NORMAL FORMS AND TYURIN DEGENERATIONS OF K3 SURFACES POLARISED BY A RANK 18 LATTICE

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Abstract. We study projective Type II degenerations of K3 surfaces polarised by a certain rank 18 lattice, where the central fibre consists of a pair of rational surfaces glued along a smooth elliptic curve. Given such a degeneration, one may construct other degenerations of the same kind by flopping curves on the central fibre, but the degenerations obtained from this process are not usually projective. We construct a series of examples which are all projective and which are all related by flopping single curves from one component of the central fibre to the other. The components of the central fibres obtained include weak del Pezzo surfaces of all possible degrees. This shows that projectivity need not impose any meaningful constraints on the surfaces that can arise as components in Type II degenerations.

1. Introduction

A Tyurin degeneration is a semistable degeneration of K3 surfaces $\mathcal{X} \to \Delta$, where Δ denotes the complex unit disc, with central fibre $X_0 = V_1 \cup_D V_2$ consisting of a pair of rational surfaces V_1, V_2 glued along a smooth elliptic curve D which is anticanonical in each. They are the simplest examples of Type II degenerations of K3 surfaces.

In this paper we will be interested in studying projective Tyurin degenerations, i.e. ones where the morphism $\mathcal{X} \to \Delta$ is projective. If one assumes that the smooth fibres of $\mathcal{X}$ are polarised by an appropriate ample divisor, then results of Shepherd-Barron [SB83] show that projectivity can always be arranged by flopping some curves between the components of the central fibre X_0.

If one further assumes that the smooth K3 fibres of $\mathcal{X}$ are lattice polarised by a lattice L, Type II degenerations occur over 1-cusps in the Baily-Borel compactification of the moduli space of L-polarised K3 surfaces. These 1-cusps have a lattice theoretic description: they correspond to isotropic sublattices of rank 2 in the orthogonal complement $L^\perp$ of L inside the K3 lattice $H \oplus H \oplus H \oplus E_8 \oplus E_8$. Thus, given a Tyurin degeneration of L-polarised K3 surfaces, one may associate to it the lattice theoretic data of a rank 2 isotropic sublattice of $L^\perp$.

The question that this paper seeks to answer is as follows. To what extent is the geometry of the central fibre $X_0 = V_1 \cup_D V_2$ of a projective Tyurin degeneration determined by this lattice theoretic data? In particular, can one deduce anything about the geometry of the rational surfaces V_i, such as their del Pezzo degrees $K_{V_i}^2$?

There is some reason to expect that something like this might hold. Indeed, given two Tyurin degenerations that are isomorphic over the punctured disc, it is

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known that they must differ by a sequence of flops. However, the process of flopping a curve is an analytic operation, and doing so will often destroy the projectivity of the Tyurin degeneration. Indeed, in the case of $\langle 2\rangle$-polarised K3 surfaces, Friedman [Fri84, Section 5] has shown that projectivity imposes a very strong restriction, and that projective Tyurin degenerations of $\langle 2\rangle$-polarised K3 surfaces are essentially unique up to labelling of the components. In other words, in the $\langle 2\rangle$-polarised case, the birational geometry of the central fibre of a projective Tyurin degeneration is completely determined by the corresponding 1-cusp in the Baily-Borel compactification, which is described by purely lattice theoretic data.

Interestingly, as we will show in this paper, such a result does not hold for general lattice polarisations. We study Tyurin degenerations of K3 surfaces polarised by the rank 18 lattice $M = H \oplus E_8 \oplus E_8$. The moduli space of M-polarised K3 surfaces is well-understood (see, for example, [CD07, CDLW09]). Its Baily-Borel compactification contains a single 1-cusp, corresponding to the unique (up to isometry) rank 2 isotropic sublattice of $M^\perp \cong H \oplus H$. We show that, given a projective Tyurin degeneration of M-polarised K3 surfaces $\mathcal{X} \to \Delta$, there are several other projective Tyurin degenerations that differ from $\mathcal{X}$ by flops. In fact we prove more than this: there is a collection of curves in the central fibre X_0 such that *all* possible flops of these curves give rise to new projective Tyurin degenerations. This answers the question above in a maximally negative way: the lattice theory associated to such a projective Tyurin degeneration provides essentially no constraints upon which surfaces V_0 and V_1 can arise. In particular, the del Pezzo degrees $K_{V_i}^2$ of the components V_i in our examples take all possible values $-9 \leq K_{V_i}^2 \leq 9$.

The approach to proving this result is explicit: we construct projective models for all of the Tyurin degenerations. As a by-product, we obtain several simple projective normal forms for M-polarised K3 surfaces which have not previously appeared in the literature.

The motivation for this paper comes from mirror symmetry. It was essentially known to Dolgachev [Dol96] (see also [DHT17, Section 4]) that there is a mirror symmetric correspondence between Type II degenerations $X \rightsquigarrow X_0$ of a K3 surface X and genus 1 fibrations $Y \to \mathbb{P}^1$ on its mirror K3 surface Y. Explicitly this works as follows. Given an L-polarised K3 surface X and an isotropic vector $e \in L^\perp$, the mirror Y is defined to be a K3 surface polarised by the mirror lattice $\check{L} = e^\perp/\mathbb{Z}e$, where the orthogonal complement of e is taken inside $L^\perp$. Such vectors e correspond to 0-cusps in the Baily-Borel compactification of the moduli space of L-polarised K3 surfaces. If the 1-cusp corresponding to a Type II degeneration is incident to such a 0-cusp, this gives an embedding of e into the rank 2 isotropic sublattice corresponding to the 1-cusp. This in turn gives rise to a class of self-intersection 0 in the mirror lattice $\check{L}$, which induces a genus 1-fibration on the mirror K3.

The DHT conjecture [DHT17] extends this idea by suggesting that, if $X_0 = V_1 \cup_D V_2$ is the central fibre in a projective Tyurin degeneration, then there exists a splitting $\mathbb{P}^1 = \Delta_1 \cup_\gamma \Delta_2$ of the base of the mirror fibration along a curve γ, such that the restriction of the fibration to Δ_i is the Landau-Ginzburg model of V_i.

The results of this paper show that information about the geometry of a projective Tyurin degeneration cannot necessarily be inferred from the lattice theoretic data of the polarising lattice L and a rank 2 isotropic sublattice of $L^\perp$. Consequently, we expect that lattice theory alone is also probably insufficient to determine the splitting of the base $\mathbb{P}^1$ on the mirror. We therefore expect that a proof of the DHT conjecture for K3 surfaces will probably require the introduction of some additional non-lattice-theoretic data, in order to uniquely specify the projective Tyurin degeneration and the mirror splitting of $\mathbb{P}^1$.

Focussing on the Tyurin degenerations of M-polarised K3 surfaces studied in this paper, the picture above can be described explicitly as follows. The Baily-Borel compactification of the moduli space of M-polarised K3 surfaces contains a unique 0-cusp, which is incident to the single 1-cusp. The corresponding mirror K3 surfaces are polarised by the lattice H, and a generic member of the mirror family admits a unique elliptic fibration $Y \to \mathbb{P}^1$ with section and 24 singular fibres of Kodaira type I_1.

Under the DHT conjecture, our result that an M-polarised K3 surface admits many different projective Tyurin degenerations should correspond to the existence of many different splittings of the base $\mathbb{P}^1$ of this fibration, all of which should satisfy the requirement that the two halves have the properties required of Landau-Ginzburg models.

There are strong hints that something like this should be true. We suspect that it should be possible to endow the base $\mathbb{P}^1$ with the structure of an integral affine sphere, with singularities corresponding to the locations of singular fibres, and that this should correspond under mirror symmetry to the integral affine sphere described in [ABE22, Section 7]. In this description, the possible splitting curves $\gamma \subset \mathbb{P}^1$ should correspond to vertical lines passing between the singularities in [ABE22, Figure 5], and performing a single flop on the Tyurin degeneration should correspond to moving the splitting curve one space to the left or right, corresponding to moving a single I_1 fibre from one side of γ to the other.

Importantly, as all of the projective Tyurin degenerations constructed in this paper are identical from a lattice-theoretic perspective (they all correspond to the same 1-cusp in the Baily-Borel compactification), we anticipate that the correct splitting curve γ in the mirror cannot be determined by lattice theory alone.

The structure of this paper is as follows. In Section 2 we discuss some necessary background from lattice theory, along with results about the geometry and moduli of M-polarised K3 surfaces.

In Section 3 we begin by introducing some background on Tyurin degenerations. We then give two different explicit approaches to the construction of projective normal forms for M-polarised K3 surfaces, which we call the *linear system approach* and the *toric approach*. Finally, we show how these two approaches can be used to construct projective Tyurin degenerations of M-polarised K3 surfaces, and we describe a collection of 11 (analytic) degenerations that can be obtained by flopping curves on the result. These degenerations are classified by certain graphs Γ, so we call them Γ-*degenerations*.

Section 4 constitutes the main body of the paper. In this section we prove our main result (Theorem 4.1), which states that all of the Γ-degenerations described in the previous section can be realised projectively. We prove this result by explicitly constructing several projective normal forms for M-polarised K3 surfaces, then degenerating them to obtain each of the 11 possibilities; the constructions are summarised in Table 4.1. Note, in particular, that the set of possible components V_i occurring in such degenerations includes rational surfaces of all del Pezzo degrees $-9 \leq K_{V_i}^2 \leq 9$, along with the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$ (which is a deformation of $\mathbb{P}^1 \times \mathbb{P}^1$) which both have degree 8.

2. Background on M-polarised K3 surfaces

2.1. **Lattice theory.** We begin with some generalities on lattice polarizations and the lattice M; for details we refer the reader to [Dol96, Section 1]. Throughout the paper we use M to denote the rank 18 lattice

$$M := H \oplus E_8 \oplus E_8,$$

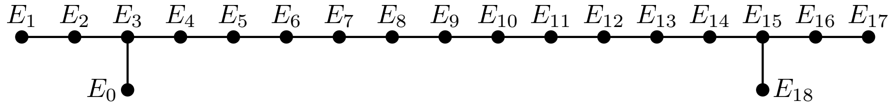

FIGURE 2.1. Dual graph of the configuration of simple roots in M.

where H denotes the usual hyperbolic plane lattice, the even unimodular indefinite lattice of rank 2, and E_8 denotes the negative definite E_8 root lattice.

The cone

$$V(M) = \{x \in M_{\mathbb{R}} : \langle x, x\rangle > 0\}$$

consists of two connected components; we pick one and call it the *positive cone* $V(M)^+$.

A class $\delta \in M$ with $\langle \delta, \delta\rangle = -2$ is called a *root.* The lattice M contains a subset $\Delta(M)^+$ consisting of roots with the following properties:

- for every root $\delta \in M$, we either have $\delta \in \Delta(M)^+$ or $-\delta \in \Delta(M)^+$, but not both;
- if δ is a root that can be written as a non-negative integer combination of classes from $\Delta(M)^+$, then $\delta \in \Delta(M)^+$.

