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Normal functions, Picard–Fuchs equations, and elliptic fibrations on $K3$ surfaces

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Abstract. Using Gauss–Manin derivatives of generalized normal functions, we arrive at results on the non-triviality of the transcendental regulator for K_m of a very general projective algebraic manifold. Our strongest results are for the transcendental regulator for K_1 of a very general $K3$ surface and its self-product. We also construct an explicit family of K_1 cycles on $H \oplus E_8 \oplus E_8$-polarized $K3$ surfaces, and show they are indecomposable by a direct evaluation of the real regulator. Critical use is made of natural elliptic fibrations, hypersurface normal forms, and an explicit parametrization by modular functions.

1. Introduction

The subject of this paper is the existence, construction, and detection of indecomposable algebraic K_1-cycle classes on $K3$ surfaces and their self-products. We begin by treating the existence of regulator-indecomposable cycles on a very general $K3$ with fixed polarization by a lattice of rank less than 20 (Section 2), as well as on their self-products in the rank one projective case (Section 4). This is intertwined with a discussion (Section 3) of homogeneous and inhomogeneous Picard–Fuchs equations for truncated normal functions – a subject of increasing interest due to their recent spectacular use in open string mirror symmetry [24] – which is further amplified by explicit examples in Section 5.

The second half of the paper takes up the question of how to use the geometry of polarized $K3$ surfaces with high Picard rank to construct indecomposable cycles (Sections 5 and 6). Elliptic fibrations yield an extremely natural source of families of cycles, whose image under the real and transcendental regulator maps have apparently not been previously studied. Our computation of their real regulator not only proves indecomposability, but turns out to be related to higher Green's functions on the modular curve $X(2)$ (cf. [20], which depends upon the present Section 6). The paper concludes (Section 7) with a discussion of the mysterious Picard-rank 20 case and its relationship to open irrationality problems. In the remainder of this

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introduction, we shall state the main existence results of Sections 2–4, and place the constructions of Section 6 in historical context.

Background on cycle class maps. Let X be a projective algebraic manifold of dimension d, and $\mathrm{CH}^r(X,m)$ the higher Chow group introduced by Bloch [4]. We are mainly interested in working modulo torsion, thus we will restrict ourselves to the corresponding group $\mathrm{CH}^r(X,m;\mathbb{Q}) := \mathrm{CH}^r(X,m)\otimes\mathbb{Q}$. We shall be especially interested in the case $m=1$, and the indecomposable cycles

$$\mathrm{CH}^r_{\mathrm{ind}}(X,1;\mathbb{Q}) := \frac{\mathrm{CH}^r(X,1;\mathbb{Q})}{\mathrm{image}(\mathbb{C}^*\otimes\mathrm{CH}^{r-1}(X;\mathbb{Q}))}.$$

An explicit description of the Bloch cycle class map to Deligne cohomology,

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}) \to J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big) \subset H^{2r-m}_{\mathcal{D}}(X,\mathbb{Q}(r)),$$

is given in [22], where

$$\begin{aligned} J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big) &:= \mathrm{Ext}^1_{\mathbb{Q}\text{-MHS}}\big(\mathbb{Q}(0), H^{2r-m-1}(X,\mathbb{Q}(r))\big)\\ &\simeq \frac{H^{2r-m-1}(X,\mathbb{C})}{F^r H^{2r-m-1}(X,\mathbb{C}) + H^{2r-m-1}(X,\mathbb{Q}(r))}\\ &\simeq \frac{\{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})\}^\vee}{H_{2d-2r+m+1}(X,\mathbb{Q}(d-r))}. \end{aligned}$$

(Note that $\mathrm{CH}^r(X,m;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q})$ for $m\geq 1$, and by convention, for singular homology, $H_i(X,\mathbb{Q}(r)) := H_i(X,\mathbb{Q})\otimes\mathbb{Q}(-r)$, which has weight $2r-i$.) Composing $\mathrm{cl}_{r,m}$ with the natural map

$$\mathrm{Ext}^1_{\mathbb{Q}\text{-MHS}}\big(\mathbb{Q}(0), H^{2r-m-1}(X,\mathbb{Q}(r))\big) \to \mathrm{Ext}^1_{\mathbb{R}\text{-MHS}}\big(\mathbb{R}(0), H^{2r-m-1}(X,\mathbb{R}(r))\big)$$

defines the real regulator

$$r_{r,m} : \mathrm{CH}^r(X,m) \to \mathrm{Ext}^1_{\mathbb{R}\text{-MHS}}\big(\mathbb{R}(0), H^{2r-m-1}(X,\mathbb{R}(r))\big), \tag{1.1}$$

where explicitly (noting $\mathbb{C} = \mathbb{R}(m-1)\oplus\mathbb{R}(m)$)

$$\begin{aligned} &\mathrm{Ext}^1_{\mathbb{R}\text{-MHS}}\big(\mathbb{R}(0), H^{2r-m-1}(X,\mathbb{R}(r))\big)\\ &\quad\simeq \frac{H^{2r-m-1}(X,\mathbb{C})}{F^r H^{2r-m-1}(X,\mathbb{C}) + H^{2r-m-1}(X,\mathbb{R}(r))}\\ &\quad\simeq \mathrm{Hom}\big(F^{d-r+1}\cap H^{2d-2r+m+1}(X,\mathbb{R}),\mathbb{R}(r-1)\big). \end{aligned}$$

We will now assume that X is a member of a family $\lambda : \mathcal{X}\to\mathcal{S}$, where $\mathcal{X}$, $\mathcal{S}$ are smooth complex quasi-projective varieties and λ is smooth and proper, and where $X := \lambda^{-1}(0)$ corresponds to $0\in\mathcal{S}$. Associated to this is the Kodaira–Spencer map $\kappa : T_0(\mathcal{S}) \to H^1(X,\Theta_X)$, whose image we will denote by $H^1_{\mathrm{alg}}(X,\Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X. The cohomology of the fibers of λ defines a variation of Hodge structure, and roughly speaking, a normal function is a "locally liftable holomorphic" cross-section: $\mathcal{S}\to\coprod_{t\in\mathcal{S}} J(H^{2r-m-1}(\lambda^{-1}(t),\mathbb{Q}(r)))$. The Hodge structure

$$H^{2r-m-1}(X,\mathbb{Q}(r)) = H^{2r-m-1}_f(X,\mathbb{Q}(r)) \bigoplus H^{2r-m-1}_v(X,\mathbb{Q}(r))$$

decomposes, where

$$H_f^{2r-m-1}(X,\mathbb{Q}(r)) := H^{2r-m-1}(X,\mathbb{Q}(r))^{\pi_1(\mathcal{S})}$$

is the fixed part of the corresponding monodromy group action on $H^{2r-m-1}(X,\mathbb{Q}(r))$, and $H_v^{2r-m-1}(X,\mathbb{Q}(r))$ is the orthogonal complement. A well-known result of Deligne [13] tells us that

$$H_f^{2r-m-1}(X,\mathbb{Q}(r)) = \mathrm{Image}\big(H^{2r-m-1}(\mathcal{X},\mathbb{Q}(r)) \to H^{2r-m-1}(X,\mathbb{Q}(r))\big).$$

Accordingly, one has a reduced cycle class map

$$\underline{\mathrm{cl}}_{r,m} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to J\big(H_v^{2r-m-1}(X,\mathbb{Q}(r))\big).$$

Such a regulator map already plays a key role in detecting interesting $\mathrm{CH}^r(X,m)$ classes, such as indecomposables (see for example [23] and [25]).

By taking a further quotient of the Jacobian, we pass to the transcendental regulator

$$\Phi_{r,m} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \frac{\{F^{d-r+m+1}H_v^{2d-2r+m+1}(X,\mathbb{C})\}^\vee}{H^v_{2d-2r+m+1}(X,\mathbb{Q}(d-r))}. \tag{1.2}$$

Since the $\mathbb{Q}$-dimension of the denominator usually exceeds twice the $\mathbb{C}$-dimension of the numerator, $\Phi_{r,m}$ is primarily of use in families, where suitable Picard–Fuchs operators kill sections of the denominator.

To give a formula for (1.2), we shall associate to $\xi \in \mathrm{CH}^r(X,m;\mathbb{Q})$ a functional

$$\widetilde{\Phi}_{r,m}(\xi) \in \mathrm{Hom}\big(F^{d-r+m+1}H_v^{2d-2r+m+1}(X,\mathbb{C}),\mathbb{C}\big),$$

whose image in the right-hand side of (1.2) yields $\Phi_{r,m}(\xi)$. Let

$$\omega \in \ker(d) \subseteq F^{d-r+m+1}A^{2d-2r+m+1}(X)$$

be a representative form. According to the moving lemma of [21, 22], we may assume the irreducible components of ξ meet the real cube $X\times[-\infty,0]^m$ properly. Viewing $|\xi| \subset X\times\mathbb{C}^m$ as a closed subset of codimension r, we have $\dim \mathrm{Pr}_X(|\xi|) \le d-r+m$. Together with the depth of ω in the Hodge filtration, this eliminates most of the terms in the [22] formula for $\mathrm{cl}_{r,m}(\xi)(\omega)$, leaving us with

$$\widetilde{\Phi}_{r,m}(\xi)(\omega) := \pm\frac{1}{(2\pi\mathrm{i})^{d-r}}\int_\Gamma \omega,$$

for Γ a choice of C^∞ $(2d-2r+m+1)$-chain on X with $\partial\Gamma = \mathrm{Pr}_X(\xi\cap\{X\times[-\infty,0]^m\})$. The ambiguity in this choice lies in the denominator of the right-hand side of (1.2).

We observe that if $m=1$, then (1.2) factors through the indecomposable classes. A class ξ for which $\Phi_{r,1}(\xi)\neq 0$ is thus called regulator-indecomposable. Of particular interest is the case $(d,r,m)=(2,2,1)$, where (1.2) and (1.1) take the form

$$\Phi_{2,1} : \mathrm{CH}^2(X,1) \to \frac{H_v^{2,0}(X)^\vee}{H_2^v(X,\mathbb{Q})}, \quad r_{2,1} : \mathrm{CH}^2(X,1) \to H^{1,1}(X,\mathbb{R}(1)).$$

Existence of regulator indecomposables. To explain the results of Sections 2–4, we first need to recall the main theorem of [5]. Recall that a point $p \in \mathcal{S}(\mathbb{C})$ is general (resp. very general) if it lies in the complement of a finite (resp. countable) union of algebraic subvarieties. When making statements about the real regulator, more analytic notions are required. By a real-analytic Zariski-open subset U of $\mathcal{S}$, we shall mean the complement of a real-analytic subvariety in $\mathcal{S}^{\mathrm{an}}_{\mathbb{C}}$ (viewed as a real analytic variety). Given some set of algebraic subvarieties of $\mathcal{S}$, an $\mathbb{R}^{\mathrm{an}}$-general member of this set is one which meets such a U.

Theorem 1.1 ([5]). *Let $\lambda : \mathcal{X} \to \mathcal{S}$ be the universal family of polarized $K3$ surfaces of genus g (≥ 2), and write $X_s := \lambda^{-1}(s)$. Then the real regulator*

$$r_{2,1;s} : \mathrm{CH}^2(X_s, 1) \otimes \mathbb{R} \to H^{1,1}(X_s, \mathbb{R}(1))$$

is surjective for s in a real-analytic Zariski-open subset $U \subset S$.

Pick $\ell \in [1, 19] \cap \mathbb{Z}$. Let $\mathcal{X}_{\mathcal{T}} \to \mathcal{T}$, with $\mathcal{T} \subset \mathcal{S}$ irreducible and $\dim(\mathcal{T}) = 20 - \ell$, be an $\mathbb{R}^{\mathrm{an}}$-general subfamily of generic Picard rank ℓ, so that $V := \mathcal{T}^{\mathrm{an}}_{\mathbb{C}} \cap U$ is non-empty. Our first result about these *maximal* families is the following.

Theorem 1.2. *Let $t \in V$ be very general in $\mathcal{T}$. Then the transcendental regulator $\Phi_{2,1;t}$ is non-zero.*

From the proof of Theorem 1.2, one may infer:

Corollary 1.3. *Let $X/\mathbb{C}$ be a very general member of a family of surfaces for which $H^1_{\mathrm{alg}}(X, \Theta_X) \otimes H^{2,0}_v(X) \to H^{1,1}_v(X)$ is surjective. If the real regulator*

$$r_{2,1} : \mathrm{CH}^2(X, 1) \to H^{1,1}_v(X, \mathbb{R}(1))$$

is non-zero, then so is the transcendental regulator $\Phi_{2,1}$.

Now consider X of dimension d as a very general member of a family $\lambda : \mathcal{X} \to \mathcal{S}$. With a little bit of effort, one can also show the following.

Theorem 1.4. *Suppose that the map*

$$H^1_{\mathrm{alg}}(X, \Theta_X) \otimes H_v^{d-r+m+1-\ell, d-r+\ell}(X) \to H_v^{d-r+m-\ell, d-r+\ell+1}(X),$$

induced by cup-product, is surjective for all $\ell = 0, \ldots, m-1$. Then $\underline{\mathrm{cl}}_{r,m} \neq 0 \Rightarrow \Phi_{r,m} \neq 0$.

Theorems 1.2, 1.4 and Corollary 1.3 will be proved in Section 2.2. We deduce from Theorem 1.4 the following corollary.

Corollary 1.5. *Let X be a very general projective $K3$ surface, so that $H^2_v(X, \mathbb{C})$ is transcendental cohomology. Then the transcendental regulator*

$$\Phi_{3,1} : \mathrm{CH}^3(X \times X, 1) \to \frac{\{F^3(H^2_v(X, \mathbb{C}) \otimes H^2_v(X, \mathbb{C}))\}^{\vee}}{H_4(X \times X, \mathbb{Q}(1))}$$

is non-zero.

We prove Corollary 1.5 in Section 4. It turns out however, that with more effort, we can actually prove the following stronger result.

Theorem 1.6. *The truncated transcendental regulator*

$$\Psi_{3,1} : \mathrm{CH}^3(X \times X, 1) \to \frac{H^{4,0}(X \times X, \mathbb{C})^\vee}{H_4(X \times X, \mathbb{Q}(1))}$$

is non-zero for a very general $K3$ surface X.

The proofs of all the above results rely on a very simple trick involving the infinitesimal invariant of a normal function associated to a family of cycles on $\mathcal{X}/\mathcal{S}$ inducing a given transcendental regulator value on X. A deeper question asks whether such a normal function is detected by a Picard–Fuchs operator. A blanket answer to this question is "yes"; but rather than explaining it here, we shall provide a complete clarification in Section 3.

Construction of regulator indecomposables. Returning to Theorem 1.2, two questions arise. First, the method of [5], which proves the existence of deformations of decomposables on Picard-rank 20 $K3$'s, to indecomposables on an $\mathbb{R}^{\mathrm{an}}$-general polarized $K3$, is highly non-explicit. How can one construct interesting *explicit* examples of cycles with non-zero $\Phi_{2,1}$ on subfamilies with Picard rank $\ell > 1$? Second, on a Picard-rank 20 $K3$, does one *expect* there to be any cycles at all which have non-zero $\Phi_{2,1}$, and which are therefore indecomposable?

The first question is our main concern for the remainder of the paper. In Section 5, we introduce the tools required for explicit computations in this setting. The notion of a polarized $K3$ surface is extended to that of a lattice polarization, and algebraic hypersurface normal forms are given for certain families of lattice polarized $K3$ surfaces of high Picard rank ℓ. We then describe a very useful "internal structure" consisting of an elliptic fibration with section(s). Explicit Picard–Fuchs operators are given and related to parametrizations of coarse moduli spaces by modular functions and their generalizations.

Starting in Section 6, we restrict our considerations to $\ell = 18$ or 19, where the literature has not been especially reliable. The article [12] considers a family of cycles $\mathcal{Z}_t \in \mathrm{CH}^2(X_t, 1)$ on a 1-parameter family of elliptically fibered $K3$'s with $\ell = 19$, and a choice of section $\{\omega_t\}$ of the relative canonical bundle. In this context, $F(t) := \Phi_{2,1}(\mathcal{Z}_t)(\omega_t)$ is a multivalued holomorphic function, and the indecomposability of $\mathcal{Z}_t$ may be detected by showing the Picard–Fuchs operator for ω does not annihilate it. Unfortunately, $\mathcal{Z}_t$ turns out to be 2-torsion,[1)] and the computation of F leaves out a part of the membrane integral which cancels the part written down. For $\ell = 18$, one can try to construct regulator-indecomposable cycles on a product $E_1 \times E_2$ of elliptic curves and then pass to the Kummer. Such a construction is attempted in [17] but this cycle, too, was shown by M. Saito to be decomposable. (That construction can, however, be corrected [27].) When $E_1 \cong E_2$, other authors (cf. [29]) have investigated "triangle cycles" supported on $E \times \{p\}$, $\{q\} \times E$, and the diagonal Δ_E, where $[p] - [q]$ is N-torsion. But this cannot produce indecomposable cycles, since the sum of the N^2 natural N-torsion translates of such a cycle (by integer multiples of $p - q$ on the two factors) is both visibly decomposable and (up to torsion) equivalent to N^2 times the original cycle.

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1) The cycle $\mathcal{Z}_t$, which is supported over $\{Z = 0\} \cup \{Z = 1\} \cup \{X = 0\} \cup \{X = \infty\}$ in the notation of [12], is in fact one-half the residue of the symbol $\{X, 1 - \frac{1}{Z}\}$.

