $\mathrm{Hl}(a, q; 1, 1, 1, 1|\bullet)$ at Step 1 of our iterative construction. Identities for the hypergeometric function are replaced by more involved identities for the Heun equation, for example relations that were found in [52, 67]. At Step 2, our iterative construction again provides a unified geometric approach to many differential equations that have been studied in isolation [13, 62, 65, 61, 12]. At Step 3, our iterative construction reproduces some of the classical examples that have been investigated in the context of mirror symmetry, for example in [51, 50]. As in Theorem 2.1, our iterative construction provides a geometric realization of all fourth-order non-symplectically rigid Calabi-Yau operators that were found in [14].

Moreover, the geometric realization of the odd case in Theorem 2.1 is not yet completely satisfactory: instead of producing families of threefolds whose Picard-Fuchs operators realize the fourth-order Calabi-Yau operators of the odd case directly, we constructed families of fourfolds instead such that the Yifan-Yang pullback of their fifth-order Picard-Fuchs operators realized the Calabi-Yau operators. The observant reader might have noticed that a very similar situation already occurred after Step 1 of our iterative construction. There, the pure-twist construction applied to any family of elliptic curves from Table 3 provided us with third-order and degree-one Calabi-Yau operators that were symmetric squares of second-oder and degree-one Calabi-Yau operators. In fact, Clausen's identity in Equation (6.20) expresses the holomorphic K3 periods as squares of Gauss hypergeometric functions. Using the hypergeometric function identity in Equation (6.21), we were able to relate the (symmetric) square root back to holomorphic solution of the Picard-Fuchs equation before the twist. Thus, carrying out a pure-twist and taking a (symmetric) square root after Step 1 turned out to be equivalent to carrying out a quadratic transformation on the parameter space of the original family of elliptic curves. Hodge-theoretically this is due to the fact that the K3 surfaces of Picard-rank 19 or 18 admit Shioda-Inose structures relating them to Kummer surfaces associated to products of elliptic curves. We have proved that at least in one case a similar Hodge-theoretic interpretation for the exterior square root after Step 4 exists as well.

Moreover, we expect that our iterative construction of the transcendental cycle for the holomorphic period can be extended to obtain a full basis of transcendental cycles. Such bases would in turn allow us to construct the integral monodromy matrices at each step of our iterative construction. The full period lattices can then be used as a powerful tool to distinguish examples of different geometric variations of Hodge structure over $\mathbb{Z}$ that are isomorphic over $\mathbb{R}$. For example, the quintic-mirror and the quintic-mirror twin family share the exact same Picard-Fuchs equation, but they have different ranks in the even dimensional cohomologies [31]. These results will be the subject of a forthcoming article.

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