The choice of such a subset $\Delta(M)^+$ defines a subset

$$\mathcal{C}(M)^+ = \{h \in V(M)^+ \cap M : \langle h, \delta\rangle > 0 \text{ for all } \delta \in \Delta(M)^+\}.$$

Definition 2.1. An *M-polarised K3 surface* is a K3 surface X along with a primitive lattice embedding $\iota\colon M \hookrightarrow \mathrm{NS}(X)$, such that $\iota(\mathcal{C}(M)^+)$ contains the class of an ample divisor.

From now on X will denote a generic M-polarised K3 surface with a fixed choice of polarization ι. As X is generic, the polarization ι induces an isomorphism $\mathrm{NS}(X) \cong M$, under which the cone $V(M)^+$ is identified with the positive cone of X, the cone $\mathcal{C}(M)^+$ is identified with the ample cone of X, and $\Delta(M)^+$ is the set of classes of effective divisors with self-intersection (-2). As a consequence of this identification, we will often use the same notation to refer to elements of M and classes in $\mathrm{NS}(X)$.

A *simple root* in $\Delta(M)^+$ is one that cannot be written as a non-negative integer combination of other roots from $\Delta(M)^+$; they correspond to classes of irreducible (-2)-curves in $\mathrm{NS}(X)$. There are 19 such roots, which we label $E_0, \ldots, E_{18}$, with Dynkin diagram as in Figure 2.1; in this diagram nodes correspond to roots and two roots E_i, E_j are joined if and only if $\langle E_i, E_j\rangle = 1$.

It is easy to show that the 19 roots E_i span M as a $\mathbb{Z}$-module. Indeed, the two E_8 factors are spanned by $\{E_0, E_1, \ldots, E_7\}$ and $\{E_{11}, E_{12}, \ldots, E_{18}\}$, and the generators of H are

$$\begin{aligned}&3E_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8,\\ &3E_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8 + E_9.\end{aligned}$$

Using this fact, we will often express elements of M and/or classes in $\mathrm{NS}(X)$ as combinations of the classes E_i. For ease of doing this, we introduce the following notation.

Notation 2.2. We will denote the class $\sum_{i=0}^{18} a_i E_i$, for $a_i \in \mathbb{Z}$, by the notation

$$(a_0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}; a_{18}).$$

We separate the end terms a_0 and a_{18} by semicolons to indicate that these correspond to the branches off the main chain of the graph.

2.2. **Geometry of M-polarised K3 surfaces.** In this subsection we will outline some of the basic geometric properties of M-polarised K3 surfaces, based mostly on ideas from [CD07].

A generic M-polarised K3 surface admits two elliptic fibrations, which Clingher and Doran [CD07] call the *standard* and *alternate* fibration. The standard fibration has two singular fibres of Kodaira type II^* and four of type I_1, plus a unique section. In terms of the divisors E_i, the II^* fibres are given by

$$(3; 2, 4, 6, 5, 4, 3, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0),$$
$$(0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 4, 2; 3),$$

and the section is E_9. The linear equivalence of these two fibres gives rise to the unique relation between the classes E_i:

$$(3; 2, 4, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -4, -2; -3) \sim 0. \tag{1}$$

The alternate fibration has one singular fibre of Kodaira type I_{12}^* and six of type I_1, plus two sections. In terms of the divisors E_i, the I_{12}^* fibre is given by

$$(1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 0; 1)$$

and the two sections are E_1 and E_{17}.

In addition to the fibration structures, a generic M-polarised K3 surface also admits a special involution.

Lemma 2.3. *Let X be a generic M-polarised K3 surface. Then the group of automorphisms which fix the M-polarization has order 2 and is generated by the fibrewise elliptic involution with respect to the standard fibration and its unique section E_9.*

Proof. Firstly, we note that the fibrewise involution with respect to the standard fibration fixes the polarization, so every M-polarised K3 surface X admits such an automorphism. We will prove that, if X is generic, then there are no others.

By [Nik80, Corollary 3.3] the group of automorphisms of X that fix the polarization is cyclic, and by [Kon92, Theorem 4.2] its order divides 12. Let σ be a generator. As σ fixes the M-polarization, it maps the curve E_9 to itself, and fixes its points of intersection with E_8 and E_{10}.

Moreover, σ preserves the class of a fibre in the standard fibration, so must act to permute the fibres of this fibration; in particular, it must permute the I_1 fibres. Identifying E_9 with $\mathbb{P}^1$ and setting the points of intersection with E_8 and E_{10} to be 0 and ∞, [Ino78, Lemma 1] shows that the I_1 fibres occur over α, α^{-1}, β, and β^{-1}, for some $\alpha, \beta \in \mathbb{C}$. It follows that σ acts on E_9 as an automorphism of $\mathbb{P}^1$ which fixes 0 and ∞ and permutes these four points. But if α and β are generic the only such automorphism is the identity. So σ acts as the identity on E_9 and therefore acts fibrewise on the standard fibration.

Finally, as the fibres of the standard fibration do not have j-invariant identically 0 or 1, the only fibrewise automorphism fixing the section E_9 is the elliptic involution, so σ must be this involution. ☐

Remark 2.4. The fibrewise elliptic involution with respect to the alternate fibration and either of its two sections also fixes the M-polarization. It follows from Lemma 2.3 that the fibrewise elliptic involutions with respect to the two fibrations are in fact the same involution. Consequently, for simplicity, we will henceforth refer to this involution simply as the *fibrewise elliptic involution*.

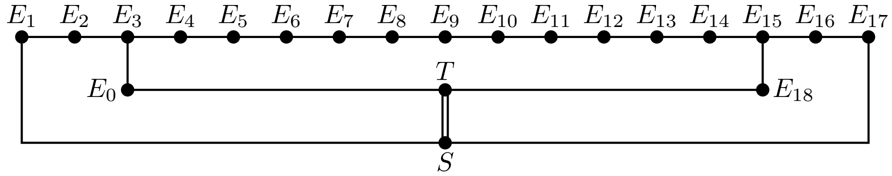

FIGURE 2.2. Dual graph of the (-2)-curves on X along with the curves S and T.

Using these ideas we define two additional curves on X. Let S denote a smooth fibre of the alternate fibration on X and let T denote the 2-torsion locus of the standard fibration with respect to the unique section E_9. As X is generic, T is a smooth curve, which forms a trisection of the standard fibration and a bisection of the alternate fibration. By construction T is part of the fixed locus of the fibrewise elliptic involution; this fixed locus consists of precisely those curves from the set

$$\{T, E_1, E_3, E_5, E_7, E_9, E_{11}, E_{13}, E_{15}, E_{17}\}.$$

The curves S and T satisfy $S^2 = 0$, $T^2 = 2$, and $S.T = 2$. We can add them into our dual graph as in Figure 2.2.

We can augment our notation for divisors on X with the classes of S and T as follows.

Notation 2.5. We will denote the $\mathbb{Q}$-divisor $\sum_{i=0}^{18} a_i E_i + sS + tT$, for $a_i, s, t \in \mathbb{Q}$, by the notation

$$(a_0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}; a_{18} \mid s, t).$$

As divisors on smooth surfaces correspond to sections of line bundles, when there is no potential for confusion we will use the same notation (with $\mathbb{Z}$-coefficients) for sections of appropriate line bundles.

The addition of S and T gives rise to two extra relations between our classes:

$$(1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 0; 1 \mid -1, 0) \sim 0, \tag{2}$$

$$(1; 1, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 1; 1 \mid 0, -1) \sim 0. \tag{3}$$

2.3. **Moduli of M-polarised K3 surfaces.** Based on earlier work of Inose [Ino78], Clingher, Doran, Lewis, and Whitcher [CD07, CDLW09] define a projective model for M-polarised K3 surfaces as the minimal resolutions of singular quartic surfaces

$$\{y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \tfrac{1}{2}(dz^2w^2 + w^4) = 0\} \subset \mathbb{P}^3[w, x, y, z], \tag{4}$$

for parameters $a, b, d \in \mathbb{C}$. The surfaces in this family are smooth whenever $d \neq 0$, and the lattice polarization jumps to $H \oplus E_8 \oplus E_8 \oplus A_1$ along the locus

$$\{a^6 + b^4 + d^2 - 2a^3b^2 - 2a^3d - 2b^2d = 0\}.$$

[CDLW09, Theorem 3.2] shows that two such quartics, with defining parameters (a_1, b_1, d_1) and (a_2, b_2, d_2), determine isomorphic M-polarised K3 surfaces if and only if

$$(a_1, b_1, d_1) = (\lambda^2 a_2, \lambda^3 b_2, \lambda^6 d_2),$$

from which it follows that the open variety

$$\{(a, b, d) \in \mathbb{WP}(2, 3, 6) : d \neq 0\}$$

forms a coarse moduli space for M-polarised K3 surfaces. This coarse moduli space can be compactified to the weighted projective space $\mathbb{WP}(2, 3, 6)$.

The Baily-Borel compactification for the moduli space of M-polarised K3 surfaces contains a unique 1-cusp, which can be identified in the compactification above as the locus

$$\{(a, b, 0) \in \mathbb{WP}(2, 3, 6) : a^3 \neq b^2\},$$

and a unique 0-cusp, which can be identified as the point $(a, b, d) = (1, 1, 0)$.

3. Tyurin degenerations

3.1. **Background.** In this subsection we summarise some basic information about degenerations of K3 surfaces; more details may be found in [FM83, Sca87].

By a *degeneration* of K3 surfaces, we mean a proper, flat, surjective morphism $\pi\colon \mathcal{X} \to \Delta$ from a Kähler manifold to the open complex unit disc, such that the restriction to the punctured disc $\hat{\pi}\colon \mathcal{X}^* \to \Delta^*$ is a smooth morphism with all fibres K3 surfaces. We denote the fibre over a point $p \in \Delta$ by X_p. A degeneration of K3 surfaces is called *semistable* if the central fibre X_0 is a simple normal crossings divisor, and is called a *Kulikov model* if it is semistable and $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$.

Kulikov degenerations can be classified into Types I, II, and III. Degenerations of Type II (resp. III) occur over 1-cusps (resp. 0-cusps) in the Baily-Borel compactification of the appropriate moduli space of K3 surfaces. In this paper we are interested in a special kind of Type II degeneration, known as a Tyurin degeneration.

Definition 3.1. A semistable degeneration of K3 surfaces is called a *Tyurin degeneration* if its central fibre X_0 consists of two rational surfaces V_1 and V_2 glued along a smooth anticanonical elliptic curve D.

By adjunction, Tyurin degenerations are automatically Kulikov models. However, they are also not unique: given a smooth family of K3 surfaces over the punctured disc, there may be multiple different ways to complete it to a Tyurin degeneration. Such non-uniqueness is common for Tyurin degenerations in the analytic category, where one can always analytically flop (-1)-curves from V_1 to V_2 (or vice versa) to obtain different Tyurin degenerations, but is less well-understood in the projective setting.