With this discouraging history, it is easy to imagine that when X is an elliptically fibered $K3$, the very natural $\mathrm{CH}^2(X,1)$ classes supported on semistable singular (Kodaira type I_n) fibers might be decomposable as well. Indeed one knows in the case of a modular elliptic fibration ($K3$ or not), that Beilinson's Eisenstein symbols [2] kill all such classes. On the other hand, using arithmetic methods to bound the rank of the dlog image, Asakura [1] demonstrated that for elliptic surfaces with general fiber $y^2 = x^3 + x^2 + t^n$ ($n \in [7,29]$ prime), the type I_1 fibers generate $n-1$ independent indecomposable K_1-classes. His paper stops short of attempting any regulator computations for such cycles, and this is what we take up in Section 6 in the context where the surface and cycle are allowed to vary.

Specifically, using an I_1 fiber in an internal elliptic fibration of the 2-parameter family $\{\mathrm{X}_{a,b}\}$ of Shioda–Inose $K3$'s ($\ell = 18$) [7], we write down a (multivalued) family of cycles $\mathcal{Z}_{a,b} \in \mathrm{CH}^2(\mathrm{X}_{a,b},1)$. Passing to the associated Kummer family with parameters α, β (and cycle $\mathrm{Z}_{\alpha,\beta}$), we find that the family of cycles becomes single-valued over the diagonal ($\ell = 19$) sublocus $\alpha = \beta$, which is the Legendre modular curve $\mathbb{P}^1 \backslash \{0,1,\infty\} \cong \mathfrak{H}/\Gamma(2)$. At this point we write down a smooth family of real closed $(1,1)$-forms η_α and compute directly the function

$$\begin{aligned}\psi(\alpha) &:= r_{2,1}(\mathrm{Z}_{\alpha,\alpha})(\eta_\alpha)\\ &= -8|\alpha+1| \operatorname{Im} \int_{\mathbb{C}} z \cdot \log\left|\frac{z+i}{z-i}\right| \frac{\{(\alpha^2-\alpha-1)z^4 + 2z^2 + (\alpha^3-\alpha^2-2\alpha+1)\}}{|z^2-\alpha|\,|1-\alpha z^2|\,|z^2+1|\,|z^2-(1+\alpha-\alpha^2)|}\\ &\qquad \times \overline{\frac{\{(\alpha^3-\alpha^2-2\alpha+1)z^4 + 2z^2 + (\alpha^2-\alpha-1)\}}{|(1+\alpha-\alpha^2)z^2-1|\,|z^4+(\alpha^3-3\alpha)z^2+1|}}\, dx \wedge dy\end{aligned}$$

to be non-zero. By Corollary 1.3 we have immediately the following result.

Theorem 1.7. *$\Phi_{2,1}(\mathcal{Z}_{a,b})$ is non-zero for (a,b) in a real-analytic Zariski-open subset of $\mathbb{C}^2$, and so $\mathcal{Z}_{a,b}$ is indecomposable for very general (a,b).*

In light of the past confusion surrounding such constructions, such a natural source of indecomposable cycles seems to us an important development. While the explicit formula above may not look promising, $\psi(\alpha)$ is in fact a very interesting function. Dividing out by the volume of the Legendre elliptic curve and pulling back by the classical modular function λ to obtain a function $\tilde{\psi}(\tau)$ on $\mathfrak{H}$, yields a "Maass cusp form with two poles". That is, $\tilde{\psi}$ is $\Gamma(2)$-invariant, is smooth away from the λ-preimage of $\alpha = \{-1,2\}$ (where it has $\log|\cdot|$ singularities), dies at the three cusps, and (away from these bad points) is an eigenfunction of the hyperbolic Laplacian $-y^2\Delta$. This is shown by the third author in the follow-up paper [20] (which, it should be noted, relies crucially on the computation here).

Finally, we turn briefly to the second question, concerning the case $\ell = 20$, in Section 7. Due to the vanishing of $H_v^{1,1}(X,\mathbb{R})$, $r_{2,1}$ is zero by definition, but this is no reason for the transcendental Abel–Jacobi map $\Phi_{2,1}$ to vanish. In the example we work out, whether or not $\Phi_{2,1}(\mathcal{Z})$ is non-torsion boils down to the irrationality of a single number (cf. (7.1)), which we do not know how to prove directly. It seems likely both that the cycle is indecomposable and that this may be shown by using the methods in [1] to compute the dlog image.

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2. Derivatives of normal functions I

2.1. Gauss–Manin derivatives. Consider a smooth projective family $\pi : \mathcal{X} \to S$ of varieties, polarized by a relatively ample line bundle L, over a polydisk S, with central fiber $X = \pi^{-1}(\underline{0})$. Write

$$F^p_{\mathcal{O}_S} R^q\pi_*\mathbb{C} := \mathbb{R}^q\pi_*\Omega^{\bullet\geq p}_{X/S} \subseteq \mathbb{R}^q\pi_*\Omega^{\bullet}_{X/S} = \mathcal{O}_S \otimes R^q\pi_*\mathbb{C}$$

for the Hodge filtration. We have the Gauss–Manin (GM) connection

$$\nabla : \mathcal{O}_S \otimes R^q\pi_*\mathbb{C} \to \Omega^1_S \otimes R^q\pi_*\mathbb{C}$$

which is a flat connection satisfying Griffiths transversality:

$$\nabla\big(F^p_{\mathcal{O}_S} R^q\pi_*\mathbb{C}\big) \subseteq \Omega^1_S \otimes F^{p-1}_{\mathcal{O}_S} R^q\pi_*\mathbb{C}.$$

Let Θ_S be the holomorphic tangent bundle of S. We can think of Θ_S as the sheaf of holomorphic linear differential operators. By identifying $\partial/\partial z_k$ with $\nabla_{\partial/\partial z_k}$, Θ_S acts on $\mathcal{O}_S \otimes R^q\pi_*\mathbb{C}$ via

$$u \cdot \omega = \nabla_u \omega$$

for $u \in H^0(\Theta_S)$, where we write $H^0(-)$ for $H^0(S, -)$. If $\langle\cdot,\cdot\rangle$ denotes the polarizing form on $\mathcal{O}_S \otimes R^q\pi_*\mathbb{C}$, then $u\langle\cdot,\cdot\rangle = \langle\nabla_u(\cdot),\cdot\rangle + (-1)^q\langle\cdot,\nabla_u(\cdot)\rangle$.

Now assume the fibers of π are (polarized) $K3$ surfaces. We fix a non-zero section $\omega \in H^0(K_{\mathcal{X}/S})$, where $K_{\mathcal{X}/S}$ is the relative canonical sheaf of $\mathcal{X}$ over S. For all $u \in H^0(\Theta_S)$ and all $\gamma \in H^2(X, \mathbb{C})$ (where $H^2(X, \mathbb{C})$ is identified with $H^0(S, R^2\pi_*\mathbb{C})$), we have

$$u\langle\gamma, \omega\rangle = \langle\gamma, \nabla_u\omega\rangle.$$

Let $\xi \in \mathrm{CH}^2(\mathcal{X}/S, 1)$ be the result of an algebraic deformation of a cycle in the central fiber X restricted to $\mathcal{X}/S$, and $\mathrm{cl}_{2,1}$ be the regulator map

$$\mathrm{cl}_{2,1} : \mathrm{CH}^2(\mathcal{X}/S, 1) \to H^0\left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{F^2_{\mathcal{O}_S} R^2\pi_*\mathbb{C} + R^2\pi_*\mathbb{Q}(2)}\right).$$

Take ν to be a lift of $\mathrm{cl}_{2,1}(\xi)$ to $H^0(\mathcal{O}_S \otimes R^2\pi_*\mathbb{C})$. Then

$$\langle\nabla_u\nu, \omega\rangle = 0 \tag{2.1}$$

since the map

$$\begin{aligned}\nabla \circ \mathrm{cl}_{2,1} : \mathrm{CH}^2(\mathcal{X}/S, 1) &\xrightarrow{\mathrm{cl}_{2,1}} H^0\left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{F^2_{\mathcal{O}_S} R^2\pi_*\mathbb{C} + R^2\pi_*\mathbb{Q}(2)}\right)\\ &\xrightarrow{\nabla} H^0\left(\Omega^1_S \otimes \left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{F^1_{\mathcal{O}_S} R^2\pi_*\mathbb{C}}\right)\right)\end{aligned} \tag{2.2}$$

induced by the GM connection is trivial. This follows from the quasi-horizontality of (higher) normal functions associated to generalized algebraic cycles.

Remark 2.1. For the non-expert reader, here is an efficient proof of this quasi-horizontality. Let $\mathcal{X}/S$ be a smooth projective family, and recall the analytic Deligne complex $0 \to \mathbb{Z}(r) \to \Omega_{\mathcal{X}}^{\bullet<r}$, which leads to an exact sequence

$$\mathbb{H}^{2r-m-1}(\Omega_{\mathcal{X}}^{\bullet<r}) \to H_{\mathcal{D}}^{2r-m}(\mathcal{X}, \mathbb{Z}(r)) \to H^{2r-m}(\mathcal{X}, \mathbb{Z}(r)).$$

We consider a relatively null-homologous cycle in $\mathrm{CH}^r(\mathcal{X}/S, m)$, which will map to zero in $H^{2r-m}(\mathcal{X}, \mathbb{Z}(r))$ (as S is a polydisk). Hence the induced normal function has a lift in $\mathbb{H}^{2r-m-1}(\Omega_{\mathcal{X}}^{\bullet<r})$, which is all we shall need.

The Leray spectral sequence for $\mathcal{X}/S$ gives us an edge map

$$\mathbb{H}^{2r-m-1}(\Omega_{\mathcal{X}}^{\bullet<r}) \to H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_{\mathcal{X}}^{\bullet<r}).$$

One has a filtering of the complex

$$\mathcal{L}^\nu \Omega_{\mathcal{X}}^{\bullet<r} := \mathrm{Image}\big(\pi^*\Omega_S^\nu \otimes \Omega_{\mathcal{X}}^{\bullet<r-\nu} \to \Omega_{\mathcal{X}}^{\bullet<r}\big),$$

with

$$Gr_{\mathcal{L}}^\nu = \pi^*\Omega_S^\nu \otimes \Omega_{\mathcal{X}/S}^{\bullet<r-\nu} \simeq \Omega_S^\nu \otimes \Omega_{\mathcal{X}/S}^{\bullet<r-\nu}.$$

There is a spectral sequence computing $\mathbb{R}^{p+q}\pi_*\Omega_{\mathcal{X}}^{\bullet<r}$ with

$$\mathcal{E}_1^{p,q} = \mathbb{R}^{p+q} Gr_{\mathcal{L}}^p = \Omega_S^p \otimes \mathbb{R}^q \rho_* \Omega_{\mathcal{X}/S}^{\bullet<r-p}.$$

So we have the composite

$$H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_{\mathcal{X}}^{\bullet<r}) \to H^0(S, \mathcal{E}_1^{0,2r-m-1}) \xrightarrow{d_1} H^0(S, \mathcal{E}_1^{1,2r-m-1}),$$

which must be zero by spectral sequence degeneration, using the fact that

$$\mathcal{E}_\infty^{0,2r-m-1} \subset \ker\big(d_1 : \mathcal{E}_1^{0,2r-m-1} \to \mathcal{E}_1^{1,2r-m-1}\big).$$

But $H^0(S, \mathcal{E}_1^{0,2r-m-1}) \xrightarrow{d_1} H^0(S, \mathcal{E}_1^{1,2r-m-1})$ is precisely the Gauss–Manin connection

$$H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_{\mathcal{X}/S}^{\bullet<r}) \xrightarrow{\nabla} H^0(S, \Omega_S^1 \otimes \mathbb{R}^{2r-m-1}\pi_*\Omega_{\mathcal{X}/S}^{\bullet<r-1}).$$

Specializing $(r, m) = (2, 1)$ now gives the vanishing asserted in (2.2).

2.2. Transcendental regulators. We continue with the notation of the last subsection, with X the central fiber of a smooth non-trivial family of algebraic $K3$ surfaces $\mathcal{X}$ over a polydisk S. Suppose that $\Phi_{2,1}(\xi)(\omega) \equiv 0$ over S, so that $\langle \nu, \omega \rangle$ is a period; that is, $\langle \nu, \omega \rangle = \langle \gamma, \omega \rangle$ for some $\gamma \in H^2(X, \mathbb{Q}(2))$. Applying (2.1),

$$\langle \gamma, \nabla_u \omega \rangle = u\langle \gamma, \omega \rangle = u\langle \nu, \omega \rangle = \langle \nabla_u \nu, \omega \rangle + \langle \nu, \nabla_u \omega \rangle = \langle \nu, \nabla_u \omega \rangle. \tag{2.3}$$

Now write

$$\kappa : \Theta_{S,0} \to H^1(X, \Theta_X)$$

for the Kodaira–Spencer map, and

$$\varepsilon : H^1(X, \Theta_X) \otimes H^{2,0}(X) \to H^{1,1}(X) \tag{2.4}$$

for the map induced by the contraction $\Theta_X \otimes \wedge^2 \Omega_X^1 \to \Omega_X^1$. Then

$$\nabla_u \omega = \varepsilon\big(\kappa(u) \cup \omega\big),$$

and we have the following elementary proposition.

Proposition 2.2. *ε is an isomorphism.*

Proof. By Serre duality, this follows from non-degeneracy of the map

$$H^1(X,\Theta_X)\otimes H^1(X,\Omega_X^1)\to H^2(X,\mathcal{O}_X) \tag{2.5}$$

induced by contraction $\Theta_X\otimes\Omega_X^1\to\mathcal{O}_X$. But since $\mathcal{O}_X=K_X$ (and $\Omega_X^1=(\Theta_X^1)^\vee\otimes K_X$) for a $K3$, (2.5) *is* the Serre pairing and hence non-degenerate. □

Note that $H^1(X,\Theta_X)$ corresponds to all deformations of X, including non-algebraic ones.

Proof of Theorem 1.2. Take S to be an open polydisk in V with center $\underline{0}$ at $t\in V$, and put $H^1_{\mathrm{alg}}(X,\Theta_X):=\kappa(\Theta_{S,0})$. Suppose that $\Phi_{2,1}(\xi)=0$ for all $\xi\in\mathrm{CH}^2(\mathcal{X}/S,1)$. Denote by

$$\varepsilon_S: H^1_{\mathrm{alg}}(X,\Theta_X)\otimes H^{2,0}(X)\to H^{1,1}_v(X)$$

the restriction of (2.4). In the setting of the theorem, $\dim H^{1,1}_v(X)=20-\ell$ and κ is injective. Thus by Proposition 2.2, ε_S is surjective, and the $\{\nabla_u\omega\}_{u\in H^0(\Theta_S)}$ together with ω generate $F^1H^2_v(X,\mathbb{C})$. Applying (2.3), we see that $\underline{\mathrm{cl}}_{2,1}(\xi)=0$ for any $\xi\in\mathrm{CH}^2(\mathcal{X}/S,1)$. But by Theorem 1.1, this is impossible since the composition of $r_{2,1}$ with the projection to $H^{1,1}_v(X,\mathbb{R}(1))$ is non-zero and factors through $\underline{\mathrm{cl}}_{2,1}$. □

This argument carries over essentially verbatim to the more general setting of Corollary 1.3.

Proof of Theorem 1.4. Let us assume that $\Phi_{r,m}(\xi)$ is zero. That means that $\mathrm{cl}_{r,m}(\xi)$ is a period with respect to (acting on forms in) $F^{d-r+m+1}H_v^{2d-2r+m+1}(X,\mathbb{C})$. Then from the surjection of

$$H^1_{\mathrm{alg}}(X,\Theta_X)\otimes H_v^{d-r+m+1-\ell,d-r+\ell}(X)\to H_v^{d-r+m-\ell,d-r+\ell+1}(X),$$

in the case where $\ell=0$, we deduce likewise that $\mathrm{cl}_{r,m}(\xi)$ is a period with respect to $F^{d-r+m}H_v^{2d-2r+m+1}(X,\mathbb{C})$. By iterating the same argument for $\ell=1,\dots,m-1$, we deduce that $\mathrm{cl}_{r,m}(\xi)$ is a period with respect to $F^{d-r+1}H_v^{2d-2r+m+1}(X,\mathbb{C})$, which implies that $\underline{\mathrm{cl}}_{r,m}(\xi)=0$. □

3. Derivatives of normal functions II

Consider the setting in Section 1, where $\lambda:\mathcal{X}\to\mathcal{S}$ is a smooth and proper map of smooth quasi-projective varieties, and where X is a very general member. In this section, we will further assume that $\mathcal{S}$ is affine. Associated to the Gauss–Manin connection ∇ and the algebraic vector fields $H^0(\mathcal{S},\Theta_{\mathcal{S}})$ is a D-module of differential operators. Suppose $\omega\in H^0(\mathcal{S},\mathcal{O}_{\mathcal{S}}\otimes R^i\lambda_*\mathbb{C})=H^0(\mathcal{S},\mathbb{R}^i\lambda_*\Omega^\bullet_{\mathcal{X}/\mathcal{S}})$ is an algebraic form, where we note that $\mathbb{R}^i\lambda_*\Omega^\bullet_{\mathcal{X}/\mathcal{S}}$ is algebraic and locally free in the Zariski topology. One can consider the ideal

$I_\omega^{\mathbb{C}(\mathcal{S})}$ of partial differential operators with coefficients in $\mathbb{C}(\mathcal{S})$ annihilating ω, which will always be non-zero using the finite dimensionality of cohomology of the fibers of λ and the fact that ∇ is algebraic, where $\mathbb{C}(\mathcal{S})$ is the field of rational functions on $\mathcal{S}$. This section addresses the following question.