In this paper we are interested in studying projective Tyurin degenerations, i.e. ones where the morphism $\pi\colon \mathcal{X} \to \Delta$ is projective. We wish to categorise the possible projective Tyurin degenerations that can occur over the 1-cusp in the moduli space of M-polarised K3 surfaces.

In order to do this, we need to understand how to place a lattice polarization on a degeneration. The following definition is adapted from [DHNT15, Definition 2.1].

Definition 3.2. A degeneration of K3 surfaces $\pi\colon \mathcal{X} \to \Delta$ is called *M-polarised* if

- there is a trivial local subsystem $\mathcal{M}$ of $R^2\hat{\pi}_*\mathbb{Z}$ such that for each $p \in \Delta^*$, the fibre $\mathcal{M}_p \subseteq H^2(X_p, \mathbb{Z})$ of $\mathcal{M}$ over p is a primitive sublattice of $\mathrm{NS}(X_p)$ that is isomorphic to M, and
- There is a line bundle $\mathcal{A}$ on $\mathcal{X}^*$ whose restriction $\mathcal{A}_p$ to any fibre X_p is ample with first Chern class $c_1(\mathcal{A}_p)$ contained in $\mathcal{M}_p$ and primitive in $\mathrm{NS}(X_p)$.

To construct projective M-polarised Tyurin degenerations, we begin by constructing a *projective model* for M-polarised K3 surfaces, then degenerate this appropriately. We use two approaches to construct projective models, which we will now illustrate by giving two derivations of the projective normal form (4).

3.2. **Linear system approach.** As usual, we let X denote an M-polarised K3 surface. In this approach we begin with a nef and big $\mathbb{Q}$-divisor D on X, then exhibit sections that generate the canonical ring

$$R(X,D)=\bigoplus_{n\geq 0} H^0(X,\mathcal{O}_X(\lfloor nD\rfloor)).$$

It follows from [May72, Corollary 5] that if kD is a Cartier divisor, then $R(X,D)$ is generated in degrees $\leq 3k$ and we have a birational morphism $X\to \operatorname{Proj} R(X,D)$ to the canonical model of (X,D). In the cases studied, $\operatorname{Proj} R(X,D)$ will be a weighted projective space. We then compute the image of this map explicitly to obtain our projective model.

For example, using Notation 2.5, we could consider the divisor

$$D=(4;3,6,9,8,7,6,5,4,3,2,1,0,0,0,0,0,0;0\mid 0,0).$$

Then D is a nef and big Cartier divisor with $D^2=4$, and by Riemann-Roch we have $h^0(X,\mathcal{O}_X(D))=4$. Using relations (1), (2), and (3), we find three independent divisors that are linearly equivalent to D:

$$(1;1,2,3,3,3,3,3,3,3,3,3,3,4,5,6,4,2;3\mid 0,0),$$
$$(0;1,1,1,1,1,1,1,1,1,1,1,1,2,3,4,3,2;2\mid 1,0),$$
$$(0;0,0,0,0,0,0,0,0,0,0,0,0,1,2,3,2,1;2\mid 0,1).$$

These four divisors form a base-point free linear system. Denote the corresponding sections of $\mathcal{O}_X(D)$ by z,w,x,y, respectively. Then w,x,y,z generate the graded ring $R(X,D)$, so we obtain a birational morphism $\varphi\colon X\to\mathbb{P}^3[w,x,y,z]$ whose image is a quartic hypersurface.

From the intersection properties of the divisors above, one can see that φ contracts $E_1,\ldots,E_{11}$ to a singularity of type A_{11} and $E_{13},\ldots,E_{18}$ to a singularity of type E_6. The curve E_0 is mapped to the line $z=w=0$ and the curve E_{12} is mapped to the line $x=w=0$.

Let $f(w,x,y,z)$ be the defining equation of $\varphi(X)$. From the description of the divisors above we have the following.

- $f(0,x,y,z)$ is a nonzero multiple of x^3z.
- $f(w,0,y,z)$ may be written as $wf_3(w,y,z)$, for some cubic f_3 defining the image of S in the hyperplane $\{x=0\}$. Moreover, the cubic f_3 intersects the line $w=0$ in the two points $w=z=0$ (with multiplicity 1) and $w=y=0$ (with multiplicity 2), so is a nonzero multiple of $zy^2+w(\text{conic})$.
- $f(w,x,0,z)$ is a quartic curve defining the image of T in the hyperplane $\{y=0\}$.
- $f(w,x,y,0)$ is a nonzero multiple of w^4, this multiple can be set to 1 by global rescaling.

This gives a candidate form for $f_4(w,x,y,z)$:

$$w^4+a_1x^3z+wz(a_2w^2+a_3wx+a_4wy+a_5wz+a_6x^2+a_7xy+a_8xz+a_9y^2),$$

where $a_i\in\mathbb{C}$ and a_1,a_9 are nonzero.

We can place further restrictions on this form. The quotient of X by the fibrewise elliptic involution is a surface containing a configuration of curves as shown in Figure 3.1, where the numbers on the vertices denote self-intersection numbers. Note that the ramification curve T is isomorphic to its image under this quotient. The birational map contracting the curves $E_{13},\ldots,E_{18}$ descends to this quotient, where it creates a singularity of type A_2 in the branch curve. We thus see that the quartic curve $f(w,x,0,z)$ should have a singularity of type A_2 at the point $x=w=0$. This forces $a_8=0$ in the form above.

As $a_{10}\neq 0$, completing the square in y allows us to eliminate the $wxyz$ and w^2yz terms, and as $a_1\neq 0$ we can complete the cube in x to eliminate the x^2wz term.

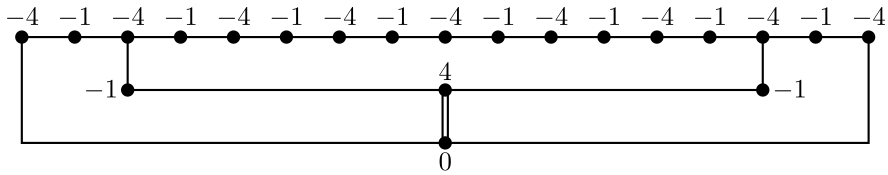

FIGURE 3.1. Dual graph of curves on the quotient by the fibrewise elliptic involution.

We are left with the form

$$f_4(w,x,y,z) = w^4 + b_1x^3z + b_2w^3z + b_3w^2xz + b_4w^2z^2 + b_5wy^2z,$$

for some $b_i \in \mathbb{C}$ with b_1, b_5 nonzero. Finally, rescaling variables allows us to obtain precisely the projective normal form (4).

3.3. **Toric approach.** For this approach we follow the ideas of [KT21, Section 3]. An M-polarised K3 surface is mirror, in the sense of Dolgachev [Dol96], to an H-polarised K3 surface. A generic H-polarised K3 surface can be expressed as the minimal resolution of a hypersurface of degree 12 in the weighted projective space $\mathbb{WP}(1,1,4,6)$. By the Batyrev mirror construction [Bat94], a generic M-polarised K3 surface can therefore be realized torically as an anticanonical hypersurface in $\mathbb{WP}(1,1,4,6)^\circ$, the polar dual of $\mathbb{WP}(1,1,4,6)$.

We derive a normal form for such an anticanonical hypersurface following the method of [CK99, Section 4.2]. Let $L := \mathbb{Z}^3$ denote the standard lattice and let $L_{\mathbb{R}} := L \otimes \mathbb{R}$. An anticanonical hypersurface in $\mathbb{WP}(1,1,4,6)$ corresponds to the reflexive polytope in $L_{\mathbb{R}}$ with vertices

$$u_0 = (-1,-1,-1), \qquad u_1 = (11,-1,-1),$$
$$u_2 = (-1,2,-1), \qquad u_3 = (-1,-1,1).$$

The rays of the fan $\Sigma^\circ \subset L_{\mathbb{R}}$ of the toric variety $\mathbb{WP}(1,1,4,6)^\circ$ are generated by the above set of vectors. These vectors generate a sublattice $L' \subset L$, whose index is given by the determinant of the matrix with rows u_1, u_2, u_3; this determinant is 6. The quotient L/L' is the group of order 6 defined by

$$G := \{(d_0,d_1,d_2,d_3) \in \mathbb{Z}_{12}^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 : d_0 + d_1 + 4d_2 + 6d_3 \equiv 0 \pmod{12}\}/\mathbb{Z}_{12},$$

where $\mathbb{Z}_{12} \subset \mathbb{Z}_{12}^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$ is embedded diagonally and $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ denotes integers modulo n.

As $u_0 + u_1 + u_2 + u_3 = 0$, a basis for L' is given by $\{u_1, u_2, u_3\}$, so if we view Σ° as a fan in $L' \otimes \mathbb{R}$, then Σ° is the standard fan for $\mathbb{WP}(1,1,4,6)$. In the overlattice L, there is a G-action on $\mathbb{WP}(1,1,4,6)$ given by

$$(x_0,x_1,x_2,x_3) \longmapsto (\omega^{d_0}x_0, \omega^{d_1}x_1, \omega^{4d_2}x_2, \omega^{6d_3}x_3),$$

where $(d_0,d_1,d_2,d_3) \in G$ and ω is a primitive twelfth root of unity. $\mathbb{WP}(1,1,4,6)^\circ$ is the quotient of $\mathbb{WP}(1,1,4,6)$ by this action.

The homogeneous coordinate ring of $\mathbb{WP}(1,1,4,6)^\circ$ is given by $\mathbb{C}[x_0,x_1,x_2,x_3]$ graded by the Chow group $A_2(\mathbb{WW}(1,1,4,6)^\circ)$. To compute this Chow group we use the exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4 \xrightarrow{\deg} \mathbb{Z} \oplus G \longrightarrow 0,$$

where the first map is given by the matrix whose rows are u_0, u_1, u_2, u_3 and the degree map is given by

$$\deg(d_0,d_1,d_2,d_3) := \big(d_0 + d_1 + 4d_2 + 6d_3, (-d_1 - 4d_2 - 6d_3, d_1, d_2, d_3)\big).$$

Thus $A_2(\mathbb{WP}(1,1,4,6)^\circ) \cong \mathbb{Z} \oplus G$ and the grading on $\mathbb{C}[x_0,x_1,x_2,x_3]$ is given by taking $x_0^{d_0}x_1^{d_1}x_2^{d_2}x_3^{d_3}$ to $\deg(d_0,d_1,d_2,d_3)$.

The anticanonical class of $\mathbb{WP}(1,1,4,6)^\circ$ has degree $\deg(1,1,1,1) = (12,0) \in \mathbb{Z} \oplus G$. A generic polynomial in $\mathbb{C}[x_0,x_1,x_2,x_3]$ of degree $(12,0)$ is given by

$$a_0x_0^{12}+a_1x_1^{12}+a_2x_2^3+a_3x_3^2+a_4x_0^6x_1^6+a_5x_0^4x_1^4x_2+a_6x_0^2x_1^2x_2^2+a_7x_0x_1x_2x_3+a_8x_0^3x_1^3x_3.$$

Using the torus action we can rescale

$$x_0 \longmapsto a_0^{-\frac{1}{12}}x_0, \quad x_1 \longmapsto a_0^{\frac{1}{12}}a_2^{\frac{1}{3}}a_3^{\frac{1}{2}}a_7^{-1}x_1, \quad x_2 \longmapsto a_2^{-\frac{1}{3}}x_2, \quad x_3 \longmapsto a_3^{-\frac{1}{2}}x_3.$$

Our equation becomes

$$x_0^{12} + b_0x_1^{12} + x_2^3 + x_3^2 + b_1x_0^6x_1^6 + b_2x_0^4x_1^4x_2 + b_3x_0^2x_1^2x_2^2 + x_0x_1x_2x_3 + b_4x_0^3x_1^3x_3,$$

for $b_0 = \frac{a_0a_1a_2^4a_3^6}{a_4^{12}}$, $b_1 = \frac{a_2^2a_3^3}{a_4^5}$, $b_2 = \frac{a_2a_3^2a_5}{a_4^4}$, $b_3 = \frac{a_3a_6}{a_4^2}$, and $b_4 = \frac{a_2a_3a_8}{a_4^3}$. Assigning weights $(1,1,4,6)$ to the variables (x_0,x_1,x_2,x_3), we see that this equation is weighted homogeneous, so defines a hypersurface in $\mathbb{WP}(1,1,4,6)$. A generic anticanonical hypersurface in $\mathbb{WP}(1,1,4,6)^\circ$ is given by the quotient of such a hypersurface by the group G.

The quotient $\mathbb{WP}(1,1,4,6)/G$ is realised by the map

$$\begin{aligned} \psi\colon \mathbb{WP}(1,1,4,6) &\longrightarrow \mathbb{P}^8 \\ (x_0,x_1,x_2,x_3) &\longmapsto (x_0^{12},x_1^{12},x_2^3,x_3^2,x_0^6x_1^6,x_0^4x_1^4x_2,x_0^2x_1^2x_2^2,x_0x_1x_2x_3,x_0^3x_1^3x_3) \end{aligned}$$

and the image of this map is isomorphic to $\mathbb{WP}(1,1,4,6)^\circ$. The anticanonical hypersurface defined by the equation above becomes the intersection of this image with the hypersurface

$$y_0 + b_0y_1 + y_2 + y_3 + b_1y_4 + b_2y_5 + b_3y_6 + y_7 + b_4y_8 = 0, \tag{5}$$

where $(y_0,\ldots,y_8)$ are homogeneous coordinates on $\mathbb{P}^8$. Working on the maximal torus given by setting $y_8 = 1$ and taking all other $y_i \in \mathbb{C}^*$, we can use relations on the image of ψ to write

$$y_0 = \frac{1}{y_1y_3^2}, \quad y_2 = \frac{y_7^3}{y_3}, \quad y_4 = \frac{1}{y_3}, \quad y_5 = \frac{y_7}{y_3}, \quad y_6 = \frac{y_7^2}{y_3}.$$

Substituting in and clearing denominators we obtain

$$1 + b_0y_1^2y_3^2 + y_1y_3y_7^3 + y_1y_3^3 + b_1y_1y_3 + b_2y_1y_3y_7 + b_3y_1y_3y_7^2 + y_1y_3^2y_7 + b_4y_1y_3^2 = 0.$$

Now set $x = y_7$, $y = y_3$, and $z = y_1y_3$ to get

$$1 + b_0z^2 + x^3z + y^2z + b_1z + b_2xz + b_3x^2z + xyz + b_4yz = 0.$$

Completing the square in y allows us to eliminate the xyz and yz terms, then completing the cube in x allows us to eliminate the x^2z term, giving

$$1 + b_0z^2 + x^3z + y^2z + c_1z + c_2xz = 0 \tag{6}$$

for some constants $c_1, c_2 \in \mathbb{C}$ defined as polynomial functions of the b_i. Adding a variable w to homogenise to $\mathbb{P}^3[x,y,z,w]$ and rescaling variables, we again obtain the projective normal form (4).

3.4. **Constructing Tyurin degenerations.** In the previous sections we have given two constructions of the projective model (4). We now show how to construct a projective M-polarised Tyurin degeneration from this projective model and derive its properties.

The projective model (4) has two singularities: a singularity of type A_{11} at the point $(w,x,y,z) = (0,0,1,0)$ and a singularity of type E_6 at the point $(w,x,y,z) = (0,0,0,1)$. It also contains the two lines $\{w = x = 0\}$, which contains both singularities, and $\{w = z = 0\}$, which passes through the A_{11} but not the E_6. These

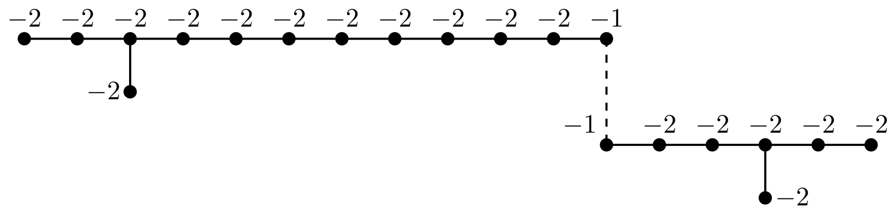

FIGURE 3.2. Dual graph of exceptional curves on V_1 (left of the dashed edge) and V_2 (right of the dashed edge). The dashed edge denotes a point in the double curve D which lies on the (-1)-curves in each component.

singularities may be crepantly resolved to give an M-polarised K3 surface and the 19 curves from Figure 2.1 arise as follows: $E_1, \ldots, E_{11}$ and $E_{12}, \ldots, E_{18}$ are the exceptional curves from the A_{11} and E_6 singularities, respectively, E_0 is the strict transform of the line $\{w = z = 0\}$, and E_{12} is the strict transform of the line $\{w = x = 0\}$.

By the discussion in Subsection 2.3, Type II degenerations of (4) occur along the locus $\{d = 0, a^3 \neq b^2\}$. To construct a Tyurin degeneration, fix values of (a, b) with $a^3 \neq b^2$ and consider the family

$$\mathcal{X} := \{y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \tfrac{1}{2}(tz^2w^2 + w^4) = 0\} \subset \mathbb{P}^3[w, x, y, z] \times \Delta,$$

where $t \in \Delta$ is a parameter on the complex disc.

This family is singular along the two curves. Computations in Singular [DGPS23] reveal that the curve $\{w = x = z = 0\}$ is a curve of A_{11} singularities, whilst the curve $\{w = x = y = 0\}$ consists of E_6 singularities for $t \neq 0$ which worsen to a singularity of Milnor number 8 in the fibre $\{t = 0\}$.

These singularities can be crepantly resolved as follows. Begin by blowing up the point $(w, x, y, z, t) = (0, 0, 0, 1, 0)$ in the ambient space. The strict transform of the threefold $\mathcal{X}$ has central fibre consisting of two components (the strict transform of the original fibre $\{t = 0\}$ plus an exceptional component) glued along a smooth elliptic curve, and is singular only along the strict transforms of the curves $\{w = x = z = 0\}$ and $\{w = x = y = 0\}$, which are now curves of A_{11} and E_6 singularities respectively. These two curves can then be resolved by blowing up in the usual way.

From the description above it is easy to see that the 19 curves E_i on a general fibre are invariant under monodromy, so the result is an M-polarised Tyurin degeneration. Its two components can be described as follows:

- The surface V_1 is the strict transform of the original fibre $\{t = 0\}$. It is obtained by blowing up $\mathbb{P}^2$ twelve times at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. The anticanonical divisor has $D^2 = -3$.
- The surface V_2 is the exceptional component of the first blow-up. It is obtained by blowing up $\mathbb{P}^2$ six times at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It is a weak del Pezzo surface of degree 3, so has $D^2 = 3$.

The configurations of exceptional curves, plus the strict transform of the tangent line at the blown up point, on the surfaces V_1 and V_2 are shown in Figure 3.2. The numbers on the vertices denote self-intersections. The (-1)-curves in these configurations meet the double curve D in the same point, denoted by the dashed edge in Figure 3.2.

It is easy to see how the M-polarization from a general fibre extends to the degenerate fibre. Indeed, the generating curves E_i degenerate as follows: $E_0, \dots, E_{11}$ become (-2)-curves in V_1 in the obvious way, whilst $E_{13}, \dots, E_{18}$ become (-2)-curves in V_2. The (-2)-curve E_{12} degenerates to a union of two (-1)-curves, one in each component.

In order to label such degenerations it will be convenient to introduce some notation for dual graphs.

Definition 3.3. An M-polarised Tyurin degeneration is called an Γ-*degeneration* if the (-2)-curves in the degenerate fibre which arise as degenerations of the E_i have dual graph Γ. In these descriptions, note that we extend the definition of the Dynkin diagram E_n to arbitrary $n \geq 6$ in the obvious way.

The degeneration constructed above is an E_6E_{12}-degeneration. In the analytic category, one can flop (-1)-curves between components to make this into Γ-degenerations with different dual graphs Γ; for example, flopping a single (-1)-curve from the left component to the right component in Figure 3.2 gives an E_7E_{11}-degeneration, whilst flopping a single (-1)-curve in the opposite direction gives a D_5E_{13}-degeneration. The aim of this paper is to determine which of these degenerations can be constructed projectively.

4. Constructions

By analytically flopping (-1)-curves in the construction from Section 3.4 we see that, up to the obvious symmetry given by left-right reflection (which corresponds to choice of labelling), there are 11 possible Γ-degenerations. The following is the main result of this paper.

Theorem 4.1. *All of the* 11 *possible* Γ*-degenerations can be realised as projective* M*-polarised Tyurin degenerations.*

We prove this result by explicitly constructing projective models for each of the 11 possible Γ-degenerations. The result is summarised in Table 4.1, which lists the possible diagrams Γ, along with the methods used to construct the Γ-degenerations (i.e. the linear system method from Section 3.2 or the toric method from Section 3.3); the projective model used in the construction, expressed as a hypersurface or complete intersection in weighted projective space, or a blow up of one of these; the singularities present in a generic projective model; and a reference to the section where the construction is given.

Note that the degenerations from E_9E_9 to A_1E_{16} are related by a single chain of flops. In A_1E_{16} there is a choice of two (-1)-curves that one may flop to proceed further: one choice gives A_1E_{17}, at which point the chain terminates, and the other gives E_{17} followed by E_{18}.