Question 3.1. If the transcendental regulator $\Phi_{r,m}(\xi)$ is non-trivial, is the normal function $\bar{\nu}(t) = \mathrm{cl}_{r,m}(\xi_t)$ associated to ξ detectable by a Picard–Fuchs operator $P \in I_\omega^{\mathbb{C}(\mathcal{S})}$, for some $\omega \in F^{d-r+m+1}H_v^{2d-2r+m+1}(X, \mathbb{C})$; namely, is $P\langle \nu, \omega\rangle \neq 0$?

Proposition 3.2. *The answer to Question* 3.1 *is affirmative under the hypotheses of Theorem* 1.4*, together with the following mild assumption.*

Assumption 3.3. *For the given choice of r and m,*

$$\{R_v^{2r-m-1}\lambda_*\mathbb{C}\} \cap \{F^r\mathcal{H}_v^{2r-m-1}\} = 0$$

as subsheaves of $\mathcal{H}_v^{2r-m-1} := \mathcal{O}_{\mathcal{S}} \otimes R_v^{2r-m-1}\lambda_*\mathbb{C}$. Equivalently,

$$\nabla : F^r\mathcal{H}_v^{2r-m-1} \to \Omega^1_{\mathcal{S}} \otimes \mathcal{H}_v^{2r-m-1}$$

is injective.

This assumption holds automatically for the families of $K3$ surfaces in Theorem 1.2 (with $(r,m) = (2,1)$), as well as for the fiber products of such in Corollary 1.5 and Theorem 1.6 (with $(r,m) = (3,1)$).

Proposition 3.2 will be proved at the end of this section.

3.1. Picard–Fuchs equations associated to regulators. This section takes inspiration from [18]. Since ∇ is algebraic, Question 3.1 reduces to a local calculation over a polydisk $S \subset \mathcal{S}_{\mathbb{C}}^{\mathrm{an}}$ in the analytic topology, cf. Proposition 3.4 below.

Recall that Θ_S is the holomorphic tangent bundle of S. We can think of Θ_S as the sheaf of holomorphic linear differential operators which generates the ring $\mathcal{D}_S$ of differential operators: in local coordinates,

$$\mathcal{D}_S = \mathcal{O}_S\left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right].$$

Identifying $\partial/\partial z_k$ with $\nabla_{\partial/\partial z_k}$, $\mathcal{D}_S$ acts on $\mathcal{O}_S \otimes R^q\pi_*\mathbb{C}$ via

$$(v_1v_2\ldots v_l)\omega = \nabla_{v_1}\nabla_{v_2}\ldots\nabla_{v_l}\omega \tag{3.1}$$

for sections $v_1, v_2, \ldots, v_l \in H^0(\Theta_S)$, where we write $H^0(-)$ for $H^0(S,-)$. For $\omega \in H^0(\mathcal{O}_S \otimes R^q\pi_*\mathbb{C})$, we let $I_\omega \subset H^0(\mathcal{D}_S)$ be the *Picard–Fuchs ideal* annihilating ω, i.e., the left ideal consisting of differential operators $P \in H^0(\mathcal{D}_S)$ satisfying $P\omega = 0$.

Proposition 3.4. *The analytic ideal I_ω, viewed as a* $\mathrm{Mer}(S)$ *vector space, is generated by the restriction of the corresponding algebraic ideal $I_\omega^{\mathbb{C}(\mathcal{S})}$ to the polydisk S, where* $\mathrm{Mer}(S)$ *is the field of the meromorphic functions on S and we extend the scalars in I_ω from $H^0(S, \mathcal{O}_S^{\mathrm{an}})$ to* $\mathrm{Mer}(S)$ *by replacing I_ω by $I_\omega \otimes_{H^0(S,\mathcal{O}_S^{\mathrm{an}})} \mathrm{Mer}(S)$.*

Proof. Let $\mathcal{S}$ be affine, and let $\omega \in H^0(\mathcal{S}, \mathbb{R}^i\lambda_*\Omega^\bullet_{\mathcal{X}/\mathcal{S}})$ be an algebraic form. By shrinking $\mathcal{S}$, we may assume that the algebraic vector bundles $\mathbb{R}^i\lambda_*\Omega^\bullet_{\mathcal{X}/\mathcal{S}}$ and $\Theta_{\mathcal{S}}$ are trivial, e.g., $\mathbb{R}^i\lambda_*\Omega^\bullet_{\mathcal{X}/\mathcal{S}} = \mathcal{O}^N_{\mathcal{S}}$. Given $v_1, v_2, \ldots, v_l \in H^0(\Theta_{\mathcal{S}})$, in the notation of (3.1) we have

$$(v_1 v_2 \ldots v_l)\omega \in H^0(\mathcal{S}, \mathcal{O}^N_{\mathcal{S}}) = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}})^N \subset \mathbb{C}(\mathcal{S})^N,$$

since ∇ is algebraic.

Now we shall pass to the generic point of $\mathcal{S}$. Given $V_1, \ldots, V_M \in \mathbb{C}(\mathcal{S})^N$ of the form $\{(v_1 v_2 \ldots v_l)\omega\}$, let Λ be the $N \times M$ matrix with jth column V_j. Evidently, solutions $A \in \mathbb{C}(\mathcal{S})^M$ to $\Lambda A = 0$ define elements of $I_\omega^{\mathbb{C}(\mathcal{S})}$, and all elements are obtained in this way for some M and $\{V_i\}$. Moreover, all elements of I_ω are obtained from solutions $\mathcal{A} \in \mathrm{Mer}(S)^M$ to $(\Lambda|_S)\mathcal{A} = 0$, and (by Gaussian elimination) the vector space of these solutions is defined over $\mathbb{C}(\mathcal{S}) \subset \mathrm{Mer}(S)$. The proposition follows. □

As in Section 2 let us again for simplicity restrict to the situation of a family of $K3$ surfaces over a polydisk S. We fix a non-zero section $\omega \in H^0(K_{\mathcal{X}/S})$. For any $u \in H^0(\Theta_S)$ and all Picard–Fuchs operators $P \in I_{\nabla_u \omega}$, it is obvious that $(Pu)\omega = 0$ and hence

$$(Pu)\langle \gamma, \omega \rangle = 0 \tag{3.2}$$

for all $\gamma \in H^2(X, \mathbb{C}) = H^0(S, R^2\pi_*\mathbb{C})$.

Let ξ and ν be as in Section 2.1. Since $Pu \in I_\omega$ "kills" all the periods $\langle \gamma, \omega \rangle$ for $\gamma \in H^2(X, \mathbb{Q}(2))$, $(Pu)\langle \nu, \omega \rangle$ is independent of the choice of lifting ν of $\mathrm{cl}_{2,1}(\xi)$. By (2.1), we have

$$Pu\langle \nu, \omega \rangle = P\big(\langle \nabla_u \nu, \omega \rangle + \langle \nu, \nabla_u \omega \rangle\big) = P\langle \nu, \nabla_u \omega \rangle,$$

and thus part (i) of the following proposition.

Proposition 3.5. *Let $\mathcal{X}/S$, ν, ω and u be given as above.*

(i) *For all $P \in I_{\nabla_u \omega}$, we have*

$$P\big(u\langle \nu, \omega \rangle - \langle \nu, \nabla_u \omega \rangle\big) = 0. \tag{3.3}$$

(ii) *There exists a $\gamma \in H^2(X, \mathbb{C}) = H^0(S, R^2\pi_*\mathbb{C})$ such that*

$$u\langle \nu, \omega \rangle - \langle \nu, \nabla_u \omega \rangle = \langle \gamma, \nabla_u \omega \rangle.$$

For part (ii), we need to check that the solutions of $Py = 0$ for $P \in I_{\nabla_u \omega}$ are generated by $\langle \gamma, \nabla_u \omega \rangle$ for all $\nabla \gamma = 0$. This is an elementary consequence of the following observation.

Lemma 3.6. *Let E be a flat holomorphic vector bundle over the polydisk S with flat connection ∇, and let I_η be the Picard–Fuchs ideal associated to an $\eta \in H^0(E)$. Then the solution set (in $\mathcal{O}(\mathcal{S})$) of the system of differential equations $\{P(\cdot) = 0 \mid P \in I_\eta\}$ is generated, as a vector space over $\mathbb{C}$, by $\{\langle \gamma, \eta \rangle \mid \gamma \in H^0(\mathcal{O}_S(E^\vee)^\nabla)\}$.*

A short proof in the case where $\mathcal{D}_S \eta = \mathcal{O}(E)$ is given in Remark 3.7 below.

Proof. We need to establish the following algebraic result: For $f_1, f_2, \ldots, f_m, f \in \mathcal{O}_S$, $I_{f_1} \cap I_{f_2} \cap \ldots \cap I_{f_m} \subset I_f$ if and only if $f = c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$ for some constants $c_1, c_2, \ldots, c_m \in \mathbb{C}$, where $I_g = \{P \in \mathcal{D}_S \mid Pg = 0\}$ for $g \in \mathcal{O}_S$.

The "if" part is trivial. Suppose that $I_{f_1} \cap I_{f_2} \cap \ldots \cap I_{f_m} \subset I_f$. Without loss of generality, we may assume that $f_1, f_2, \ldots, f_m$ are linearly independent over $\mathbb{C}$.

Let $n = \dim S$. For $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, we write

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{a_1 + a_2 + \cdots + a_n}}{\partial z_1^{a_1} \partial z_2^{a_2} \ldots \partial z_n^{a_n}}$$

and then every $P \in \mathcal{D}_S$ can be written as

$$P = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \frac{\partial^\alpha}{\partial z^\alpha}, \tag{3.4}$$

where $p_\alpha \in \mathcal{O}_S$ vanishes except for finitely many $\alpha \in \mathbb{N}^n$. Then we can identify

$$\mathcal{D}_S \simeq \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_S$$

as $\mathcal{O}_S$-modules by sending P in (3.4) to (p_α). For every $g \in \mathcal{O}_S$, we have a homomorphism $\varphi_g : \mathcal{D}_S \to \mathcal{O}_S$ of $\mathcal{O}_S$-modules given by

$$\varphi_g(P) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \frac{\partial^\alpha g}{\partial z^\alpha}$$

for P given in (3.4). Clearly, I_g is the kernel of φ_g, i.e., $I_g = \ker(\varphi_g)$. Therefore,

$$I_{f_1} \cap I_{f_2} \cap \ldots \cap I_{f_m} = \ker(\varphi_{f_1, f_2, \ldots, f_m}),$$

where $\varphi_{f_1, f_2, \ldots, f_m}$ is the map $\mathcal{D}_S \to \mathcal{O}_S^{\oplus m}$ given by

$$\varphi_{f_1, f_2, \ldots, f_m} = (\varphi_{f_1}, \varphi_{f_2}, \ldots, \varphi_{f_m}).$$

Let F be the vector space spanned by $f_1, f_2, \ldots, f_m$ over $\mathbb{C}$. We can choose $f_1, f_2, \ldots, f_m$ to be a basis of F such that there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{N}^n$ with the property that

$$\frac{\partial^{\alpha_i} f_j}{\partial z^{\alpha_i}}(0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for all $1 \leq i, j \leq m$. Then

$$\det \left[\frac{\partial^{\alpha_i} f_j}{\partial z^{\alpha_i}} \right]_{m \times m} \in \mathcal{O}_S^\times. \tag{3.5}$$

It follows that $\varphi_{f_1, f_2, \ldots, f_m}$ is surjective. Combining this with the hypothesis that

$$I_{f_1} \cap I_{f_2} \cap \ldots \cap I_{f_m} = \ker(\varphi_{f_1, f_2, \ldots, f_m}) \subset I_f = \ker(\varphi_f),$$

we see that the map $\varphi_f : \mathcal{D}_S \to \mathcal{O}_S$ factors through $\varphi_{f_1, f_2, \ldots, f_m}$. Namely, there exists $\nu : \mathcal{O}_S^{\oplus m} \to \mathcal{O}_S$ such that

$$\varphi_f = \nu \circ \varphi_{f_1, f_2, \ldots, f_m}. \tag{3.6}$$

Suppose that ν is given by

$$\nu(g_1, g_2, \ldots, g_m) = c_1 g_1 + c_2 g_2 + \cdots + c_m g_m$$

for some $c_1, c_2, \ldots, c_m \in \mathcal{O}_S$. It then follows from (3.6) that

$$\frac{\partial^\alpha f}{\partial z^\alpha} = c_1 \frac{\partial^\alpha f_1}{\partial z^\alpha} + c_2 \frac{\partial^\alpha f_2}{\partial z^\alpha} + \cdots + c_m \frac{\partial^\alpha f_m}{\partial z^\alpha} \tag{3.7}$$

for all $\alpha \in \mathbb{N}^n$. Differentiating both sides of (3.7), we obtain

$$\Big(\frac{\partial c_1}{\partial z_k}\Big)\Big(\frac{\partial^\alpha f_1}{\partial z^\alpha}\Big) + \Big(\frac{\partial c_2}{\partial z_k}\Big)\Big(\frac{\partial^\alpha f_2}{\partial z^\alpha}\Big) + \cdots + \Big(\frac{\partial c_m}{\partial z_k}\Big)\Big(\frac{\partial^\alpha f_m}{\partial z^\alpha}\Big) = 0 \tag{3.8}$$

for all $\alpha \in \mathbb{N}^n$ and $1 \le k \le n$.

Combining (3.5) and (3.8), we conclude that

$$\frac{\partial c_i}{\partial z_k} = 0$$

for all $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. That is, $c_1, c_2, \ldots, c_m$ are constants. Thus, $f = c_1 f_1 + c_2 f_2 + \cdots + c_m f_m \in F$ by (3.7). We are done. □

Remark 3.7. Suppose given a left $\mathcal{D}_S$-module of the form $\mathcal{E} = \mathcal{D}_S g \cong \mathcal{D}_S / I_g$, and another $\mathcal{D}_S$-module $\mathcal{F} \ni f$, with $I_g f = 0$. Then the morphism $\mathcal{D}_S \to \mathcal{F}$ of $\mathcal{D}_S$-modules sending $1_S \mapsto f$ clearly factors through $\mathcal{D}_S / I_g$, producing $h \in \mathrm{Hom}_{\mathcal{D}_S}(\mathcal{E}, \mathcal{F})$ with $h(g) = f$. Taking $\mathcal{F} := \mathcal{O}_S$, $\mathcal{E} := \mathcal{O}_S(E)$ and $g = \eta$, we have

$$\mathrm{Hom}_{\mathcal{D}_S}(\mathcal{E}, \mathcal{F}) = H^0(\mathcal{O}_S(E^\vee)^\nabla),$$

so that $h = \gamma$ (and $f = \langle \gamma, \eta \rangle$) as in the statement of Proposition 3.5.

However, this quick proof does not seem to easily extend to the more general setting (of Lemma 3.6) where η does not generate $\mathcal{O}(E)$ as a $\mathcal{D}_S$-module.