4.1. E_8E_{10} and E_9E_9-degenerations.

We construct these degenerations using the linear system approach of Section 3.2. We first note that, given an elliptically fibred K3 surface X with fibre F and (-2)-section S, the $\mathbb{Q}$-divisor $D = \frac{1}{2}S + F$ is nef and big and the canonical ring $R(X, D)$ is generated in degrees ≤ 6. A straightforward Riemann-Roch calculation then shows that $\operatorname{Proj} R(X, D)$ is isomorphic to the weighted projective space $\mathbb{WP}(1, 1, 4, 6)$. The birational morphism $\varphi\colon X \to \operatorname{Proj} R(X, D)$ contracts S along with those components of fibres that do not meet S; the image of this morphism is therefore the surface obtained by contracting the section in the Weierstrass model of X. By standard results in the theory of elliptic surfaces (see, for example, [Mir89]), we obtain that the image $\varphi(X)$ is given by an equation of the form

$$w^2 = z^3 + a_8(x, y)z + b_{12}(x, y),$$

Γ	Construction	Model	Singularities	Section
E_9E_9	Linear system	$X_{2,6} \subset \mathbb{WP}(1,1,1,2,3)$	$E_8E_8A_1$	4.1
E_8E_{10}	Linear system	$X_{12} \subset \mathbb{WP}(1,1,4,6)$	$E_8E_8A_1$	4.1
E_7E_{11}	Toric	$X_6 \subset \mathbb{WP}(1,1,1,3)$	$D_{10}E_7$	4.3
E_6E_{12}	Both	$X_4 \subset \mathbb{P}^3$	$A_{11}E_6$	3.4
D_5E_{13}	Toric	$X_8 \subset \mathbb{WP}(1,1,2,4)$	$D_{12}D_5$	4.3
A_4E_{14}	Toric	$X_{16} \subset \mathbb{WP}(1,2,5,8)$	$D_{13}A_4$	4.4
$A_1A_2E_{15}$	Toric	$X_{10} \subset \mathbb{WP}(1,1,3,5)$	$D_{14}A_2A_1$	4.3
A_1E_{16}	Linear system	$\mathrm{Bl}(X_{12} \subset \mathbb{WP}(1,1,4,6))$	$D_{15}A_1$	4.7
A_1E_{17}	Linear system	$X_{12} \subset \mathbb{WP}(1,1,4,6)$	$D_{16}A_1$	4.2
E_{17}	Linear system	$\mathrm{Bl}(X_8 \subset \mathbb{WP}(1,1,2,4))$	D_{16}	4.6
E_{18}	Toric	$X_6 \subset \mathbb{WP}(1,1,1,3)$	A_{17}	4.5

Table 4.1. The Γ-degenerations, along with the methods used to construct them, the projective models used and their singularities, and section references for the constructions.

where (x,y,z,w) are variables of weights $(1,1,4,6)$, respectively, and a_8 and b_{12} are homogeneous of degrees 8 and 12.

Remark 4.2. As elliptic fibrations with section on a K3 surface X give rise to primitive embeddings of H into the Picard lattice of X, with H generated by the classes of a section and fibre, this is effectively the statement that an H-polarised K3 can be expressed as the minimal resolution of a hypersurface of degree 12 in $\mathbb{WP}(1,1,4,6)$, which was already used in Section 3.3.

To construct a projective model for a M-polarised K3 surfaces we apply this approach to the standard fibration. Specifically, we take

$$D = (3;2,4,6,5,4,3,2,1,\tfrac{1}{2},0,0,0,0,0,0,0,0,0;0 \mid 0,0).$$

Then we have the following sections, which generate $R(X,D)$:

$$\begin{aligned}
x &= (3;2,4,6,5,4,3,2,1,0,0,0,0,0,0,0,0,0,0;0 \mid 0,0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
y &= (0;0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,4,2;3 \mid 0,0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
z &= (5;4,7,10,8,6,4,2,0,0,0,2,4,6,8,10,7,4;5 \mid 1,0) \in H^0(\mathcal{O}_X(4D),\\
w &= (8;5,10,15,12,9,6,3,0,0,0,3,6,9,12,15,10,5;8 \mid 0,1) \in H^0(\mathcal{O}_X(6D)).
\end{aligned}$$

From the intersection properties of D one can see that φ contracts $E_0,\ldots,E_7$ and $E_{11},\ldots,E_{18}$ to a pair of E_8 singularities, which occur in the fibres $\{x=0\}$ and $\{y=0\}$, and contracts the section E_9 to an A_1 singularity. The curves E_8 and E_{10} are taken to $\{x=0\}$ and $\{y=0\}$ respectively. Standard results on the classification of fibre types in Weierstrass models show that, after a coordinate change, a generic surface satisfying these conditions is given by

$$\{w^2 = z^3 + a_1x^4y^4z + a_2x^5y^7 + a_3x^6y^6 + x^7y^5\} \subset \mathbb{WP}(1,1,4,6), \tag{7}$$

for constants $a_i \in \mathbb{C}$. It is easy to show that a minimal resolution for such a surface is a K3 containing the 19 curves E_i which generate M, so this is a projective model for an M-polarised K3 surface.

A Tyurin degeneration is obtained by fixing values of $a_1, a_3 \in \mathbb{C}$ and letting $a_2 = t$ be a parameter on the complex disc Δ. The resulting degeneration has three curves of singularities: $\{x = y = w^2 - z^3 = 0\}$, which is a curve of A_1's, $\{w = y = z = 0\}$, which is a curve of E_8's, and $\{w = x = z = 0\}$ which consists of E_8's for $t \neq 0$ and worsens to an elliptic singularity of type $\widetilde{E}_8$ in the fibre $\{t = 0\}$.

These singularities can be crepantly resolved as follows. Blow up the point $(w, x, z, t) = (0, 0, 0, 0)$ with weights $(3, 1, 2, 1)$ respectively. The strict transform then has central fibre consisting of two components glued along a smooth elliptic curve, and is singular only along curves of E_8, E_8, and A_1 singularities, which can be resolved in the usual way. The components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is obtained by blowing up $\mathbb{P}^2$ ten times at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. It has $D^2 = -1$. The nine (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_9, \ldots, E_{18}$.
- V_2 is the exceptional component of the weighted blow up. It is a weak del Pezzo surface of degree 1 obtained by blowing up $\mathbb{P}^2$ eight times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = 1$. The seven (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, \ldots, E_7$.

The curve E_8 degenerates to the union of the exceptional (-1)-curves in each component (i.e. those coming from the final blowup). This is therefore an E_8E_{10}-degeneration.

A second Tyurin degeneration, which differs from the E_8E_{10}-degeneration above by a single flop, can be constructed from the projective model (7) as follows. Perform a Veronese embedding of degree 2 to obtain the complete intersection

$$\{w^2 - z^3 - a_1u_2^4z - a_2u_2^5u_3 - a_3u_2^6 - u_1u_2^5 = u_1u_3 - u_2^2 = 0\} \subset \mathbb{WP}(1, 1, 1, 2, 3),$$

where $\mathbb{WP}(1, 1, 1, 2, 3)$ has coordinates $u_1 = x^2$, $u_2 = xy$, $u_3 = y^2$, z and w. Now consider the degeneration given by

$$\{w^2 - z^3 - a_1u_2^4z - a_2u_2^5u_3 - a_3u_2^6 - u_1u_2^5 = u_1u_3 - tu_2^2 = 0\} \subset \mathbb{WP}(1, 1, 1, 2, 3) \times \Delta.$$

The central fibre of this degeneration consists of two components, one with $u_1 = 0$ and one with $u_3 = 0$, and the threefold total space is singular along three curves of singularities: two of type E_8 and one of type A_1. After crepantly resolving each of these curves we obtain a Tyurin degeneration with a symmetric central fibre: both V_1 and V_2 are obtained by blowing up $\mathbb{P}^2$ nine times at an inflection point of a smooth cubic curve C, and the anticanonical divisor D is the strict transform of C. We have $D^2 = 0$ in both components. The eight (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, \ldots, E_8$ respectively $E_{10}, \ldots, E_{18}$ in the two components, and the curve E_9 degenerates to a (-1)-curve in each component. This is therefore an E_9E_9-degeneration.

4.2. **A_1E_{17}-degeneration.** We can obtain another projective model by applying the approach of Section 4.1 to the alternate fibration, with E_1 as the distinguished (-2)-section. We take

$$D = (1; \tfrac{1}{2}, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1; 1 \mid 0, 0).$$

Then we have the following sections, which generate $R(X, D)$:

$$\begin{aligned}
x &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 0; 1 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
y &= (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0 \mid 1, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
z &= (1; 0, 0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 8, 2; 7 \mid 0, 0) \in H^0(\mathcal{O}_X(4D),\\
w &= (2; 0, 0, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 8, 1; 8 \mid 0, 1) \in H^0(\mathcal{O}_X(6D)).
\end{aligned}$$

The morphism φ contracts $E_0, E_3, \ldots, E_{16}, E_{18}$ to a D_{16} singularity, which occurs at the point $(x, y, z, w) = (0, 1, 0, 0)$, and contracts the section E_1 to an A_1 singularity, which occurs at $(x, y, z, w) = (0, 0, 1, 1)$. The curves E_2 and E_{17} are taken to $\{x = w^2 - z^3 = 0\}$ and $\{z = w = 0\}$ respectively. Standard results on the classification of fibre types in Weierstrass models show that, after a coordinate change, a generic surface satisfying these conditions is given by

$$\{w^2 = z^3 + (a_1x^4 + a_2x^3y + xy^3)z^2 + a_3x^8z\} \subset \mathbb{WP}(1, 1, 4, 6), \tag{8}$$

for constants $a_i \in \mathbb{C}$. It is easy to show that a minimal resolution for such a surface is a K3 containing the 19 curves E_i which generate M, so this is a projective model for an M-polarised K3 surface.

A Tyurin degeneration is obtained by fixing values of $a_1, a_2 \in \mathbb{C}$ and letting $a_3 = t$ be a parameter on the complex disc Δ. This family contains a curve of generically A_1 singularities given by $\{t = z = w = 0\}$. Blowing up this curve once, one obtains a degeneration whose central fibre contains two components: the strict transform of the original central fibre is a double cover of $\mathbb{WP}(1, 1, 4)$ ramified along the smooth quartic curve $\{z + a_1x^4 + a_2x^3y + xy^3 = 0\}$, which contains a singularity of type A_1 over the $\frac{1}{4}(1, 1)$ orbifold point, and the exceptional component is a double cover of the Hirzebruch surface $\mathbb{F}_4$ ramified in two curves in the classes $(s + 4f)$, and $(s + 8f)$ (where s and f denote the classes of the (-4)-section and fibre on $\mathbb{F}_4$ respectively), meeting at a singularity of type D_{16}. These components meet along a smooth elliptic curve given as a double cover of the line $\{z = 0\} \subset \mathbb{P}(1, 1, 4)$ and the (-4)-section in $\mathbb{F}_4$.