3.2. Non-triviality of Picard–Fuchs operators. Suppose that any Picard–Fuchs operator in I_ω annihilates $\mathrm{cl}_{2,1}(\xi)(\omega)$. According to Lemma 3.6, $\langle \nu, \omega \rangle = \langle \gamma, \omega \rangle$ for some $\gamma \in H^2(X, \mathbb{C})$. By (3.2) and (3.3), it follows that

$$P\langle \nu, \nabla_u \omega \rangle = 0 \tag{3.9}$$

for all $u \in H^0(\Theta_S)$ and $P \in I_{\nabla_u \omega}$ and (again applying Lemma 3.6)

$$\langle \nu, \nabla_u \omega \rangle = \langle \gamma, \nabla_u \omega \rangle \tag{3.10}$$

for some $\gamma \in H^2(X, \mathbb{C})$. This is just (2.3), and the same proof as in Section 2.2 now shows that $\underline{\mathrm{cl}}_{2,1}(\xi) = 0$ (hence $\Phi_{2,1}(\xi) = 0$), assuming $\mathcal{X} \to S$ is one of the families of Theorem 1.2. Briefly, from the surjection $H^1(X, \Theta_X) \otimes H^{2,0}(X) \to H^{1,1}_v(X)$ and (3.10) we have

$$\nu \in H^0(F^1 \mathcal{H}^2_v)^\perp + H^2(X, \mathbb{C}). \tag{3.11}$$

To arrive at a more general treatment, we consider infinitesimal and topological invariants of normal functions. Let $\lambda : \mathcal{X} \to \mathcal{S}$ be a smooth and proper morphism as in Section 1, with $\mathcal{S}$ affine. Choose (r, m) and impose Assumption 3.3. The short exact sequence

$$0 \to J\big(H_f^{2r-m-1}(X, \mathbb{Q}(r))\big) \to \mathrm{Ext}^1_{V\mathrm{MHS}}\big(\mathbb{Q}(0), R^{2r-m-1}\lambda_*\mathbb{Q}(r)\big)$$
$$\xrightarrow{\delta} \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^1(\mathcal{S}, R^{2r-m-1}\lambda_*\mathbb{Q}(r))\big) \to 0$$

induces injections

$$\mathrm{Ext}^1_{V\mathrm{MHS}}\big(\mathbb{Q}(0), R_v^{2r-m-1}\lambda_*\mathbb{Q}(r)\big) \hookrightarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^1(\mathcal{S}, R_v^{2r-m-1}\lambda_*\mathbb{Q}(r))\big),$$

and

$$\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^1(\mathcal{S}, R_v^{2r-m-1}\lambda_*\mathbb{Q}(r))\big) \hookrightarrow \nabla\Gamma J,$$

where

$$\nabla\Gamma J := \frac{\ker\big\{\nabla : H^0(\mathcal{S}, \Omega^1_{\mathcal{S}} \otimes F^{r-1}\mathcal{H}_v^{2r-m-1}) \to H^0(\mathcal{S}, \Omega^2_{\mathcal{S}} \otimes F^{r-2}\mathcal{H}_v^{2r-m-1})\big\}}{\nabla H^0(\mathcal{S}, F^r\mathcal{H}_v^{2r-m-1})}.$$

Now let $\xi \in \mathrm{CH}^r(\mathcal{X}/\mathcal{S}, m)$ be a relative higher Chow cycle. Denoting by

$$\bar{\nu} \in \mathrm{Ext}^1_{V\mathrm{MHS}}\big(\mathbb{Q}(0), R^{2r-m-1}\lambda_*\mathbb{Q}(r)\big)$$

the (higher) normal function associated to $\mathrm{cl}_{r,m}(\xi)$, $\delta\bar{\nu}$ gives the topological invariant of $\bar{\nu}$. Next, consider the sheaf

$$\nabla J := \frac{\ker\big\{\nabla : \Omega^1_{\mathcal{S}} \otimes F^{r-1}\mathcal{H}_v^{2r-m-1} \to \Omega^2_{\mathcal{S}} \otimes F^{r-2}\mathcal{H}_v^{2r-m-1}\big\}}{\nabla(F^r\mathcal{H}_v^{2r-m-1})},$$

with corresponding $\Gamma\nabla J := H^0(\mathcal{S}, \nabla J)$ and Griffiths infinitesimal invariant $\delta_G\bar{\nu} \in \Gamma\nabla J$. Moreover, the natural map $\nabla\Gamma J \to \Gamma\nabla J$ is an isomorphism. Indeed, by Assumption 3.3, this follows from the short exact sequence

$$0 \to F^r\mathcal{H}_v^{2r-m-1} \xrightarrow{\nabla} \big(\Omega^1_{\mathcal{S}} \otimes F^{r-1}\mathcal{H}_v^{2r-m-1}\big)^{\nabla} \to \nabla J \to 0.$$

So if we have $\delta_G\bar{\nu} = 0$, then $\delta\bar{\nu} = 0$ and $\bar{\nu}$ lies in $J(H_f^{2r-m-1}(X, \mathbb{Q}(r)))$, with trivial image in $\mathrm{Ext}^1_{\mathrm{VMHS}}(\mathbb{Q}(0), R_v^{2m-r-1}\lambda_*\mathbb{Q}(r))$, rendering $\underline{\mathrm{cl}}_{r,m}(\xi)$ (hence $\Phi_{r,m}(\xi)$) trivial.

Proof of Proposition 3.2. Impose the hypothesis of Theorem 1.4, and write

$$\nu \in H^0(\mathcal{O}_S \otimes R^{2r-m-1}\lambda_*\mathbb{Q}(r))$$

for a local lifting of $\mathrm{cl}_{r,m}(\xi)$ over a polydisk.

Suppose that we have $P\langle\nu, \omega\rangle = 0$ for all $\omega \in H^0(\mathcal{S}, F^{d-r+m+1}\mathcal{H}_v^{2d-2r+m+1})$ and $P \in I_\omega^{\mathbb{C}(\mathcal{S})}$. Then from the surjection of

$$H^1_{\mathrm{alg}}(X, \Theta_X) \otimes H_v^{d-r+m+1-\ell, d-r+\ell}(X) \to H_v^{d-r+m-\ell, d-r+\ell+1}(X),$$

in the case $\ell = 0$, we deduce exactly as in (3.9)–(3.11) that

$$\nu \in \big[H^0(F^{d-r+m}\mathcal{H}_v^{2d-2r+m+1})\big]^{\perp} + H^{2d-2r+m+1}(X, \mathbb{C}).$$

By iterating the same argument for $\ell = 1, \dots, m-1$, we deduce that

$$\nu \in \big[H^0(F^{d-r+1}\mathcal{H}_v^{2d-2r+m+1})\big]^{\perp} + H^{2d-2r+m+1}(X, \mathbb{C}),$$

which implies that the associated normal function has (everywhere locally) zero infinitesimal invariant, and so $\Phi_{r,m}(\xi) \equiv 0$. □

4. Proof of Theorem 1.6

In this section we restrict to the case where X is a projective $K3$ surface. We recall the real regulator

$$r_{3,1} : \mathrm{CH}^3(X \times X, 1) \to H^{2,2}(X \times X, \mathbb{R}(2)).$$

The image of $r_{3,1}$ thus contains

$$r_{3,1}(\mathrm{CH}^1(X) \otimes \mathrm{CH}^2(X,1)) \otimes \mathbb{R} = H^{1,1}(X, \mathbb{Q}(1)) \otimes H^{1,1}(X, \mathbb{R}(1))$$

for X general and it also contains the class $[\Delta_X]$ of the diagonal. So it is natural to look at the reduced real regulator

$$\underline{r}_{3,1} : \mathrm{CH}^3(X \times X, 1) \xrightarrow{r_{3,1}} H^{2,2}(X \times X, \mathbb{R}) \xrightarrow{\text{projection}} V_X,$$

where

$$\begin{aligned} V_X = H^{2,2}(X \times X, \mathbb{R}) &\cap \left(H^{1,1}(X, \mathbb{Q}(1)) \otimes H^{1,1}(X, \mathbb{R}(1))\right)^{\perp} \\ &\cap \left(H^{1,1}(X, \mathbb{R}(1)) \otimes H^{1,1}(X, \mathbb{Q}(1))\right)^{\perp} \cap [\Delta_X]^{\perp}. \end{aligned}$$

It was proven in [6] that

$$\mathrm{Im}(\underline{r}_{3,1}) \otimes \mathbb{R} \neq 0. \tag{4.1}$$

Of course, this implies that the indecomposables $\mathrm{CH}^3_{\mathrm{ind}}(X \times X, 1) \otimes \mathbb{Q}$ are non-zero for a general projective $K3$ surface X [6, Corollary 1.3].

Now let us look at the transcendental part of $\mathrm{cl}_{3,1}$:

$$\Phi_{3,1} : \mathrm{CH}^3(X \times X, 1) \to \frac{\left\{F^3(H^2_v(X, \mathbb{C}) \otimes H^2_v(X, \mathbb{C}))\right\}^{\vee}}{H_4(X \times X, \mathbb{Q}(1))},$$

where now X is a very general $K3$ and $H^2_v(X, \mathbb{C})$ is the transcendental cohomology. The proof of Corollary 1.5 is a stepping stone to the proof of the stronger Theorem 1.6.

4.1. The transcendental regulator $\Phi_{3,1}$. It is instructive to explain precisely how Theorem 1.4 leads to Corollary 1.5, viz., to the non-triviality of $\Phi_{3,1}$ for $Y := X \times X$, where X is a very general projective $K3$ surface. In this case Y takes the role of X in the proof of Theorem 1.2, with $(d, r, m, \ell) = (4, 3, 1, 1)$, $H^1_{\mathrm{alg}}(Y, \Theta_Y)$ will be identified with $H^1_{\mathrm{alg}}(X, \Theta_X) \simeq \mathbb{C}^{19}$, and $H^{2d-2r+m+1}_v(Y, \mathbb{Q}) = H^4_v(Y, \mathbb{Q})$ will be replaced by

$$[\Delta_X]^{\perp} \cap \left\{H^2_v(X, \mathbb{Q}) \otimes H^2_v(X, \mathbb{Q})\right\},$$

where $[\Delta_X]$ is the diagonal class. The pairing in Theorem 1.4 amounts to studying the properties of the pairing

$$H^1(\Theta_X) \otimes H^{3,1}(X \times X) \to H^{2,2}(X \times X),$$

which amounts to a Gauss–Manin derivative calculation. So let $\mathcal{X}/S$ be a smooth projective family of $K3$ surfaces over a polydisk S (arising from a universal family), $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$, $X = \mathcal{X}_0$ be a very general fiber of $\mathcal{X}/S$, $Y = X \times X$ and $\pi_{\mathcal{X}}$ be the projection $\mathcal{X} \to S$. Let

∇ be the GM connection associated to $\mathcal{X}/S$ and let $\alpha \in H^1(\Theta_X)$ be a tangent vector of S at 0. For $\omega \in H^0((\pi_{\mathcal{X}})_* \wedge^2 \Omega_{\mathcal{X}/S})$ and $\eta \in H^0(R^1(\pi_{\mathcal{X}})_* \Omega_{\mathcal{X}/S})$, i.e., for $\omega \in H^{2,0}(X)$ and $\eta \in H^{1,1}(X)$ when restricted to X, we claim that

$$\bigcap_{\alpha,\omega,\eta} \big((\nabla_\alpha(\omega \otimes \eta))^\perp \cap (\nabla_\alpha(\eta \otimes \omega))^\perp\big) \cap [\Delta_X]^\perp = \{0\} \tag{4.2}$$

in $H^{2,2}(Y)$ and hence the condition on the cup product pairing in Theorem 1.4 holds. Note that

$$[\nabla_\alpha(\omega \otimes \eta)] = [\nabla_\alpha \omega] \otimes \eta + \omega \otimes [\nabla_\alpha \eta],$$

where $[\nabla_\alpha(\omega \otimes \eta)]$, $[\nabla_\alpha \omega]$ and $[\nabla_\alpha \eta]$ are the projections of $\nabla_\alpha(\omega \otimes \eta)$, $\nabla_\alpha \omega$ and $\nabla_\alpha \eta$ onto $H^{2,2}(Y)$, $H^{1,1}(X)$ and $H^{0,2}(X)$, respectively. We know that

$$[\nabla_\alpha \omega] = \langle \alpha, \omega \rangle \quad \text{and} \quad [\nabla_\alpha \eta] = \langle \alpha, \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the pairing

$$H^1(\Theta_X) \otimes (H^{1,1}(X) \oplus H^{2,0}(X)) \to H^{0,2}(X) \oplus H^{1,1}(X).$$

We write $\langle \alpha, \omega \rangle = \delta_\alpha \omega$ and $\langle \alpha, \eta \rangle = \delta_\alpha \eta$. Then (4.2) follows directly from the following statement.

Proposition 4.1. *For every complex $K3$ surface X,*

$$\bigcap_{\alpha,\omega,\eta} \big((\delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta)^\perp \cap (\delta_\alpha \eta \otimes \omega + \eta \otimes \delta_\alpha \omega)^\perp\big) \cap [\Delta_X]^\perp = \{0\} \tag{4.3}$$

in $H^{2,2}(X \times X, \mathbb{C})$, where $\alpha \in H^1(\Theta_X)$, $\omega \in H^{2,0}(X)$ and $\eta \in H^{1,1}(X)$.

Proof. Combining Proposition 2.2 with the fact that

$$\langle \delta_\alpha \omega, \eta \rangle + \langle \omega, \delta_\alpha \eta \rangle = 0, \tag{4.4}$$

we obtain

$$\langle [\Delta_X], \delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta \rangle = 0$$

and hence

$$\begin{aligned} &\mathrm{Span}\{\delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta\} \\ &\quad = [\Delta_X]^\perp \cap \big(H^{1,1}(X) \otimes H^{1,1}(X) \oplus H^{2,0}(X) \otimes H^{0,2}(X)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathrm{Span}\{\delta_\alpha \eta \otimes \omega + \eta \otimes \delta_\alpha \omega\} \\ &\quad = [\Delta_X]^\perp \cap \big(H^{0,2}(X) \otimes H^{2,0}(X) \oplus H^{1,1}(X) \otimes H^{1,1}(X)\big) \end{aligned}$$

and (4.3) follows easily. □

Note that $H^2_f(X, \mathbb{C}) = H^{1,1}(X, \mathbb{Q}(1)) \otimes \mathbb{C}$ and $\pi_1(\mathcal{S})$ acts on $H^2_v(X, \mathbb{C})$ irreducibly. It is then not hard to see that

$$H^4_f(Y, \mathbb{C}) \cap H^2_v(X, \mathbb{C}) \otimes H^2_v(X, \mathbb{C}) \cap [\Delta_X]^\perp = \{0\}$$

and hence

$$H^4_f(Y, \mathbb{C}) \subset V_X^\perp.$$

Since $\underline{r}_{3,1}(\xi) \neq 0$, this shows that $\Phi_{3,1}$ is non-trivial.

4.2. The truncated transcendental regulator $\Psi_{3,1}$. We now turn our attention to the proof of Theorem 1.6. More explicitly, we fix a non-vanishing holomorphic two-form $\omega \in H^{2,0}(X)$ and look at

$$\langle \mathrm{cl}_{3,1}(\xi), \omega \otimes \omega \rangle$$

modulo the periods $\int_\gamma \omega \otimes \omega$ for $\gamma \in H_4(X \times X, \mathbb{Q}(1))$. We claim $\Psi_{3,1}$ is non-trivial, or equivalently, $\langle \mathrm{cl}_{3,1}(\xi), \omega \otimes \omega \rangle$ is not a period for some $\xi \in \mathrm{CH}^3(X \times X, 1)$. Here we go slightly beyond the range of ℓ in Theorem 1.4, namely we allow $\ell = -1$, 0. More specifically we consider

$$\left\{ \begin{array}{r} H^1_{\mathrm{alg}}(Y, \Theta_Y) \to \mathrm{Hom}\big(H^{4,0}(Y), H^{3,1}(Y)\big), \\ H^1_{\mathrm{alg}}(Y, \Theta_Y)^{\otimes 2} \to \mathrm{Hom}\big(H^{4,0}(Y), H^{2,2}(Y)\big), \end{array} \right. \tag{4.5}$$

where again $Y = X \times X$ is a self product of a very general projective $K3$ surface X, and $H^1_{\mathrm{alg}}(Y, \Theta_Y)$ is identified with the first order deformation space of a universal family of projective $K3$'s. Of course if the former map in (4.5) were surjective, then the latter map could be replaced by

$$H^1_{\mathrm{alg}}(Y, \Theta_Y) \to \mathrm{Hom}\big(H^{3,1}(Y), H^{2,2}(Y)\big).$$

Let us assume for the moment that both maps in (4.5) are surjective. Then by the same reasoning as in the previous section, one could argue that $\Psi_{3,1}$ is non-trivial. However by a dimension count, it is clear that both maps in (4.5) are not surjective. We remedy this by passing to the symmetric product $\hat{Y} = Y/\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the symmetric group of order 2 acting on $Y = X \times X$. In fact, instead of working directly on $\hat{Y}$, we will work with the equivariant cohomologies $H^4(Y, \mathbb{Q})^\sigma$ and $\mathrm{CH}^3(Y, 1)^\sigma$. That is, they consist of classes fixed under σ. Note that $H^4(Y, \mathbb{Q})^\sigma$ is still a Hodge structure. With the same setup for $\Phi_{3,1}$ and following the same argument by differentiating, we consider the orthogonal complements

$$(\nabla_\alpha(\omega \otimes \omega))^\perp \text{ and } (\nabla_\beta \nabla_\alpha(\omega \otimes \omega))^\perp,$$

following the situation in (4.5). In particular, we are interested in the subspace

$$\bigcap_{\alpha,\beta} \big(\delta_\alpha\delta_\beta\omega \otimes \omega + \delta_\alpha\omega \otimes \delta_\beta\omega + \delta_\beta\omega \otimes \delta_\alpha\omega + \omega \otimes \delta_\alpha\delta_\beta\omega\big)^\perp$$
$$\cap \bigcap_\alpha (\delta_\alpha\omega \otimes \omega + \omega \otimes \delta_\alpha\omega)^\perp \cap (\omega \otimes \omega)^\perp \cap [\Delta_X]^\perp$$

when restricted to Y. Note that

$$\langle \delta_\alpha\omega, \delta_\beta\omega \rangle + \langle \omega, \delta_\alpha\delta_\beta\omega \rangle = \langle \delta_\alpha\omega, \delta_\beta\omega \rangle + \langle \omega, \delta_\beta\delta_\alpha\omega \rangle = 0 \tag{4.6}$$

by (4.4) and hence

$$\delta_\alpha\delta_\beta\omega \otimes \omega + \delta_\alpha\omega \otimes \delta_\beta\omega + \delta_\beta\omega \otimes \delta_\alpha\omega + \omega \otimes \delta_\alpha\delta_\beta\omega \in [\Delta_X]^\perp \tag{4.7}$$

for all $\alpha, \beta \in H^1(\Theta_X)$. Similarly,

$$\delta_\alpha\omega \otimes \omega + \omega \otimes \delta_\alpha\omega \in [\Delta_X]^\perp \tag{4.8}$$

for all $\alpha \in H^1(\Theta_X)$. Although we do not need it, (4.6) also implies that $\delta_\alpha\delta_\beta = \delta_\beta\delta_\alpha$ and hence the map

$$H^1(\Theta_X) \otimes H^1(\Theta_X) \to \hom(H^{2,0}(X), H^{0,2}(X)) \tag{4.9}$$

induced by $H^1(\Theta_X) \otimes H^1(\Theta_X) \otimes H^{2,0}(X) \to H^{0,2}(X)$ is a symmetric non-degenerate pairing. Obviously,

$$\operatorname{Span}\{\delta_\alpha\delta_\beta\omega \otimes \omega + \delta_\alpha\omega \otimes \delta_\beta\omega + \delta_\beta\omega \otimes \delta_\alpha\omega + \omega \otimes \delta_\alpha\delta_\beta\omega\} = [\Delta_X]^\perp \cap H^{2,2}(Y)^\sigma$$

and

$$\operatorname{Span}\{\delta_\alpha\omega \otimes \omega + \omega \otimes \delta_\alpha\omega\} = [\Delta_X]^\perp \cap H^{3,1}(Y)^\sigma$$

by (4.7), (4.8) and the non-degeneracy of (4.9). Therefore,

$$\bigcap_{\alpha,\beta}\big(\delta_\alpha\delta_\beta\omega \otimes \omega + \delta_\alpha\omega \otimes \delta_\beta\omega + \delta_\beta\omega \otimes \delta_\alpha\omega + \omega \otimes \delta_\alpha\delta_\beta\omega\big)^\perp$$
$$\cap \bigcap_{\alpha}(\delta_\alpha\omega \otimes \omega + \omega \otimes \delta_\alpha\omega)^\perp \cap (\omega \otimes \omega)^\perp \cap [\Delta_X]^\perp \cap H^4(Y,\mathbb{C})^\sigma = \{0\}.$$

Thus, in order to prove Theorem 1.6, we just have to find ξ such that $\underline{r}_{3,1}(\xi) \neq 0$ and $\mathrm{cl}_{3,1}(\xi) \in H^4(Y,\mathbb{C})^\sigma$. The obvious way to do this is to find an equivariant higher Chow class $\xi \in \mathrm{CH}^3(Y,1)^\sigma$ with $\underline{r}_{3,1}(\xi) \neq 0$. Namely, we need a slightly stronger statement than (4.1). That is,

Theorem 4.2. *There exists $\xi \in \mathrm{CH}^3(X \times X, 1)^\sigma$ such that $\underline{r}_{3,1}(\xi) \neq 0$ for a general projective $K3$ surface X.*

Proof. This is a consequence of the explicit construction of the cycle in [6]. □

5. Intermezzo: Lattice polarized $K3$ surfaces, hypersurface normal forms, and modular parametrization

At this point it is natural to ask how one might construct explicit families of $K3$ surfaces satisfying the conditions of Theorem 1.2, with enough "internal structure" to make it possible to construct explicit cycles with non-zero $\Phi_{2,1}$. In light of Section 3, it would also be highly desirable to have a means of explicitly constructing the Picard–Fuchs operators for these families.

Families of the sort required by Theorem 1.2 with a fixed generic Néron–Severi lattice are known as *lattice polarized $K3$ surfaces* [14], and defined with specific reference to a polarizing lattice as follows. Let X be an algebraic $K3$ surface over the field of complex numbers. If M is an even lattice of signature $(1, \ell - 1)$ (with $\ell > 0$), then an *M-polarization* on X is a primitive lattice embedding

$$i : M \hookrightarrow \mathrm{NS}(X)$$

such that the image $i(M)$ contains a pseudo-ample class. There is also a coarse moduli space $\mathcal{M}_M$ for equivalence classes of pairs (M, i), which satisfies a version of the global Torelli theorem. Moreover, surjectivity of the period map holds for families which are maximal in the sense of Theorem 1.2.

An elliptic $K3$ surface with section consists of a triple (X, ϕ, S) of a $K3$ surface X, an elliptic fibration $\phi : X \to \mathbb{P}^1$, and a smooth rational curve $S \subset X$ forming a section of ϕ. This "internal structure" of an elliptic fibration with section on a $K3$ surface X is equivalent to a lattice polarization of X by the even rank two hyperbolic lattice

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [8, Theorem 2.3] for details). The moduli space $\mathcal{M}_H$ of H-polarized $K3$ surfaces has complex dimension 18, and the generic elliptic $K3$ surface with section has 24 singular fibers of Kodaira type I_1. Instead of working with a very general member of this family, which will have Picard rank $\ell = 2$, one can enhance the lattice polarization by considering a higher rank lattice M, with H as a sublattice. For each distinct embedding of H into M, up to automorphisms of the ambient lattice M, we find an elliptic surface structure with section on all M-polarized $K3$ surfaces. There is a decomposition of the Néron–Severi lattice

$$\mathrm{NS}(X) = H \oplus W_X,$$

where W_X is the negative definite sublattice of $\mathrm{NS}(X)$ generated by classes associated to algebraic cycles orthogonal to both the elliptic fiber and the section. The sublattice

$$W_X^{\mathrm{root}} := \{r \in W_X \mid \langle r, r \rangle = -2\}$$

is called the *ADE type* of the elliptic fibration with section, as it decomposes naturally into the sum of ADE type sublattices spanned by c_1 of the irreducible (rational) components of the singular fibers of the elliptic fibration (see [8, Section 6]).

For the explicit computations in Sections 6 and 7 we will make essential use of one particular elliptic fibration with section on a family of $K3$ surfaces polarized by the lattice $H \oplus E_8 \oplus E_8$. It is not, in fact, the "standard" fibration, which corresponds to $W_X = E_8 \oplus E_8$, but the "alternate fibration" for which $W_X = D_{16}^+$ (the other even negative definite rank 16 lattice). Up to ambient lattice automorphisms, these are the only two distinct embeddings of the lattice H into $H \oplus E_8 \oplus E_8$. As a result, we know that these are the only two elliptic fibrations with section on a very general member of this family of $K3$ surfaces [7].

5.1. Normal forms and elliptic fibrations. The natural setting for Theorem 1.2 is families of lattice-polarized $K3$ surfaces which cover their corresponding coarse moduli spaces. In order to effectively compute, we first need to construct such maximal families of $K3$ surfaces.

The most classical construction of $K3$ surfaces is as smooth quartic (anticanonical) hypersurfaces in $\mathbb{P}^3$. A very general member of this family will have a 4-polarization and Picard rank $\ell = 1$. It is possible, however, to construct subfamilies of smooth quartics with natural polarization by lattices of much higher rank. For example, consider the "Fermat quartic pencil"

$$X_t := \{x^4 + y^4 + z^4 + w^4 + t \cdot xyzw = 0\} \subset \mathbb{P}^3. \tag{5.1}$$

For generic $t \in \mathbb{P}^1$, the group $G := (\mathbb{Z}/4\mathbb{Z})^2$ acts on X_t by

$$x \mapsto \lambda \cdot x, \quad y \mapsto \mu \cdot y, \quad z \mapsto \lambda^{-1}\mu^{-1} \cdot z,$$

where λ and μ are fourth roots of unity.

The induced action of this group on the cohomology of X_t fixes the holomorphic two-form ω_t (i.e., it acts symplectically). Nikulin's classification of symplectic actions on $K3$ surfaces then implies that there is a rank 18 negative definite sublattice in the Néron–Severi group of X_t, which together with the (fixed) 4-polarization class means that the Picard rank of X_t is at least 19. As the family is not isotrivial, the Picard rank is not generically equal to 20, and we conclude that the family X_t, $t \in \mathbb{P}^1$ satisfies the conditions of Theorem 1.2 with $\ell = 19$. (See [28] for a general set of tools to bound the Picard rank of pencils of hypersurfaces with a high degree of symmetry.) This is an example of a *normal form* for the corresponding class of lattice polarized $K3$ surfaces, in this case providing a natural generalization of the Hesse pencil normal form for cubic curves in $\mathbb{P}^2$.

There is another family Y_t of $K3$ surfaces with $\ell = 19$ easily derivable from the X_t in (5.1) by quotienting each X_t by the group G and simultaneously resolving the resulting singularities in the family. The family Y_t, known as the "quartic mirror family," has rank 19 lattice polarization by the lattice $M_2 := H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$.

Another way to construct families of 4-polarized $K3$ surfaces with an enhanced lattice polarization is to consider *singular* quartic hypersurfaces in $\mathbb{P}^3$. By introducing ordinary double point singularities of ADE type, it is a simple matter to engineer (upon minimal resolution) $K3$ surfaces with large negative definite sublattices of ADE type in their Néron–Severi groups. One feature that both the smooth and singular quartic hypersurface constructions enjoy is that for each line lying on the surface there is a corresponding elliptic fibration structure, defined by taking the pencil of planes passing through the line and considering the excess intersection of each (a pencil of cubic curves). In this way, suitably nice quartic normal forms readily admit the structure of elliptic fibrations with section corresponding to various embeddings of the hyperbolic lattice H into their polarizing lattices.

Let us illustrate this with the key example for the constructions in Sections 6 and 7, the singular quartic normal form for $K3$ surfaces polarized by the lattice

$$M := H \oplus E_8 \oplus E_8$$

(see [7]). Let (X, i) be an M-polarized $K3$ surface. Then there exists a triple $(a, b, d) \in \mathbb{C}^3$, with $d \neq 0$ such that (X, i) is isomorphic to the minimal resolution of the quartic surface

$$Q_M(a, b, d) : y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0.$$

Two such quartics $Q_M(a_1, b_1, d_1)$ and $Q_M(a_2, b_2, d_2)$ determine via minimal resolution isomorphic M-polarized $K3$ surfaces if and only if

$$(a_2, b_2, d_2) = (\lambda^2 a_1, \lambda^3 b_1, \lambda^6 d_1)$$

for some $\lambda \in \mathbb{C}^*$. Thus the coarse moduli space for M-polarized $K3$ surfaces is the open variety

$$\mathcal{M}_M = \left\{[a, b, d] \in \mathbb{WP}(2, 3, 6) \mid d \neq 0\right\}$$

with fundamental invariants

$$\frac{a^3}{d} \quad \text{and} \quad \frac{b^2}{d}.$$

On the singular quartic hypersurface $Q_M(a, b, d) \subset \mathbb{P}^3$ there are two distinct lines

$$\{x = w = 0\} \quad \text{and} \quad \{z = w = 0\},$$

and the points

$$P_1 := [0, 1, 0, 0] \quad \text{and} \quad P_2 := [0, 0, 1, 0]$$

are rational double point singularities on $Q_M(a, b, d)$ of ADE types A_{11} and E_6, respectively. The standard fibration is induced by the projection to $[z, w]$, and the alternate fibration is induced by the projection to $[x, w]$. Moreover, among the exceptional rational curves in the resolution of P_1 are sections of both elliptic fibrations on $X(a, b, d)$; among the exceptional rational curves in the resolution of P_2 is a second section of the alternate fibration on $X(a, b, d)$.

It is useful to note that both the quartic mirror normal form Y_t for M_2-polarized $K3$ surfaces and the M-polarized normal form $X(a, b, d)$ admit natural reinterpretations as the generic anticanonical hypersurfaces in certain toric Fano varieties [11,15,16]. In both cases we build the toric Fano variety from the normal fan of a reflexive polytope. For the M_2-polarized case, the polytope is the convex hull of

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\} \subset \mathbb{R}^3,$$

polar to the Newton polytope for $\mathbb{P}^3$. For the M-polarized case, the polytope is the convex hull of

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -4, -6)\},$$

polar to the Newton polytope for $\mathbb{WP}(1, 1, 4, 6)$. What is more, the two elliptic fibrations with section on a very general $X(a, b, d)$ are themselves induced by ambient toric fibrations on the toric variety in which it sits as a hypersurface. Combinatorially, these correspond to reflexive "slices" of the corresponding polytope, i.e., planes in $\mathbb{R}^3$ which slice the reflexive polytope in a reflexive polygon.

5.2. Picard–Fuchs equations and modular parametrization. There is a reverse nesting of moduli spaces corresponding to embeddings of the polarizing lattices. In the context of the families Y_t and $X(a, b, d)$ above, the usual embedding

$$H \oplus E_8 \oplus E_8 \hookrightarrow H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$$

corresponds to an algebraic parametrization

$$a(t) = (t + 16)(t + 256), \quad b(t) = (t - 512)(t - 8)(t + 64), \quad d(t) = 2^{12}\, 3^6\, t^3$$

of a genus zero *modular curve*. To see the connection with classical modular curves, and indeed the Hodge-theoretic evidence for the underlying geometry, it is instructive to consider the Picard–Fuchs systems annihilating periods on the $K3$ surfaces involved.

Let $f(t)$ denote a period of the holomorphic two-form on $X(a, b, d)$. The Griffiths–Dwork method for producing Picard–Fuchs systems yields (in an affine chart, where we have set $a = 1$)

$$\Big(\frac{\partial^2}{\partial b^2} - 4d\frac{\partial^2}{\partial d^2} - 4\frac{\partial}{\partial d}\Big) f(b, d) = 0$$

and

$$\Big((-1 + b^2 + d)\frac{\partial^2}{\partial b^2} + 2b\frac{\partial}{\partial b} + 4bd\frac{\partial^2}{\partial b \partial d} + 2d\frac{\partial}{\partial d} + \frac{5}{36}\Big) f(b, d) = 0$$

(see [11]). By reparametrizing in terms of variables j_1 and j_2,

$$b^2 = \frac{(j_1 - 1)(j_2 - 1)}{j_1 j_2}, \quad d = \frac{1}{j_1 j_2},$$

we find that the Picard–Fuchs system completely decouples as

$$72 j_1 \Big(2(j_1 - 1) j_1 \frac{\partial^2}{\partial j_1^2} + (2j_1 - 1) \frac{\partial}{\partial j_1} \Big) f(j_1, j_2) - 5 f(j_1, j_2) = 0$$

and

$$72 j_2 \Big(2(j_2 - 1) j_2 \frac{\partial^2}{\partial j_2^2} + (2j_2 - 1) \frac{\partial}{\partial j_2} \Big) f(j_1, j_2) - 5 f(j_1, j_2) = 0.$$

This implies that the periods of the M-polarized $K3$ surfaces split naturally as products

$$f(j_1, j_2) = f_1(j_1) \cdot f_2(j_2).$$

At this point it is natural to ask whether the second order ordinary differential equation satisfied by $f(j)$ is itself a Picard–Fuchs equation for a family of elliptic curves. One can check for a family of elliptic curves over $\mathbb{P}_t^1$ in Weierstrass normal form

$$\{E_t\} := \big\{ y^2 z - 4x^3 + g_2(t) x z^2 + g_3(t) z^3 = 0 \big\} \subset \mathbb{P}^2$$

that the periods of a suitably normalized holomorphic one-form on E_t,

$$g_2(t)^{\frac{1}{4}} \frac{dx}{y},$$

satisfy Picard–Fuchs equations of the form of the second order equations above. Thus, by the Hodge conjecture, we expect there to be an algebraic correspondence between M-polarized $K3$ surfaces and abelian surfaces (with principal polarization) which split as a product of a pair of elliptic curves. This correspondence was made explicit in [7]; we recall the necessary features for our higher K-theory computations in Section 6 below.

What then is the meaning of the special subfamily Y_t in terms of these split abelian surfaces? When specialized to the subfamily $Y_t = X(a(t), b(t), c(t))$, the Griffiths–Dwork method produces the following Picard–Fuchs differential equation

$$f^{(\mathrm{iii})}(t) + \frac{3(3t + 128)}{2t(t + 64)} f''(t) + \frac{13t + 256}{4t^2(t + 64)} f'(t) + \frac{1}{8t^2(t + 64)} f(t) = 0.$$

On a general parametrized disk in the moduli space $\mathcal{M}_M$, the Picard–Fuchs ODE will have rank 4, just as the full Picard–Fuchs system. The drop in rank indicates a special relationship between the two elliptic curves E_{τ_1} and E_{τ_2} corresponding to Y_t. A differential algebraic characterization of the curves in $\mathcal{M}_M$ on which the Picard–Fuchs ODE drops in rank was given in [11, Theorem 3.4]. In fact, in the M_2-polarized case, the relationship is simply the existence of a two-isogeny between the two elliptic curves, i.e., $\tau_2 = 2 \cdot \tau_1$. More generally, the M_n-polarized case corresponds to a cyclic n-isogeny, i.e., $\tau_2 = n \cdot \tau_1$.

Given that M-polarized $K3$ surfaces correspond to abelian surfaces which are the products of a pair of elliptic curves, the natural modular parameters on the (rational) coarse moduli space $\mathcal{M}_M$ are the elementary symmetric polynomials in the two j-invariants $j_1 = j(\tau_1)$ and $j_2 = j(\tau_2)$,

$$\sigma := j_1 + j_2 \quad \text{and} \quad \pi := j_1 \cdot j_2.$$

In this notation, it is easy to identify explicit rational curves in $\mathcal{M}_M$ over which the Picard–Fuchs differential equation has maximal rank ($= 4$). One such locus, which arises in the context of the construction of $K3$ surface fibered Calabi–Yau threefolds realizing hypergeometric variations, is specified by simply setting $\sigma = 1$; see [26]. The Picard–Fuchs ODE has fourth order and takes the following form:

$$f^{(\mathrm{iv})}(s) + \frac{2(4s^2 - 3s - 2)}{s(s-1)(s+1)} f^{(\mathrm{iii})}(s) + \frac{1031s^3 - 553s^2 - 1175s - 167}{72s^2(s-1)(s+1)^2} f''(s)$$
$$+ \frac{167s^2 - 239s - 118}{36s^2(s-1)(s+1)^2} f'(s) + \frac{385(s-1)^2}{20736s^4(s+1)^2} f(s) = 0$$

which splits as a tensor product of two very closely related factor second order ODEs

$$f_1''(s) + \frac{3s+1}{2s(s+1)} f_1'(s) + \frac{5}{144s(s+1)} f_1(s) = 0$$

and

$$f_2''(s) + \frac{3s+1}{2s(s+1)} f_2'(s) + \frac{5}{144s^2(s+1)} f_2(s) = 0$$

corresponding to the two families of elliptic curves satisfying $j_1(s) + j_2(s) = 1$. Examples such as this provide a source of families of explicit non-maximal families of $K3$ surfaces to explore.