The resulting degeneration is singular only in the curves of D_{16} and A_1 singularities, which can be crepantly resolved in the usual way. The resulting components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is isomorphic to the Hirzebruch surface $\mathbb{F}_2$ and the anticanonical divisor D is a smooth member of the linear system $|2s + 4f|$, where s and f denote the classes of the (-2)-section and a fibre respectively. It has $D^2 = 8$. The (-2)-section is the degeneration of E_1.
- V_2 is the exceptional component of the first blow-up. It is the surface obtained by blowing up $\mathbb{P}^2$ a total of 17 times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -8$. The sixteen (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, E_3, \ldots, E_{18}$.

To describe the degeneration of the exceptional curve E_2 we need to introduce some notation. Let F_1 denote a fibre in the ruling on $V_1 \cong \mathbb{F}_2$ which lies tangent to D. Then let F_2 and F_3 denote the final two exceptional curves in the chain of blow-ups on V_2, so that $F_2^2 = -1$ and $F_3^2 = -2$ (note that F_3 is therefore the degeneration of E_0). Then the curve E_2 degenerates to the divisor $F_1 + 2F_2 + F_3$. This is therefore an A_1E_{17}-degeneration.

4.3. E_7E_{11}**,** D_5E_{13}**, and** $A_1A_2E_{15}$**-degenerations.** These degenerations are all closely related and it makes sense to construct them together. We apply the toric approach of Section 3.3.

With notation as in Section 3.3, we work on the maximal torus $y_4 = 1$ and take all other $y_i \in \mathbb{C}^*$, then use relations on the image of ψ to write

$$y_1 = \frac{1}{y_0}, \quad y_2 = y_5^3, \quad y_3 = y_8^2, \quad y_6 = y_5^2, \quad y_7 = y_5y_8.$$

Substituting into Equation (5) and multiplying by y_0^2 we obtain

$$y_0^3 + b_0y_0 + y_0^2y_5^3 + y_0^2y_8^2 + b_1y_0^2 + b_2y_0^2y_5 + b_3y_0^2y_5^2 + y_0^2y_5y_8 + b_4y_0^2y_8 = 0.$$

Now set $y = y_5$, $z = y_0$, and $w = y_0y_8$ to get

$$z^3 + b_0z + y^3z^2 + w^2 + b_1z^2 + b_2yz^2 + b_3y^2z^2 + wyz + b_4wz = 0.$$

Completing the square in w allows us to eliminate the wyz and wz terms, the completing the cube in y allows us to eliminate the y^2z^2 term, giving

$$z^3 + a_1z + y^3z^2 + w^2 + a_2z^2 + a_3yz^2 = 0, \tag{9}$$

for some constants $a_i \in \mathbb{C}$.

Now we make an important observation: by adjusting the weights, there are three different ways to homogenize this equation to define a hypersurface in weighted projective space. Specifically, if we take (y, z, w) to have weights $(1, n, n + 2)$, respectively, for $n \in \{1, 2, 3\}$, then we can homogenize by adding a variable x of weight 1 to obtain a hypersurface

$$\{w^2 + xz(a_1x^{n+3} + x^{3-n}z^2 + a_2x^3z + a_3x^2yz + y^3z) = 0\} \subset \mathbb{WP}(1, 1, n, n + 2).$$

This hypersurface contains a D_{8+2n} singularity at $(w, x, y, z) = (0, 1, 0, 0)$ and an E_{9-2n} singularity at $(w, x, y, z) = (0, 0, 1, 0)$ (where we use the convention that $E_5 = D_5$ and $E_3 = A_2 + A_1$). The resolution is an M-polarised K3 surface, where $E_0, E_2, \ldots, E_{8+2n}$ are the exceptional curves coming from the D_{8+2n} singularity, $E_{10+2n}, \ldots, E_{18}$ are the exceptional curves coming from the E_{9-2n} singularity, and E_1 and E_{9+2n} are the strict transforms of the lines $\{z = w = 0\}$ and $\{x = w = 0\}$, respectively.

Remark 4.3. If $\varphi_n \colon X \to \mathbb{WP}(1, 1, n, n + 2)$ is the projective model for an M-polarised K3 surface X described above (for $n \in \{1, 2, 3\}$), we may describe the pull-back $\varphi_n^*\mathcal{O}(1)$ as a $\mathbb{Q}$-divisor D_n on X and give generating sections of the canonical ring $R(X, D_n)$ corresponding to the coordinates (x, y, z, w). Using this, one may check the description above using the linear system method from Section 3.2. Explicitly, one obtains

$$\begin{aligned} D_1 &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 4, 5, 6, 4, 2; 3 \mid 0, 0), \\ D_2 &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \tfrac{5}{2}, 3, 2, 1; \tfrac{3}{2} \mid 0, 0), \\ D_3 &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \tfrac{4}{3}, \tfrac{1}{3}; 1 \mid 0, 0). \end{aligned}$$

Then $R(X, D_1)$ is generated by

$$\begin{aligned} x &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 4, 5, 6, 4, 2; 3 \mid 0, 0) \in H^0(\mathcal{O}_X(D)), \\ y &= (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 3, 2; 2 \mid 1, 0) \in H^0(\mathcal{O}_X(D)), \\ z &= (4; 2, 5, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0, 0, 0, 0, 0; 0 \mid 0, 0) \in H^0(\mathcal{O}_X(D)), \\ w &= (5; 1, 5, 9, 8, 7, 6, 5, 4, 3, 2, 1, 3, 5, 7, 9, 6, 3; 5 \mid 0, 1) \in H^0(\mathcal{O}_X(3D)); \end{aligned}$$

$R(X, D_2)$ is generated by

$$
\begin{aligned}
x &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 1; 1 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
y &= (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1; 0 \mid 1, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
z &= (5; 2, 6, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0, 0, 0; 0 \mid 0, 0) \in H^0(\mathcal{O}_X(2D),\\
w &= (6; 1, 6, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 2, 1; 2 \mid 0, 1) \in H^0(\mathcal{O}_X(4D));
\end{aligned}
$$

and $R(X, D_3)$ is generated by

$$
\begin{aligned}
x &= (1; 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 0; 1 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
y &= (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0 \mid 1, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
z &= (6; 2, 7, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0; 0 \mid 0, 0) \in H^0(\mathcal{O}_X(3D),\\
w &= (7; 1, 7, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0; 1 \mid 0, 1) \in H^0(\mathcal{O}_X(5D)).
\end{aligned}
$$

A Tyurin degeneration is obtained by fixing values of $a_2, a_3 \in \mathbb{C}$ and letting $a_1 = t$ be a parameter on the complex disc Δ. This family contains a curve of generically A_1 singularities given by $\{t = z = w = 0\}$. Blowing up this curve once, one obtains a degeneration whose central fibre contains two components: the strict transform of the original central fibre is a double cover of $\mathbb{WP}(1, 1, n)$ ramified along the quartic curve $\{x(x^{2-n}z + a_2x^3 + a_3x^2y + y^3) = 0\}$, which contains a singularity of type E_{9-2n}, and the exceptional component is a double cover of the Hirzebruch surface $\mathbb{F}_n$ ramified in three curves in the classes f, $(s + nf)$, and $(s + (n + 3)f)$ (where s and f denote the classes of the $(-n)$-section and fibre on $\mathbb{F}_n$ respectively), meeting at a singularity of type D_{8+2n}. These components meet along a smooth elliptic curve given as a double cover of the line $\{z = 0\} \subset \mathbb{P}(1, 1, n)$ and the $(-n)$-section in $\mathbb{F}_n$.

The resulting degeneration is singular only in the curves of D_{8+2n} and E_{9-2n} singularities, which can be resolved in the usual way. The resulting components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is a weak del Pezzo surface of degree $2n$ obtained by blowing up $\mathbb{P}^2$ a total of $(9 - 2n)$ times at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. It has $D^2 = 2n$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_{10+2n}, \ldots, E_{18}$.
- V_2 is the exceptional component of the first blow-up. It is the surface obtained by blowing up $\mathbb{P}^2$ a total of $(9 + 2n)$ times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -2n$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, \ldots, E_{8+2n}$.

The curve E_{9+2n} degenerates to the union of the exceptional (-1)-curves in each component (i.e. those coming from the final blowup). We therefore obtain an E_7E_{11}-degeneration for $n = 1$, a D_5E_{13}-degeneration for $n = 2$, and a $A_1A_2E_{15}$-degeneration for $n = 3$.

Remark 4.4. The approach from this section can also be applied with $n = 4$ to yield the A_1D_{16}-degeneration from Section 4.2. However, given the relationship between the A_1D_{16}-degeneration and the alternate fibration, and because it is geometrically of a slightly different character to the degenerations in this section (owing to the fact that the branch divisor is not divisible by x), we chose to keep it in its own section.

4.4. A_4E_{14}**-degeneration.** The approach from Section 4.3 can be used to construct one further case. Starting from the affine equation (9),

$$z^3 + a_1 z + y^3 z^2 + w^2 + a_2 z^2 + a_3 y z^2 = 0,$$

we can take (y, z, w) to have weights $(2, 5, 8)$, respectively, then homogenize by adding a variable x of weight 1 to obtain a hypersurface

$$\{w^2 + z(a_1 x^{11} + a_2 x^6 z + a_3 x^4 y z + x z^2 + y^3 z) = 0\} \subset \mathbb{WP}(1, 2, 5, 8).$$

This hypersurface contains a D_{13} singularity at $(x, y, z, w) = (0, 1, 0, 0)$ and an A_4 singularity at $(x, y, z, w) = (0, 0, 1, 0)$. The resolution is an M-polarised K3 surface, where $E_0, E_2, \ldots, E_{13}$ are the exceptional curves coming from the D_{13} singularity, $E_{15}, \ldots, E_{18}$ are the exceptional curves from the A_4, and E_1 and E_{14} are the strict transforms of the curves $\{z = w = 0\}$ and $\{x = w^2 + y^3 z = 0\}$ respectively.

Remark 4.5. If $\varphi\colon X \to \mathbb{WP}(1, 2, 5, 8)$ is the projective model for an M-polarised K3 surface X described above, we may describe the pull-back $\varphi^*\mathcal{O}(1)$ as a $\mathbb{Q}$-divisor D on X and give generating sections of the canonical ring $R(X, D)$ corresponding to the coordinates (x, y, z, w). Using this, one may check the description above using the linear system method from Section 3.2. Explicitly, one obtains

$$D = (\tfrac{1}{2}; 0, \tfrac{1}{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \tfrac{6}{5}, \tfrac{4}{5}, \tfrac{2}{5}; \tfrac{3}{5} \mid 0, 0)$$

and we have the following sections, which generate $R(X, D)$:

$$\begin{aligned}
x &= (0; 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0; 0 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),\\
y &= (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0 \mid 1, 0) \in H^0(\mathcal{O}_X(\lfloor 2D \rfloor)),\\
z &= (5; 2, 5, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0, 0; 0 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor 5D \rfloor),\\
w &= (6; 1, 6, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0, 0; 0 \mid 0, 1) \in H^0(\mathcal{O}_X(\lfloor 8D \rfloor)).
\end{aligned}$$

A Tyurin degeneration is obtained by fixing values of $a_2, a_3 \in \mathbb{C}$ and letting $a_1 = t$ be a parameter on the complex disc Δ. This family contains a curve of generically A_1 singularities given by $\{t = z = w = 0\}$. Blowing up this curve once, one obtains a degeneration whose central fibre contains two components: the strict transform of the original central fibre is a double cover of $\mathbb{WP}(1, 2, 5)$ ramified along the degree 6 curve $\{a_2 x^6 + a_3 x^4 y + x z + y^3 = 0\}$, which contains a singularity of type A_4 over the point $(0, 0, 1)$.

The exceptional component is most simply described as a double cover of a *half Hirzebruch surface*. The following definition is fairly well-known.

Definition 4.6. The half Hirzebruch surface $\mathbb{F}_{n-\frac{1}{2}}$ is defined as follows. First blow up the Hirzebruch surface $\mathbb{F}_n$ at a point on the $(-n)$-section, then blow up again at the intersection between the two (-1)-curves in the reducible fibre that results; the resulting surface has a $\mathbb{P}^1$-fibration with a single reducible fibre consisting of three curves with self-intersections $(-2, -1, -2)$ and multiplicities $(1, 2, 1)$. Finally, contract the two (-2)-curves to a pair of A_1 singularities. The resulting singular surface is $\mathbb{F}_{n-\frac{1}{2}}$.

If s and f are the strict transforms of the $(-n)$-section and a generic fibre from $\mathbb{F}_n$, then the Weil divisor class group of $\mathbb{F}_{n-\frac{1}{2}}$ is generated by s and $\frac{1}{2}f$, with intersection numbers $s^2 = -n + \frac{1}{2}$, $s.f = 1$, and $f^2 = 0$.

With this definition, the exceptional component is a double cover of the half Hirzebruch surface $\mathbb{F}_{\frac{5}{2}}$ ramified over two curves in the classes $(s + \frac{5}{2}f)$ and $(s + \frac{11}{2}f)$, chosen so that the double cover has a singularity of type D_{13}. This component meets the strict transform of the original central fibre along a smooth elliptic curve, given as a double cover of the line $\{z = 0\} \subset \mathbb{P}(1, 2, 5)$ and the $(-\frac{5}{2})$-section in $\mathbb{F}_{\frac{5}{2}}$.

The resulting degeneration is singular only in the curves of D_{13} and A_4 singularities, which can be resolved in the usual way. The resulting components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is a weak del Pezzo surface of degree 5 obtained by blowing up $\mathbb{P}^2$ four times at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. It has $D^2 = 5$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_{15}, \ldots, E_{18}$.
- V_2 is the exceptional component of the first blow-up. It is the surface obtained by blowing up $\mathbb{P}^2$ fourteen times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -5$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, \ldots, E_{13}$.

The curve E_{14} degenerates to the union of the exceptional (-1)-curves in each component (i.e. those coming from the final blowup). We therefore obtain an A_4E_{14}-degeneration.

Remark 4.7. We note that this method can also be used to construct an alternative model for the E_6E_{12}-degeneration from Sections 3.2, 3.3, and 3.4, by assigning weights $(2, 3, 6)$ to (y, z, w) and compactifying to $\mathbb{WP}(1, 2, 3, 6)$. We chose to retain the construction given in those sections for compatibility with the existing literature on M-polarised K3 surfaces, which makes extensive use of the normal form (4).

4.5. **E_{18}-degeneration.** To construct this degeneration we return to the affine equation (6):

$$1 + b_0z^2 + x^3z + y^2z + c_1z + c_2xz = 0.$$

In Section 3.3 we obtained a projective model by compactifying this to a hypersurface in $\mathbb{P}^4$. However, as we saw in Section 4.3, we can obtain different projective models by compactifying to different weighted projective spaces.

We can compactify the equation above to a sextic hypersurface in $\mathbb{WP}(1, 1, 1, 3)$ by assigning weights $(1, 1, 3)$ to (x, y, z) and introducing a new weight 1 variable w:

$$w^6 + b_0z^2 + x^3z + wy^2z + c_1w^3z + c_2w^2xz = 0.$$

By the results of Sections 2.3 and 3.3, it is easy to see that the parameters (b_0, c_1, c_2) in the equation above are related to the modular parameters (a, b, d) describing an M-polarised K3 surface by

$$b_0 = d, \qquad c_1 = 2b, \qquad c_2 = -3a,$$

from which it follows that $b_0 \neq 0$ for any M-polarised K3 surface. We can therefore make a coordinate change $z \mapsto \frac{z}{b_0}$ and clear denominators to obtain

$$b_0w^6 + z^2 + x^3z + wy^2z + c_1w^3z + c_2w^2xz = 0.$$

Completing the square in z and rearranging gives the projective model

$$\{z^2 + ((b_1 + c_1)w^3 + x^3 + wy^2 + c_2w^2x)((b_1 - c_1)w^3 - x^3 - wy^2 - c_2w^2x) = 0\}$$

in $\mathbb{WP}(1, 1, 1, 3)$, where $b_1 \in \mathbb{C}$ is a square root of $4b_0$.

This model is a double cover of $\mathbb{P}^2$ ramified over the two cubic curves:

$$\{w^2y + x^3 + c_2w^2x + (c_1 + b_1)w^3 = 0\},$$
$$\{w^2y + x^3 + c_2w^2x + (c_1 - b_1)w^3 = 0\}.$$

These two curves meet in the single point $(w, x, y) = (0, 0, 1)$, over which the double cover has a singularity of type A_{17}. The resolution is an M-polarised K3 surface, where $E_1, \ldots, E_{17}$ are the exceptional curves coming from the resolution of the A_{17} singularity and E_0, E_{18} are the curves $\{w = z + x^3 = 0\}$ and $\{w = z - x^3 = 0\}$, respectively, so this is a projective model.

A Tyurin degeneration is obtained by fixing values of $c_1, c_2 \in \mathbb{C}$ and letting $b_1 = t$ be a parameter on the complex disc Δ. The central fibre of this family is a union of the two surfaces

$$\overline{V}_1 := \{z + c_1w^3 + x^3 + wy^2 + c_2w^2x = 0\} \subset \mathbb{WP}(1, 1, 1, 3),$$
$$\overline{V}_2 := \{z - c_1w^3 - x^3 - wy^2 - c_2w^2x = 0\} \subset \mathbb{WP}(1, 1, 1, 3),$$

each of which is isomorphic to $\mathbb{P}^2$ and is a non-$\mathbb{Q}$-Cartier divisor in the total space of the degeneration. The curves E_0 and E_{18} degenerate to $\{w = z + x^3 = 0\} \subset \overline{V}_1$ and $\{w = z - x^3 = 0\} \subset \overline{V}_2$, respectively.

To resolve the singularities of this degeneration, first blow-up the locus $\{t = z + c_1w^3 + x^3 + wy^2 + c_2w^2x = 0\}$ in the total space of the family; this corresponds to blowing along a non-$\mathbb{Q}$-Cartier divisor. The strict transform V_1 of $\overline{V}_1$ is the blow-up of $\overline{V}_1$ at the single point $\{w = x = z = 0\}$, and the strict transform V_2 of $\overline{V}_2$ is isomorphic to $\overline{V}_2$.

After performing this blow-up, the family is singular only along a smooth curve of A_{17} singularities, which intersect the central fibre in a point of V_1 away from the double curve. Resolving this curve in the usual way, the resulting components of the central fibre are:

- V_1 is the surface obtained by blowing up $\mathbb{P}^2$ a total of 18 times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -9$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_0, \ldots, E_{17}$.
- V_2 is isomorphic to $\mathbb{P}^2$ and the anticanonical divisor D is a smooth cubic curve. It has $D^2 = 9$ and contains no (-2)-curves.

To describe the degeneration of the exceptional curve E_{18} we need to introduce some notation. Let F_1 denote the curve $\{w = z - x^3 = 0\} \subset V_2$, which is a line in $\mathbb{P}^2$ lying inflectionally tangent to D. Then let F_2, F_3, and F_4 denote the final three exceptional curves in the chain of blow-ups on V_1, so that $F_2^2 = -1$ and $F_3^2 = F_4^2 = -2$ (note that F_3 and F_4 are therefore the degenerations of E_{17} and E_{16} respectively). Then the curve E_{18} degenerates to the divisor $F_1 + 3F_2 + 2F_3 + F_4$. This is therefore an E_{18}-degeneration.

4.6. **E_{17}-degeneration.** The final two cases are somewhat more technical, as they are not constructed directly as hypersurfaces in weighted projective space; instead, we start with a hypersurface in weighted projective space and perform a single blow-up.

To construct the E_{17}-degeneration we use the linear system approach from Section 3.2. We begin with the nef and big $\mathbb{Q}$-divisor

$$D = (1; 1, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, \tfrac{3}{2}, 0; \tfrac{3}{2} \mid 0, 0).$$

By Riemann-Roch, $H^0(\mathcal{O}_X(\lfloor D \rfloor))$ is generated by the two sections

$$x = (1; 1, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1, 0; 1 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)),$$
$$y = (1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0; 0 \mid 1, 0) \in H^0(\mathcal{O}_X(\lfloor D \rfloor)).$$

Now, $2D$ is a Cartier divisor and $h^0(\mathcal{O}_X(2D)) = 4$, but we only obtain three generators $\{x^2, xy, y^2\}$ from above. Moreover, we cannot obtain any extra generators of $H^0(\mathcal{O}_X(2D))$ using the relations (2) and (3). However, as the linear system $|2D|$ is not an elliptic pencil and D is nef and big, by [May72, Propositions 1 and 8] we have that the generic member of $|2D|$ is a smooth, irreducible curve. Let Z be any such curve and let $z \in H^0(\mathcal{O}_X(2D))$ be its defining section. The curve Z intersects our divisors E_i, S and T as follows:

$$Z.E_0 = 2, \quad Z.E_{17} = 3, \quad Z.S = 2, \quad Z.T = 5, \quad Z.E_i = 0 \text{ for } i \neq 0, 17.$$

Then $\{x^2, xy, y^2, z\}$ generate $H^0(\mathcal{O}_X(2D))$, and one can show that the canonical ring is generated by x, y, z and

$$w = (0; 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 8, 1; 8 \mid 0, 1) \in H^0(\mathcal{O}_X(4D)).$$

It follows that $\operatorname{Proj} R(X, D)$ is isomorphic to $\mathbb{WP}(1, 1, 2, 4)$ and the image $\varphi(X)$ is defined by an equation of degree 8.