Instead of looking at superlattices of $H \oplus E_8 \oplus E_8$ such as M_n, one can consider sublattices such as $N := H \oplus E_7 \oplus E_8$ and $S := H \oplus E_7 \oplus E_7$; see [9, 10]. Moduli spaces of $K3$ surfaces polarized by these sublattices are themselves parametrized by modular functions (and contain $\mathcal{M}_M$ as a natural sublocus). For example, there is a normal form for N-polarized $K3$ surfaces extending the singular quartic normal form for M-polarized $K3$ surfaces with one additional monomial deformation

$$Q_N(a,b,c,d) : y^2zw - 4x^3z + 3axzw^2 + bzw^3 + cxz^2w - \frac{1}{2}(dz^2w^2 + w^4) = 0.$$

The associated coarse moduli space $\mathcal{M}_N$ is again an open subvariety of a weighted projective space

$$\mathcal{M}_N = \big\{[a,b,c,d] \in \mathbb{WP}(2,3,5,6) \mid c \neq 0 \text{ or } d \neq 0\big\}$$

with modular parametrization

$$[a,b,c,d] = [\mathcal{E}_4, \mathcal{E}_6, 2^{12}3^5\mathcal{C}_{10}, 2^{12}3^6\mathcal{C}_{12}],$$

where $\mathcal{E}_4$ and $\mathcal{E}_6$ are genus-two Eisenstein series of weights 4 and 6, and $\mathcal{C}_{10}$ and $\mathcal{C}_{12}$ are Igusa's cusp forms of weights 10 and 12; see [10, Theorem 1.5].

The connection to genus two curve moduli here is suggestive of the fundamental geometric fact that N-polarized $K3$ surfaces are Shioda–Inose surfaces coming from principally-polarized abelian surfaces. The hypersurface normal form once again has two natural elliptic fibration structures with section, just as in the M-polarized case, and the Nikulin involution which gives rise to the Shioda–Inose structure can be seen most naturally as the operation of "translation by 2-torsion" in the alternate elliptic fibration [9]. There is a further extension to a normal form for S-polarized $K3$ surfaces. In this case, most of the related geometric structures are still present, and we find a still more general modular parametrization of $\mathcal{M}_S$. For all these families of lattice-polarized $K3$ surfaces in normal form, Picard–Fuchs equations can be obtained via the Griffiths–Dwork method applied directly to the singular quartic equations or in their realization as anticanonical hypersurfaces in Gorenstein toric Fano threefolds.

The explicit computations which follow in Sections 6 and 7 offer a glimpse of the range of phenomena surrounding Theorem 1.2 which become accessible when we work with modular parametrizations of hypersurface normal forms for lattice polarized $K3$ surfaces equipped with well-chosen elliptic fibrations. Both generalization to related higher-dimensional moduli spaces and manipulation of the associated explicit Picard–Fuchs systems now become possible.

6. Explicit K_1-class on a family of Shioda–Inose $K3$ surfaces

We will now turn to a direct computation on the modular 2-parameter family $\mathrm{X}_{a,b}$ of $M := H \oplus E_8 \oplus E_8$-polarized (Picard-rank 18) $K3$'s introduced by Clingher and Doran [7]. Here $\mathrm{X}_{a,b}$ $(a, b \in \mathbb{C})$ is the minimal desingularization of

$$\left\{Y^2Z - P(\theta)W^2Z - \frac{1}{2}Z^2W - \frac{1}{2}W^3 = 0\right\} \subset \mathbb{P}^2_{[Y:Z:W]} \times \mathbb{P}^1_\theta,$$

where $P(\theta) := 4\theta^3 - 3a\theta - b$. Consider the real regulator map

$$r_{2,1} : \mathrm{CH}^2(\mathrm{X}_{a,b}, 1) \to \mathrm{Hom}_{\mathbb{R}}(H_v^{1,1}(\mathrm{X}_{a,b}, \mathbb{R}), \mathbb{R}). \tag{6.1}$$

The results of [5] already tell us that generically $\mathrm{span}_{\mathbb{R}}\{\mathrm{image}(r_{2,1})\}$ equals the right-hand side of (6.1), making $\Phi_{2,1}$ non-zero for very general (a, b). (We note that for those $\mathrm{X}_{a,b}$ with Picard rank 18, $H_v^{1,1} = H_{\mathrm{tr}}^{1,1}$.) The proof is based on *non*-explicit deformations of decomposable classes on Picard-rank 20 $K3$'s.

What we felt was missing here and in the literature are concrete indecomposable cycles on which $r_{2,1}$ and $\Phi_{2,1}$ are non-zero, particularly those which arise naturally in the context of an internal elliptic fibration. In our example, the projection $\mathrm{X}_{a,b} \to \mathbb{P}^1_\theta$ produces the so-called *alternate fibration* with six fibers of Kodaira type I_1 and one fiber of type I_{12}^*. If D is an I_1 fiber, with $\mathbb{P}^1_z \cong \tilde{D} \twoheadrightarrow D$ its normalization (attaching $z = 0$ and $z = \infty$), then (D, z) (or (D, z^{-1})) generates $\mathrm{CH}^2_D(\mathrm{X}_{a,b}, 1) \cong \mathbb{Z}$, and we may consider its image under $\mathrm{CH}^2_D(\mathrm{X}_{a,b}, 1) \to \mathrm{CH}^2(\mathrm{X}_{a,b}, 1)$. Clearly then, the I_1 fibers provide *the* most natural source of classes in $\mathrm{CH}^2(\mathrm{X}_{a,b}, 1)$ provided one can show their real regulators are non-zero.

This turns out to require some serious and interesting work, by first passing to a *Kummer* $K3$ family $\mathrm{K}_{\alpha,\beta}$ which is the minimal resolution of both the quotient of $\mathrm{X}_{a,b}$ by the Nikulin involution and the quotient of a product of elliptic curves $E_\alpha \times E_\beta$ by $(-\mathrm{id}, -\mathrm{id})$. This "intermediate" setting seems to be the one place where *both* the normalization of the rational curves supporting the family of K_1-classes (namely, a Néron 2-gon) and the closed $(1, 1)$-form, against which we integrate its regulator current to compute $r_{2,1}$, are tractable. In fact, the form has some singularities, even after pulling back the rational curves, and so the computation requires careful additional justification.

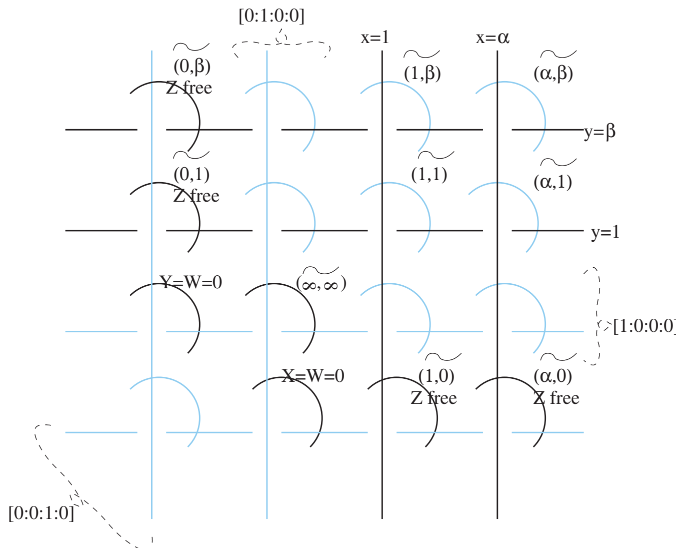

Figure 1. $\widetilde{(\infty,\infty)}$ stands for $\{W = 0, XY = Z^2\}$.

6.1. Kummer $K3$ geometry. We begin with a review of special features of the Kummer family from [7], which has two parameters $\alpha, \beta \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$:

$$\check{\mathrm{K}}'_{\alpha,\beta} := \left\{Z^2 XY = (X - W)(X - \alpha W)(Y - W)(Y - \beta W)\right\} \subset \mathbb{P}^3$$

is the singular model, with affine equation

$$z^2 xy = (x-1)(x-\alpha)(y-1)(y-\beta) \quad \left(x = \tfrac{X}{W},\ y = \tfrac{Y}{W},\ z = \tfrac{Z}{W}\right)$$

and $\mathrm{K}_{\alpha,\beta}$ shall denote its minimal desingularization. Recall that a Kummer is usually constructed by taking a pair of elliptic curves, in this case

$$\left\{u^2 = x(x-1)(x-\alpha)\right\} =: E_\alpha \circlearrowright \jmath_\alpha : (x,u) \mapsto (x,-u),$$

$$\left\{v^2 = y(y-1)(y-\beta)\right\} =: E_\beta \circlearrowright \jmath_\beta : (y,v) \mapsto (y,-v),$$

then taking the quotient $\check{\mathrm{K}}_{\alpha,\beta}$ of $E_\alpha \times E_\beta$ by the automorphism $\jmath_\alpha \times \jmath_\beta$. This is singular at the image of the 16 products of 2-torsion points – ordinary double points whose resolution yields 16 exceptional $\mathbb{P}^1$'s , and produces $\mathrm{K}_{\alpha,\beta}$.

Figure 1 shows a diagram of rational curves on $\mathrm{K}_{\alpha,\beta}$. The exceptional divisors are represented by arcs; while the proper transforms of the quotients of $E_\alpha \times \{\text{2-torsion point}\}$ resp. $\{\text{2-torsion point}\} \times E_\beta$ are represented by horizontal resp. vertical lines. The projective model $\check{\mathrm{K}}'_{\alpha,\beta}$ is the blow-down of $\mathrm{K}_{\alpha,\beta}$ along the 13 rational curves depicted more faintly. Notice that the configuration shown in Figure 2 has Dynkin diagram D_{10}, hence Kodaira type I_6^*.

We now describe an elliptic fibration of $\mathrm{K}_{\alpha,\beta}$ which shall have:

- this I_6^* as its singular fiber at ∞;
- the lines $y = 1$, $y = \beta$, $x = 1$, $x = \alpha$ as sections;
- the lines marked $\widetilde{(1,0)}$, $\widetilde{(\alpha,0)}$, $\widetilde{(0,1)}$, $\widetilde{(0,\beta)}$ as bi-sections;

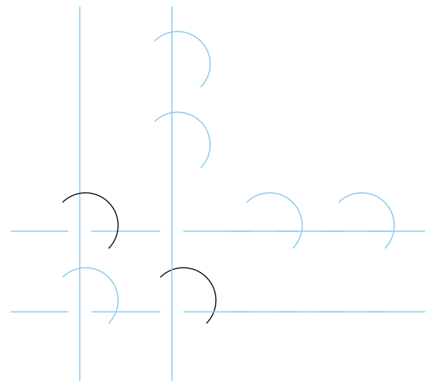

Figure 2. Sub-configuration of Figure 1.

- the line marked $\widetilde{(\infty,\infty)}$ as a 4-section; and
- six I_2 singular fibers, four of which have one of the lines marked $\widetilde{(1,\beta)}$, $\widetilde{(\alpha,\beta)}$, $\widetilde{(1,1)}$, or $\widetilde{(\alpha,1)}$ as one component.

Write

$$R(X,Y,W) := -\frac{X^2}{\alpha} - \frac{Y^2}{\beta} + \frac{\alpha+1}{\alpha}XW + \frac{\beta+1}{\beta}YW - W^2.$$

Then the fibration, which is really nothing but the pencil $|I_6^*|$, is given on the (singular) projective model by

$$\check{\mathrm{K}}'_{\alpha,\beta} \to \mathbb{P}^1, \quad [X:Y:Z:W] \mapsto [R(X,Y,W):XY] =: [\mu:1].$$

In either case, the smooth elliptic fibers $\mathcal{E}_\mu$ (resp. $\check{\mathcal{E}}'_\mu$) are double covers of the smooth conic curves

$$\mathcal{C}_\mu := \{R(X,Y,W) = \mu XY\} \subset \mathbb{P}^2,$$

branched over

$$(x,y) = \big(1,(1-\mu)\beta+1\big),\ \big(\alpha,(1-\mu\alpha)\beta+1\big),\ \big((1-\mu)\alpha+1,1\big),\ \big((1-\mu\beta)\alpha+1,\beta\big).$$

By [7], $\mathcal{E}_\mu$ is singular if and only if one of the following holds:

- $\mu=\infty$: then $\mathcal{E}_\infty = I_6^*$;
- $\mu \in \{1,\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\alpha\beta}\}$: then two of the branch points collide, making $\check{\mathcal{E}}'_\mu$ into an I_1. The fiber $\mathcal{E}_\mu$ is then the (Kodaira type I_2) union of its proper transform $(\check{\mathcal{E}}'_1)^\sim$ with the exceptional divisor over the collision point – for example, for $\mu=1$, $\mathcal{E}_1 = (\check{\mathcal{E}}'_1)^\sim \cup \widetilde{(1,1)}$; or
- $\mu \in \{\frac{\alpha\beta+1}{\alpha\beta},\frac{\alpha+\beta}{\alpha\beta}\}$: then the rational curve $\mathcal{C}_\mu$ acquires a node, so $\mathcal{E}_\mu$ has two nodes (again of type I_2).

This is all in case $J(E_\alpha) \neq J(E_\beta)$, i.e., $\beta \notin \{\alpha,\frac{1}{\alpha},1-\alpha,\frac{1}{1-\alpha},\frac{\alpha}{\alpha-1},\frac{\alpha-1}{\alpha}\}$. Below we will eventually specialize to the case $\beta=\alpha$, for which generically $\mathcal{E}_1$ is still an I_2 but $\mathcal{E}_{1/\alpha=1/\beta}$ becomes an I_4.

6.2. Normalization of $(\check{\mathcal{E}}_1')^{\sim}$. We will build our higher Chow cycle on $\mathcal{E}_1$. One can see right away that the cycle must have order-two monodromies about the components of $(\mathbb{P}^1 \times \{0,1,\infty\}) \cup (\{0,1,\infty\} \times \mathbb{P}^1)$, since the tangent vectors of the I_1 fiber $\check{\mathcal{E}}_1'$ at its singular point $(x,y,z) = (1,1,0)$ are

$$\Big(1, -\frac{\beta}{\alpha}, \pm\sqrt{\frac{\beta}{\alpha}(1-\alpha)(1-\beta)}\Big).$$

Notice that with $\alpha = \beta$, the branches of the square root become single-valued hence the monodromy will disappear; this will have consequences later.

In order to compute, we need to parametrize $\check{\mathcal{E}}_1'$ by a $\mathbb{P}^1$. The first step is to do this for $\mathcal{C}_1$ using stereographic projection. Putting $x = \Gamma + 1$, $y = \xi\Gamma + 1$ in its equation

$$\begin{aligned} 0 &= -\frac{x^2}{\alpha} - \frac{y^2}{\beta} + \frac{\alpha+1}{\alpha}x + \frac{\beta+1}{\beta}y - 1 - xy \\ &= \cdots = -\Big(\frac{1}{\alpha} + \frac{\xi^2}{\beta} + \xi\Big)\Gamma^2 - \Big(\frac{1}{\alpha} + \frac{\xi}{\beta}\Big)\Gamma \end{aligned} \tag{6.2}$$

and solving for Γ, yields

$$\big(x(\xi), y(\xi)\big) = \Big(\frac{\alpha\xi^2 + \alpha(\beta-1)\xi}{\Delta(\xi)}, \frac{\beta(\alpha-1)\xi + \beta}{\Delta(\xi)}\Big),$$

where $\Delta(\xi) := \alpha\xi^2 + \alpha\beta\xi + \beta$.