Remark 4.8. It follows from this computation that the divisor

$$(0; 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 8, 2; 7 \mid 0, 0) \in H^0(\mathcal{O}_X(\lfloor 3D \rfloor)),$$

obtained by applying the relation (1) to x^3, can be written as a linear combination of the generators $\{x^3, x^2y, xy^2, y^3, xz, xy\}$. We denote this linear combination by $f_3(x, y, z)$.

From the intersection properties of D one can see that φ contracts the curves $E_1, \ldots, E_{16}, E_{18}$ to a D_{17} singularity, which occurs at the point $(x, y, z, w) = (0, 0, 1, 0)$. The curve E_{17} is taken to $\{w = f_3(x, y, z) = 0\}$, where f_3 is the same as in Remark 4.8, and E_0 is taken to the intersection of the hyperplane $\{x = 0\}$ with our projective model.

After completing the square in w, we may assume that our projective model is a double cover of $\mathbb{WP}(1, 1, 2)$ given by an equation of the form

$$w^2 = f_3(x, y, z)g_5(x, y, z),$$

where g_5 is an equation of degree 5 defining the image of the divisor T. Moreover, the intersection properties of D show that the two curves $\{f_3(x, y, z) = 0\}$ and $\{g_5(x, y, z) = 0\}$ are both tangent to $\{x = 0\}$ at $(x, y, z) = (0, 0, 1)$, and they do not intersect away from this point.

After a coordinate change in z, we may assume that $f_3(x, y, z) = xz + cy^3$, for some nonzero $c \in \mathbb{C}$. From the intersection condition between f_3 and g_5, it follows that g_5 must have the form

$$g_5(x, y, z) = (xz + cy^3)(a_0 z + a_1 y^2 + a_2 xy + a_3 x^2) + a_4 x^5,$$

for $a_i \in \mathbb{C}$ constants with $a_0, a_4 \neq 0$. After rescaling x, y, z we may assume that $c = a_0 = 1$, so our projective model is given by

$$w^2 = (xz + y^3)\big((xz + cy^3)(a_0 z + a_1 y^2 + a_2 xy + a_3 x^2) + a_4 x^5\big).$$

One may verify that the minimal resolution of such a surface is a K3 containing the 19 curves E_i which generate M.

To construct an E_{17}-degeneration we begin by partially resolving the D_{17} singularity. To do this we blow-up the model above once at the point $(0, 0, 1, 0)$, so that it becomes a double cover of the Hirzebruch surface $\mathbb{F}_2$ instead of the cone $\mathbb{WP}(1, 1, 2)$. The Hirzebruch surface $\mathbb{F}_2$ can be realised explicitly as the rational scroll $\mathbb{F}(2, 0)$, which is constructed as the quotient of $\big(\mathbb{A}^2 \setminus \{0\}\big)^2$ by the action of $(\mathbb{C}^*)^2$ defined by

$$\begin{aligned} (\lambda, 1)\colon (x, y; u, v) &\longmapsto (\lambda x, \lambda y; \lambda^{-2} u, v) \\ (1, \mu)\colon (x, y; u, v) &\longmapsto (x, y; \mu u, \mu v), \end{aligned}$$

where $(x, y; u, v)$ are coordinates on $\big(\mathbb{A}^2 \setminus \{0\}\big)^2$ and $(\lambda, \mu) \in (\mathbb{C}^*)^2$. For generalities on rational scrolls, we refer the reader to [Rei97, Chapter 2].

After this blow-up, our projective model becomes a double cover of $\mathbb{F}_2$ ramified over the (-2)-section s and two curves in the linear systems $|s+3f|$ and $|s+5f|$, where f is the class of a fibre of the ruling on $\mathbb{F}_2$. In the bihomogeneous coordinates $(x, y; u, v)$ introduced in the rational scroll description above, this branch curve is given explicitly by

$$u(vx+uy^3)\big((vx+uy^3)(v+a_1uy^2+a_2uxy+a_3ux^2)+a_4u^2x^5\big).$$

Note that it has a singularity of type D_{16} at the point $(x, y; u, v) = (0, 1; 0, 1)$.

A Tyurin degeneration is obtained by fixing values of $a_1, a_2, a_3 \in \mathbb{C}$ and letting $a_4 = t$ be a parameter on the complex disc Δ. This family contains a curve of generically A_1 singularities given by $\{t = w = vx+uy^3 = 0\}$. Blowing up this curve once, one obtains a degeneration whose central fibre contains two components: the strict transform of the original central fibre is a double cover of $\mathbb{F}_2$ ramified over the (-2)-section $\{u = 0\}$ and the curve $\{v+a_1uy^2+a_2uxy+a_3ux^2 = 0\} \in |s+2f|$, and the exceptional component is a double cover of the Hirzebruch surface $\mathbb{F}_4$ ramified in three curves in the classes f, $(s+4f)$, and $(s+7f)$ (where s and f denote the classes of the (-4)-section and fibre on $\mathbb{F}_4$ respectively), meeting at a singularity of type D_{16}. These components meet along a smooth elliptic curve given as a double cover of the curve $\{vx+uy^3 = 0\} \subset \mathbb{F}_2$ and the (-4)-section in $\mathbb{F}_4$.

The resulting degeneration is singular only in the curve of D_{16} singularities, which can be resolved in the usual way. The resulting components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is isomorphic to the Hirzebruch surface $\mathbb{F}_1$ obtained by blowing up $\mathbb{P}^2$ once at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. It has $D^2 = 8$.
- V_2 is the exceptional component of the first blow-up. It is the surface obtained by blowing up $\mathbb{P}^2$ a total of 17 times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -8$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_2, \ldots, E_{18}$.

The curve E_1 degenerates to the union of the exceptional (-1)-curves in each component (i.e. those coming from the final blowup). Finally, to describe the degeneration of the exceptional curve E_0 we need to introduce some notation. Let F_1 denote the fibre of the ruling on $V_1 \cong \mathbb{F}_1$ passing through the point where D intersects the (-1)-section (in coordinates, F_1 is the double cover of $\{x = 0\} \subset \mathbb{F}_2$). Then let F_2 and F_3 denote the final two exceptional curves in the chain of blow-ups on V_2, so that $F_2^2 = -1$ and $F_3^2 = -2$ (note that F_3 is therefore the degeneration of E_2). Then the curve E_0 degenerates to the divisor $F_1 + 2F_2 + F_3$. This is therefore an E_{17}-degeneration.

4.7. A_1E_{16}**-degeneration.** To construct the final A_1E_{16}-degeneration, we begin with the projective model from Section 4.2, given by Equation (8):

$$\{w^2 = z^3 + (a_1x^4 + a_2x^3y + xy^3)z^2 + a_3x^8z\} \subset \mathbb{WP}(1, 1, 4, 6).$$

Recall that this model is a double cover of $\mathbb{WP}(1, 1, 4)$ branched along the divisor

$$B = \{z^3 + (a_1x^4 + a_2x^3y + xy^3)z^2 + a_3x^8z\} \subset \mathbb{WP}(1, 1, 4),$$

which contains a singularity of type D_{16} at the point $(x, y, z) = (0, 1, 0)$.

As in Section 4.6, to construct an A_1E_{16}-degeneration we begin by partially resolving the D_{16} singularity. Explicitly, this is given by performing a weighted blow-up of $\mathbb{WP}(1,1,4)$ along the ideal $\langle xy^3+z, z^2\rangle$; i.e. z has weight 1 and xy^3-z has weight 2. Our projective model is a double cover of the resulting surface S, branched along the strict transform of the divisor B.

In the affine chart $y=1$ we can describe this blow-up explicitly as follows. Introduce a new variable $s := x+z$ and use it to eliminate x. The equation of B becomes

$$\{z(a_1(s-z)^4 + a_2(s-z)^3 + s)z + a_3(s-z)^8) = 0\} \subset \mathbb{A}^2[s,z].$$

Now perform a weighted blow-up along the ideal $\langle s, z^2\rangle$. The resulting surface S is given by the equation

$$\{sv^2 = z^2u\} \subset \mathbb{A}^2[s,z] \times \mathbb{WP}(1,2),$$

where (u,v) are variables of weights $(2,1)$ on $\mathbb{WP}(1,2)$, and the strict transform of B is the intersection of this surface with

$$\big\{(s-z)(su-2zu+v^2)\big(a_1(s-z)v^2+a_2v^2+a_3z(s-z)^3(su-2zu+v^2)\big)+uv^2=0\big\}.$$

The double cover branched along B has a singularity of type D_{15} over the point $s=z=v=0$.

A Tyurin degeneration is obtained by fixing values of $a_1, a_2 \in \mathbb{C}$ and letting $a_3 = t$ be a parameter on the complex disc Δ. This family contains a curve of generically A_1 singularities, given in the chart above by $\{t=z=v=0\}$. Blowing up this curve once, one obtains a degeneration whose central fibre contains two components: the strict transform of the original central fibre is a double cover of S ramified over a curve given in the chart above by $\{(s-z)(su-2zu+v^2)(a_1(s-z)+a_2)+u=0\}$, and the exceptional component is a double cover of the half Hirzebruch surface $\mathbb{F}_{\frac{7}{2}}$ ramified in two curves in the classes $(s+\frac{7}{2}f)$ and $(s+\frac{13}{2}f)$ (where s and f denote the classes of the $(-\frac{7}{2})$-section and a generic fibre on $\mathbb{F}_{\frac{7}{2}}$ respectively), chosen so that the double cover has a singularity of type D_{15}. These components meet along a smooth elliptic curve given as a double cover of the curve $\{z=v=0\}\subset S$ and the $(-\frac{7}{2})$-section in $\mathbb{F}_{\frac{7}{2}}$.

The resulting degeneration is singular only in the curve of D_{15} singularities, which can be resolved in the usual way. The resulting components of the central fibre are:

- V_1 is the strict transform of the original central fibre. It is isomorphic to the del Pezzo surface of degree 7 obtained by blowing up $\mathbb{P}^2$ twice at an inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_1$ is the strict transform of C. It has $D^2 = 7$. The (-2)-curve obtained from this blow-up is the degeneration of E_1.
- V_2 is the exceptional component of the first blow-up. It is the surface obtained by blowing up $\mathbb{P}^2$ a total of 16 times at the inflection point of a smooth cubic curve C, and the anticanonical divisor $D \subset V_2$ is the strict transform of C. It has $D^2 = -7$. The (-2)-curves obtained from this blow-up, along with the (-2)-curve obtained as the strict transform of the inflectional tangent line, are the degenerations of $E_3, \ldots, E_{18}$.

To describe the degeneration of the curves E_0 and E_2, we introduce a little notation. Let F_1 denote the unique exceptional (-1)-curve in V_2, let F_2 denote the (-1)-curve in V_1 that is exceptional for the final blow-up, and let F_3 denote the (-1)-curve in V_1 that arises as the strict transform of the inflectional tangent line. Then E_0 degenerates to F_1+F_3 and E_2 degenerates to F_1+F_2. This is therefore an A_1E_{16}-degeneration.

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