The second step is to pull the affine equation of $\check{\mathrm{K}}_{\alpha,\beta}'$ back along $\xi \mapsto (x(\xi), y(\xi))$ and again use an analogue of stereographic projection:

$$z^2 = \frac{(x-1)(x-\alpha)(y-1)(y-\beta)}{xy} = \cdots = \frac{(\alpha\xi+\beta)^2(\xi+\beta)(\alpha\xi+1)}{(\Delta(\xi))^2}.$$

So the equation of the I_1 fiber $\check{\mathcal{E}}_1'$ is

$$(\Delta(\xi))^2 z^2 = (\xi+\beta)(1+\alpha\xi)(\beta+\alpha\xi)^2, \tag{6.3}$$

which regarded as a curve in $\mathbb{P}^1_\xi \times \mathbb{P}^1_z$ has bidegree $(4,2)$ and three nodes (hence of course genus 0). A curve of bidegree $(2,1)$ must meet $\check{\mathcal{E}}_1'$ in eight points with multiplicity; so taking it to pass through the nodes

$$\Big(-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \frac{\beta}{\alpha}}, \infty\Big), \quad \Big(-\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \frac{\beta}{\alpha}}, \infty\Big), \quad \Big(-\frac{\beta}{\alpha}, 0\Big)$$

and the smooth point $(-\beta, 0)$, it must pass through one more point of $\check{\mathcal{E}}_1'$. Explicitly, these curves are of the form

$$\Delta(\xi)z = (\alpha\xi+\beta)(\xi+\beta)\gamma, \tag{6.4}$$

where $\gamma \in \mathbb{C}$ is a constant. To find the ξ-coordinate of the residual point we square the right-hand side of (6.4) and set equal to the right-hand side of (6.3), which yields

$$\xi(\gamma) = \frac{1-\beta\gamma^2}{\gamma^2-\alpha}. \tag{6.5}$$

γ^2	ξ	(x, y)
0	$-1/\alpha$	$(\alpha(1-\beta)+1, \beta)$
∞	$-\beta$	$(\alpha, \beta(1-\alpha)+1)$
δ	$-\beta/\alpha$	$(1, 1)$
$1/\beta$	0	$(0, 1)$
α	∞	$(1, 0)$
$-\alpha\beta+\alpha+1$	$1-\beta$	$(0, \beta)$
$\frac{1}{1+\beta-\alpha\beta}$	$\frac{1}{1-\alpha}$	$(\alpha, 0)$
roots of $\Delta(\xi(\gamma^2))$	roots of $\Delta(\xi)$	(∞, ∞)

Table 1

Thinking of $\mathbb{P}^1_\gamma$ as $(\check{\mathcal{E}}'_1)^\sim$ and $\mathbb{P}^1_\xi$ as $\mathcal{C}_1$, (6.5) gives the branched double cover

$$(\check{\mathcal{E}}'_1)^\sim \twoheadrightarrow \check{\mathcal{E}}'_1 \twoheadrightarrow \mathcal{C}_1,$$

where the first map just identifies a pair of points – namely, those with $\gamma^2 = \delta := \frac{\alpha\beta-\alpha}{\beta-\alpha\beta}$. Table 1 illustrates the relationship between functions on $\check{\mathcal{E}}'_1$. The rows starting with 0 and ∞ correspond to the branch points of $\check{\mathcal{E}}'_1 \to \mathcal{C}_1$.

The third and last step is to find a coordinate $\mathfrak{z}$ on $(\check{\mathcal{E}}'_1)^\sim$ $(\cong \mathbb{P}^1)$ which is 0 and ∞ (rather than $\pm\sqrt{\delta}$) at the two points mapping to the node of $\check{\mathcal{E}}'_1$, and ± 1 at the two branch points of $(\check{\mathcal{E}}'_1)^\sim \to \mathcal{C}_1$. This is given by

$$\mathfrak{z} = \frac{\gamma + \sqrt{\delta}}{\gamma - \sqrt{\delta}} \quad \longleftrightarrow \quad \gamma = \sqrt{\delta}\, \frac{\mathfrak{z}+1}{\mathfrak{z}-1}.$$

Our higher Chow cycle in $\mathrm{CH}^2(\mathrm{K}_{\alpha,\beta}, 1)$ will then simply be

$$\mathrm{Z}_{\alpha,\beta} := \big((\check{\mathcal{E}}'_1)^\sim, \mathfrak{z}\big) + \big(\widetilde{(1,1)}, g\big),$$

where g has zero and pole cancelling with those of $\mathfrak{z}$. (Note that while $\mathfrak{z}$ is the "preferred" coordinate on the $\mathbb{P}^1$, we will work mainly in γ below since this simplifies computations.) We remark that $\mathrm{Z}_{\alpha,\beta}$ is defined as long as $\alpha, \beta \notin \{0, 1, \infty\}$ and $1 \notin \{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}, \frac{\alpha\beta+1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta}\}$, but not quite well-defined: there is the issue of sign in $\mathfrak{z}^{\pm 1}$ (or equivalently, $\pm\sqrt{\delta}$) which leads to the predicted order-2 monodromies.

6.3. The (1, 1)-current. On $E_\alpha \times E_\beta$ there is the closed, real-analytic $(1,1)$-form

$$\omega = \frac{dx}{u} \wedge \overline{\left(\frac{dy}{v}\right)} = \frac{dx}{\sqrt{x(x-1)(x-\alpha)}} \wedge \overline{\left(\frac{dy}{\sqrt{y(y-1)(y-\beta)}}\right)}, \tag{6.6}$$

and $\omega + \bar{\omega}$, $i(\omega - \bar{\omega})$ obviously span $H^{1,1}_{\mathrm{tr},\mathbb{R}}$. Clearly ω is invariant under $\jmath_\alpha \times \jmath_\beta$, hence is the pullback of a $(1,1)$-current on $\check{\mathrm{K}}_{\alpha,\beta}$, whose pullback[2)] ω_{K} to $\mathrm{K}_{\alpha,\beta}$ has integrable singularities

[2)] Technically these observations should be expressed in terms of push-forwards, but the computations are better done as formal pullbacks.

along the exceptional divisors: if locally the equation of one looks like $w = 0$, then there is a term of the form $(dw \wedge d\bar{w})/|w|$. Now we could argue that this current ω_{K} is closed and represents a class in $H^{1,1}_{\mathrm{tr}}(\mathrm{K}_{\alpha,\beta}, \mathbb{C})$; but this approach runs into trouble because $\widetilde{(1,1)}$, where part of the cycle is supported, is an exceptional divisor. (The current's singularity along this divisor makes the pairing "improper", even though it "formally pulls back" to zero there.) Therefore, we will simply carry out an *ad hoc* pairing between[3] $r_{2,1}(\mathrm{Z}_{\alpha,\beta})$ and ω_{K} on $(\check{\mathcal{E}}'_1)^{\sim}$, then interpret it on $E_\alpha \times E_\beta$ where ω is smooth.

So taking ι_1 to denote the inclusion $(\check{\mathcal{E}}'_1)^{\sim} \hookrightarrow \mathrm{K}_{\alpha,\beta}$, we must compute $\iota_1^*\omega_{\mathrm{K}}$. This is done by "formally" pulling back the above form (6.6) under $\xi \mapsto (x(\xi), y(\xi))$: after some calculation, we obtain

$$\frac{-(\alpha\xi^2 + 2\beta\xi + \beta(\beta-1))\overline{(\alpha(\alpha-1)\xi^2 + 2\alpha\xi + \beta)}\, d\xi \wedge d\bar{\xi}}{|\Delta(\xi)|\,|\xi|\,|\alpha\xi+\beta|\,|\xi+(\beta-1)|\,|(\alpha-1)\xi+1|\sqrt{(\xi+\beta)\overline{(\alpha\xi+1)}}},$$

a sort of multivalued form on $\mathcal{C}_1$. Pulling this back (again "formally") to $(\check{\mathcal{E}}'_1)^{\sim} \cong \mathbb{P}^1_\gamma$ via $\gamma \mapsto \xi(\gamma)$ then yields (with apologies to the reader)

(6.7)
$$\iota_1^*\omega_{\mathrm{K}} = \frac{-4|\alpha\beta-1|}{|\beta|\,|1-\alpha|} \cdot \frac{\{(\alpha\beta^2-\beta^2-\beta)\gamma^4 + 2\beta\gamma^2 + (\alpha^2\beta^2-\alpha^2\beta+\alpha-2\alpha\beta)\}\,\gamma d\gamma}{|\gamma^2-\alpha|\,|1-\beta\gamma^2|\,|\gamma^2-\delta|\,|\gamma^2-(1+\alpha-\alpha\beta)|}$$
$$\wedge \frac{\overline{\{(\alpha^2\beta^2-\alpha\beta^2+\beta-2\alpha\beta)\gamma^4 + 2\alpha\gamma^2 + (\alpha^2\beta-\alpha^2-\alpha)\}}\, d\bar{\gamma}}{|(1+\beta-\alpha\beta)\gamma^2-1|\,|\beta\gamma^4+(\alpha^2\beta^2-3\alpha\beta)\gamma^2+\alpha|}.$$

While complicated, the 14 poles of this $(1,1)$-current are all of the integrable form mentioned above, and their locations are precisely the points where $\check{\mathcal{E}}'_1$ hits the exceptional divisors: $\widetilde{(1,1)}$, $\widetilde{(1,0)}$, $\widetilde{(\alpha,0)}$, $\widetilde{(0,1)}$, $\widetilde{(0,\beta)}$ twice each; $\widetilde{(\infty,\infty)}$ four times.

Along the locus $\alpha = \beta$, this form simplifies a little:

$$\iota_1^*\omega_{\mathrm{K}} = -4|\alpha+1| \cdot \frac{\{(\alpha^2-\alpha-1)\gamma^4 + 2\gamma^2 + (\alpha^3-\alpha^2-2\alpha+1)\}\,\gamma d\gamma}{|\gamma^2-\alpha|\,|1-\alpha\gamma^2|\,|\gamma^2+1|\,|\gamma^2-(1+\alpha-\alpha^2)|} \tag{6.8}$$
$$\wedge \frac{\overline{\{(\alpha^3-\alpha^2-2\alpha+1)\gamma^4 + 2\gamma^2 + (\alpha^2-\alpha-1)\}}\, d\bar{\gamma}}{|(1+\alpha-\alpha^2)\gamma^2-1|\,|\gamma^4+(\alpha^3-3\alpha)\gamma^2+1|}.$$

6.4. The pairing. The next step is simply to integrate $\log|\mathfrak{z}|$ against $\iota_1^*\omega_{\mathrm{K}}$ on $(\check{\mathcal{E}}'_1)^{\sim}$. As $\log|\mathfrak{z}| = \log|(\gamma+\sqrt{\delta})/(\gamma-\sqrt{\delta})|$, this integral will have a multivalued behavior as indicated above. It is singular but absolutely convergent: the worst behavior is at $\gamma = \pm\sqrt{\delta}$ where it locally takes the form

$$\int_{D_\epsilon} \frac{\log|z|}{|z|} dz \wedge d\bar{z},$$

which is equivalent to $\int_0^\epsilon (\log r)dr$.

[3] Pairing the regulator with $\omega_{\mathrm{K}} + \overline{\omega_{\mathrm{K}}}$ and $i(\omega_{\mathrm{K}} - \overline{\omega_{\mathrm{K}}})$ to get two real numbers is equivalent to pairing it with ω_{K} to get a single complex number.

But setting $\alpha = \beta$ (which makes $\delta = -1$) kills this monodromy, allowing for a well-defined choice of $\mathrm{Z}_{\alpha,\alpha} \in \mathrm{CH}^2(\mathrm{X}_{a,b}, 1)$ over $\mathbb{P}^1\backslash\{0, 1, \infty, -1, 2\}$ (see the end of Section 6.2). On a smooth compactification of the total space $\mathcal{X} \xrightarrow{\rho} \mathbb{P}^1_\alpha$, the "total cycle" is easily seen to have residues (i.e., $\log|\mathfrak{z}|$ blows up) along $\mathrm{X}_{-1,-1} \cup \mathrm{X}_{2,2}$ *only* (cf. the proof of [20, Theorem 3.7]). By the localization sequence for higher Chow groups, it can in fact be extended to all of $\rho^{-1}(\mathbb{P}^1\backslash\{-1, 2\})$. Most importantly, eliminating the monodromy makes the integrals

$$\psi(\alpha) = \int_{\mathbb{P}^1} \log\left|\frac{\gamma + i}{\gamma - i}\right| \Re(\imath_1^*\omega_{\mathrm{K}}), \quad \eta(\alpha) = \int_{\mathbb{P}^1} \log\left|\frac{\gamma + i}{\gamma - i}\right| \Im(\imath_1^*\omega_{\mathrm{K}}) \tag{6.9}$$

real-analytic functions of $\alpha \in \mathbb{P}^1\backslash\{0, 1, \infty, -1, 2\}$.

Now on $E_\alpha \times E_\alpha$, by considering pullbacks to the diagonal, one sees immediately that $i(\omega - \bar{\omega})$ is the algebraic class whilst $\omega + \bar{\omega}$ is the transcendental one. Clearly the same story holds on $\mathrm{K}_{\alpha,\alpha}$. So to check generic indecomposability of $\mathrm{Z}_{\alpha,\alpha}$ we need to demonstrate that $\psi(\alpha)$ (rather than $\eta(\alpha)$) is generically non-zero.[4] Clearly it will suffice to show that $\lim_{\alpha\to 1} \psi(\alpha) \neq 0$.

Setting $\alpha = 1$ in (6.8) yields

$$\begin{aligned} \imath_1^*\omega_{\mathrm{K}} &= \frac{-8|\gamma^2 - 1|^4\, \gamma d\gamma \wedge d\bar{\gamma}}{|\gamma^2 - 1|^6 |\gamma^2 + 1|} \\ &= \frac{-8\gamma d\gamma \wedge d\bar{\gamma}}{|\gamma^2 - 1|^2 |\gamma^2 + 1|} = \frac{16r\{i \cos\theta - \sin\theta\} dx \wedge dy}{|\gamma^2 - 1|^2 |\gamma^2 + 1|}, \end{aligned}$$

where $\gamma = x + iy = re^{i\theta}$. Because of the cancellations in the second step, it requires some analysis to prove that $\int_{\mathbb{P}^1} \log|\mathfrak{z}| \Re(\imath_1^*\omega_{\mathrm{K}})$ *at* $\alpha = 1$ actually computes the limit of ψ. This is done in the appendix to this section, and so we have

$$-\frac{1}{16} \lim_{\alpha\to 1} \psi(\alpha) = \int_{\mathbb{P}^1} \frac{\log\left|\frac{\gamma+i}{\gamma-i}\right| r \sin\theta}{|\gamma^2 - 1|^2 |\gamma^2 + 1|} dx \wedge dy. \tag{6.10}$$

Now simply notice that

- the integral over $\mathbb{P}^1$ in (6.10) is double that over the upper half plane, since $\log\left|\frac{\gamma+i}{\gamma-i}\right|$ and $\sin\theta$ are both odd in γ; and
- the integrand is (where non-singular) strictly positive on the upper half plane.

We conclude that (6.10) is a positive real number, finishing this part of the argument.

Remark 6.1. It is more natural to normalize ω_K, and hence ψ, by dividing out by

$$\left| \int_{E_\alpha} \frac{dx}{y} \wedge \overline{\left(\frac{dx}{y}\right)} \right|.$$

One can show – either using formula (6.9) or from general principles to be explained in [20] – that this modified ψ is asymptotic to a constant times $\log|\alpha + 1|$ (resp. $\log|\alpha - 2|$) as $\alpha \to -1$ (resp. 2), and goes to zero as $\alpha \to 0, 1, \infty$. The first approach is indicated in the appendix.

[4] In fact, a simple change of coordinates to $\tilde{z} = \frac{1}{z}$ shows that $\eta(\alpha)$ is identically zero.

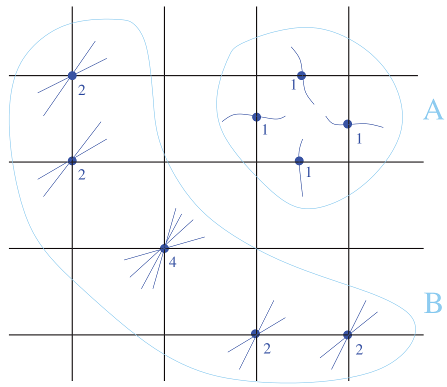

Figure 3

6.5. Interpretation of the integrals. From the generic non-triviality of $\psi(\alpha)$, we know that

$$\int_{(\check{\mathcal{E}}_1')^{\sim}} (\log|\mathfrak{z}|)\iota_1^* \omega_{\mathrm{K}} \tag{6.11}$$

is non-zero for generic α, β. We will show that this integral has meaning as an invariant of $\mathrm{Z}_{\alpha,\beta}$ in roundabout fashion, by first exhibiting it as an invariant of a related cycle on $E_\alpha \times E_\beta$.

For generic μ, the image $\check{\mathcal{E}}_\mu$ of $(\check{\mathcal{E}}_\mu')^{\sim}$ in $\check{\mathrm{K}}_{\alpha,\beta}$ is a curve with intersection numbers shown in Figure 3, where the horizontal and vertical lines have the same meaning as in Figure 1. Obviously its normalization is elliptic, with four smooth branch points over the conic $\mathcal{C}_\mu$ at the points of type (A) in the figure. Its preimage $\mathcal{D}_\mu$ in $E_\alpha \times E_\beta$ is an irreducible curve with singularities at the points of type (B); and its normalization can be thought of as a double cover of the normalization of $\check{\mathcal{E}}_\mu$, branched at the points lying over these singularities. An easy Riemann–Hurwitz calculation shows that $\widetilde{\mathcal{D}_\mu}$ has genus 7.

As $\mu \to 1$, $\mathcal{D}_\mu$ and $\check{\mathcal{E}}_\mu$ each acquire a new node, one mapping to the other: $\mathcal{O} \mapsto (1,1)$. The local description (at the nodes) of the map $\mathcal{D}_1 \twoheadrightarrow \check{\mathcal{E}}_1$ is "$z \mapsto z^2$" on each branch separately. (Note that $\widetilde{\mathcal{D}_1}$ has genus 6.) Therefore, the pullback $\tilde{\mathfrak{z}} \in \mathbb{C}(\widetilde{\mathcal{D}_1})^*$ of the function $\mathfrak{z}$ on $(\check{\mathcal{E}}_1)^{\sim}$ pushes forward to $\mathcal{D}_1$ to yield a K_1-class: its double-zero and double-pole cancel at $\mathcal{O}$. That is, $\mathrm{W}_{\alpha,\beta} := \pi_{2,*}(\widetilde{\mathcal{D}_1}, \tilde{\mathfrak{z}})$ belongs to $\mathrm{CH}^2(E_\alpha \times E_\beta, 1)$. Further, the real regulator current $\log|\tilde{\mathfrak{z}}|\delta_{\mathcal{D}_1}$ pairs against $\omega \in \Gamma(E_\alpha \times E_\beta, A^{1,1})_{d\text{-closed}}$ from (6.6) to yield

(a) an honest invariant of this K_1-class $\mathrm{W}_{\alpha,\beta}$; and

(b) twice the value of the integral (6.11), since ω and $\tilde{\mathfrak{z}}$ are both invariant under the involution flipping $\widetilde{\mathcal{D}_1}$ over $(\check{\mathcal{E}}_1)^{\sim}$.

Consider the diagram

$$\begin{tikzcd}
 & \tilde{\mathrm{K}}_{\alpha,\beta} \arrow[dl, two heads, "\pi_2"'] \arrow[dr, two heads, "\pi_1"] & & \tilde{\mathrm{K}}'_{\alpha,\beta} \arrow[dl, two heads, "\pi_1'"'] \arrow[dr, two heads, "\pi_2'"] & \\
E_\alpha \times E_\beta \arrow[dr, two heads, "2:1"] & & \mathrm{K}_{\alpha,\beta} \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, two heads] & & \mathrm{X}_{a,b} \arrow[dl, two heads, "2:1"] \\
 & \check{\mathrm{K}}_{\alpha,\beta} & \check{\mathrm{K}}'_{\alpha,\beta} & \check{\mathrm{K}}''_{\alpha,\beta} &
\end{tikzcd}$$

in which $\mathrm{X}_{a,b}$ is the Shioda–Inose $K3$, $\check{\mathrm{K}}''_{\alpha,\beta}$ its quotient by the Nikulin involution, and the relationship between the two sets of parameters is given by

$$J(E_\alpha) + J(E_\beta) = a^3 - b^2 + 1, \quad J(E_\alpha) \cdot J(E_\beta) = a^3.$$

The preimage of $\mathcal{D}_1$ under π_2 consists of $\widetilde{\mathcal{D}_1}$ and $\mathcal{W}$ (an exceptional $\mathbb{P}^1$ with coordinate "w") meeting at $w = 0$ and $w = \infty$ on $\mathcal{W}$. The map π_1 pushes this down to $\mathcal{E}_1 = (\check{\mathcal{E}}_1')^\sim \cup \widetilde{(1,1)}$, where the map from $\mathcal{W}$ to $\widetilde{(1,1)}$ is given by $w \mapsto w^2$. Setting

$$\tilde{\mathrm{Z}}_{\alpha,\beta} := (\widetilde{\mathcal{D}_1}, \mathfrak{z}) + (\mathcal{W}, w^2) \in \mathrm{CH}^2(\tilde{\mathrm{K}}_{\alpha,\beta}, 1), \tag{6.12}$$

we have $\pi_{1,*}(\tilde{\mathrm{Z}}_{\alpha,\beta}) = 2\mathrm{Z}_{\alpha,\beta}$ and $\pi_{2,*}(\tilde{\mathrm{Z}}_{\alpha,\beta}) = \pi_{2,*}(\widetilde{\mathcal{D}_1}, \mathfrak{z}) = \mathrm{W}_{\alpha,\beta}$. By (a), (b), and functoriality of $r_{2,1}$, it now follows that the pairing (6.11) indeed computes the regulator of $\mathrm{Z}_{\alpha,\beta}$.

What about cycles on $\mathrm{X}_{a,b}$? The 2:1 birational correspondence provided by π_1' and π_2' identifies its alternate fibration with the elliptic fibration of $\mathrm{K}_{\alpha,\beta}$ (generically in 2:1 étale fashion). More precisely, we have a diagram

$$\begin{tikzcd} \mathrm{K}_{\alpha,\beta} \arrow[d] & \mathrm{X}_{a,b} \arrow[l, dashed, "2:1"'] \arrow[d] \\ \mathbb{P}^1_\mu & \mathbb{P}^1_\theta \arrow[l] \end{tikzcd}$$

in which (a) the bottom map is of the form $\theta \mapsto q\theta + p$ (with p and q constants dependent on α, β), and (b) the six I_1 fibers of $\mathrm{X}_{a,b} \to \mathbb{P}^1$ exactly match the six I_2 fibers of $\mathrm{K}_{\alpha,\beta} \to \mathbb{P}^1_\mu$; see [7]. On $\tilde{\mathrm{K}}'_{\alpha,\beta}$ there is a K_1-cycle $\tilde{\mathrm{Z}}'_{\alpha,\beta}$ supported on an I_2, $\pi'_{1,*}$ of which is $2\mathrm{Z}_{\alpha,\beta}$. By (b), its push-forward

$$\mathcal{Z}_{a,b} := \pi'_{2,*}(\tilde{\mathrm{Z}}'_{\alpha,\beta}) \in \mathrm{CH}^2(\mathrm{X}_{a,b}, 1)$$

is supported on an I_1 fiber D. Imitating the argument around (6.12), one sees that $\mathcal{Z}_{a,b}$ is both indecomposable and the image of a generator of $\mathrm{CH}^2_D(\mathrm{X}_{a,b}, 1)$, hence the cycle we seek.

Summing up, we have the following theorem.

Theorem 6.2. *The real and imaginary parts of the (multivalued) integral* (6.11) *compute $r_{2,1}$ (cf.* (6.1)*) for the three (multivalued)*[5] *families of cycles $\mathrm{W}_{\alpha,\beta} \in \mathrm{CH}^2(E_\alpha \times E_\beta, 1)$, $\mathrm{Z}_{\alpha,\beta} \in \mathrm{CH}^2(\mathrm{K}_{\alpha,\beta}, 1)$, and $\mathcal{Z}_{a,b} \in \mathrm{CH}^2(\mathrm{X}_{a,b}, 1)$. For (α, β) (resp. (a, b)) in a real-analytic Zariski-open subset of $\mathbb{C}^2$, this integral is non-zero, and the cycles are therefore regulator-indecomposable. The same result holds along the locus $\alpha = \beta$ (resp. $4a^3 = (a^3 - b^2 + 1)^2$).*

Appendix to Section 6. Here we perform the analytic estimate which establishes the limiting assertion in Section 6.4, for $\alpha \to 1$. It will suffice to consider the behavior of the integral in a fixed neighborhood of one of the points (we use $\gamma = +1$) where zeroes and poles collide. Write $\chi = \alpha - 1$, $\gamma^2 = \zeta + 1$, and let $D_r(c)$ denote the open disk about c of radius r.

We may leave out the polynomial factors with no zero or pole approaching $\zeta = 0$, and approximate the locations of zeroes and poles to the lowest order required to distinguish them.

[5] Again, the multivaluedness arises from the action of monodromy sending $\mathfrak{z} \mapsto \mathfrak{z}^{-\pm 1}$.

The problem is then to show that

$$\int_{|\zeta|<\frac{1}{2}} \frac{(\zeta-3\chi)\overline{(\zeta+3\chi)}(\zeta+\chi)\overline{(\zeta-\chi)}\log|\mathfrak{z}|d\zeta\wedge d\bar{\zeta}}{|\zeta-(\chi+\chi^2)|\,|\zeta-(\chi-\chi^2)|\,|\zeta+(\chi+\chi^2)|\,|\zeta+(\chi-\chi^2)|\,|\zeta-i\sqrt{3}\chi|\,|\zeta+i\sqrt{3}\chi|} \tag{6.13}$$

limits to

$$\int_{|\zeta|<\frac{1}{2}} \log|\mathfrak{z}|\frac{d\zeta\wedge d\bar{\zeta}}{|\zeta|^2}$$

as $\chi\to 0^+$ along the real axis. Given $\epsilon>0$, and taking $0<\chi<\epsilon/3$, it is obvious that the integrand in (6.13) converges uniformly on $\epsilon<|\zeta|<1/2$. We claim that the remaining part $\int_{|\zeta|<\epsilon}$ of the integral, independently of $\chi\in(0,\frac{\epsilon}{3})$, is bounded by $1000\pi\epsilon$. This will prove the desired convergence.

To verify the claim, we first remark that $\log|\mathfrak{z}|$ is zero for all $\zeta\in\mathbb{P}^1(\mathbb{R})$; in fact, we shall just use that $|\log|\mathfrak{z}||<|\zeta|$. Next, note that on the complement in $D_\epsilon(0)$ of the four disks $D_{\chi/2}(\chi)$, $D_{\chi/2}(-\chi)$, $D_{\chi/2}(i\sqrt{3}\chi)$, $D_{\chi/2}(-i\sqrt{3}\chi)$,

$$\frac{|\zeta+3\chi|\,|\zeta+\chi|}{|\zeta+\chi+\chi^2|\,|\zeta+\chi-\chi^2|}=\frac{|\lambda+2\chi|\,|\lambda|}{|\lambda+\chi^2|\,|\lambda-\chi^2|}=\frac{|1+\frac{2\chi}{\lambda}|}{|1+\frac{\chi^2}{\lambda}|\,|1-\frac{\chi^2}{\lambda}|}$$

(where $\lambda:=\zeta+\chi$) is bounded by 6, since $\left|\frac{2\chi}{\lambda}\right|\le 4$, $\left|\frac{\chi^2}{\lambda}\right|\le 2\chi$ and we are assuming χ is small. The same is true for

$$\frac{|\zeta-3\chi|\,|\zeta-\chi|}{|\zeta-\chi+\chi^2|\,|\zeta-\chi-\chi^2|};$$

and similarly,

$$\frac{|\zeta|^2}{|\zeta-i\sqrt{3}\chi|\,|\zeta+i\sqrt{3}\chi|}$$

is bounded by 9. So the integral over $D_\epsilon(0)\backslash\{4 \text{ disks}\}$ is bounded by

$$\int_{|\zeta|<\epsilon} 9\cdot 6^2\cdot\frac{|d\zeta\wedge d\bar{\zeta}|}{|\zeta|}=324\cdot 2\pi\int_0^\epsilon\frac{r\,dr}{r}<650\pi\epsilon.$$

Now consider (say) the right half of $D_{\chi/2}(-\chi)$: here the absolute value of the integrand, apart from the $\frac{1}{|\lambda-\chi^2|}$, is

$$\frac{|\lambda-4\chi|\,|\lambda-2\chi|}{|\lambda-(2\chi+\chi^2)|\,|\lambda-(2\chi-\chi^2)|}\cdot\frac{|\lambda|}{|\lambda+\chi^2|}\cdot\frac{|\lambda+2\chi|\,|\lambda-\chi|}{|\lambda-i\sqrt{3}\chi|\,|\lambda+i\sqrt{3}\chi|}\le 6\cdot 1\cdot\frac{10}{3}\le 20.$$

We have then

$$20\int_{D_{\chi/2}(0)\cap\Re(\lambda)>0}\frac{|d\lambda\wedge d\bar{\lambda}|}{|\lambda-\chi^2|}\le 20\int_{D_\chi(0)}\frac{|d\lambda\wedge d\bar{\lambda}|}{|\lambda|}=40\pi\chi<\frac{40}{3}\pi\epsilon,$$

together with similar estimates on three other half-disks. The estimates for $D_{\chi/2}(\pm i\sqrt{3}\chi)$ are each $\frac{250}{3}\pi\epsilon$. Adding everything from inside and outside the four disks, we are safely under $1000\pi\epsilon$.

We briefly address the situation at the other four points where poles in (6.8) collide. The most striking case is that of $\alpha \to 2$. Substituting $\alpha = 2$ in $\int_{\mathbb{P}^1} \log|\mathfrak{z}|\Re(\imath^*\omega_{\mathrm{K}})$ yields the convergent integral

$$-24\int_{\mathbb{P}^1} \frac{\log\left|\frac{\gamma+i}{\gamma-i}\right| r\sin(\theta)}{|\gamma^2+1|\,|\gamma^2-2|\,|2\gamma^2-1|}dx\wedge dy.$$

Writing $\chi = \alpha - 2$, $\gamma^2 = \zeta - 1$, to show this is $\lim_{\alpha\to 2}\psi(\alpha)$ one must check (in analogy to the above argument for (6.13)) that

$$\int_{|\zeta|<\frac{1}{2}} \frac{|\zeta+3i\sqrt{\chi}|^2|\zeta-3i\sqrt{\chi}|^2\log|\zeta|d\zeta\wedge d\bar{\zeta}}{|\zeta|\,|\zeta+3\chi|\,|\zeta-3\chi|\,|\zeta-3\sqrt{\chi}|\,|\zeta+3\sqrt{\chi}|} \tag{6.14}$$

limits to

$$\int_{|\zeta|<\frac{1}{2}} \log|\zeta|\frac{d\zeta\wedge d\bar{\zeta}}{|\zeta|}$$

as $\chi \to 0$. But this *fails*, due to the rapid convergence to ($\zeta =$) 0 of two of the poles; in fact, (6.14) diverges logarithmically.

For $\alpha \to -1$, the limiting of the factor $|\alpha+1| \to 0$ in (6.8) is no match for the convergence of seven poles each to ($\gamma =$) i and $-i$, again resulting in a logarithmic divergency for $\psi(\alpha)$. On the other hand, analyses similar to (but simpler than) that for $\alpha \to 1$ show $\lim_{\alpha\to 0}\psi(\alpha)$ and $\lim_{\alpha\to\infty}\psi(\alpha)$ to be convergent.

7. The transcendental regulator for a Picard-rank 20 $K3$

Here we specialize to the case (cf. Section 6.5)

$$\alpha = \frac{1}{2} = \beta, \quad a = 1, \quad b = 0,$$

in which case $E_\alpha, E_\beta \cong \mathbb{C}/\mathbb{Z}\langle 1, i\rangle$ are CM and $p = 3$, $q = -2$ (cf. [7]). The singular fibres are at $\theta = \pm\frac{1}{2}$ (type I_2) and ± 1 (type I_1) in $\mathrm{X} := \mathrm{X}_{1,0}$, and at $\mu = 2, 4$ (type I_4) and $1, 5$ (type I_2) in $\mathrm{K}_{1/2,1/2}$. Recalling that our original cycle on $\mathrm{K}_{\alpha,\beta}$ was supported over $\mu = 1$, which in this specialization has *remained* an I_2 fiber (hence preserving the cycle), its transform $\mathcal{Z} := \mathcal{Z}_{1,0}$ is supported over $\theta = 1$ (an I_1 fiber) in X.

To take a closer look at the fibration structure of X, we use its affine equation

$$2y^2 = w\underbrace{\left(w^2 + 2\{4\theta^3 - 3\theta\}w + 1\right)}_{=:Q_\theta(w)}$$

to sketch the families of branch points of the elliptic fibers shown in Figure 4. Here $r_\pm(\theta)$ are the roots of $Q_\theta(w)$, which are both negative real for $\theta \in [1,\infty)$, with $r_- = r_+^{-1}$. For purposes of constructing transcendental cycles, one should imagine all the branch points coalescing at $\theta = \infty$ since that fiber, an I_{12}^*, has trivial H_1.

In particular, considering the fiber over $\theta = 1$, the membrane Γ we use for the transcendental regulator computation must bound on the indicated cycle $\partial\Gamma = T_{\mathcal{Z}}$ (see Figure 5),

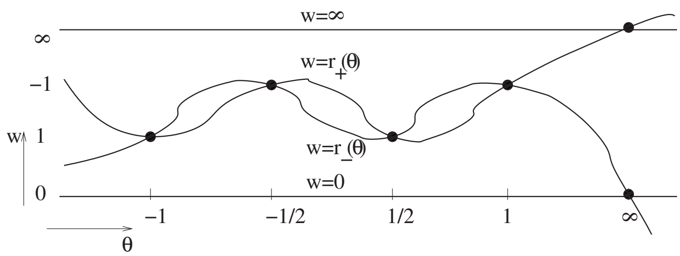

Figure 4

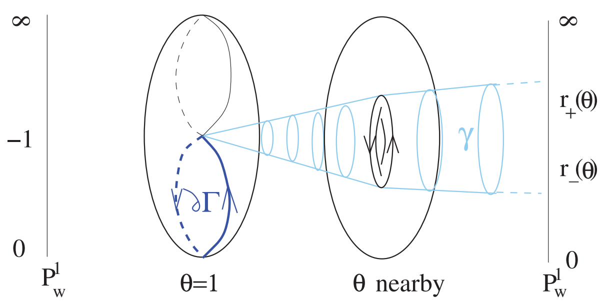

Figure 5

which is a double cover of the path $[-1, 0] \subset \mathbb{P}^1_w$. The transcendental 2-cycle γ is the family of double covers of $[r_-, r_+]$ as θ goes from 1 to ∞.

By basic residue theory the holomorphic $(2, 0)$-form on X is given by

$$\omega_0 = \frac{dw \wedge d\theta}{y}$$

in the affine coordinates. If

$$\int_\gamma \omega_0 = 2\sqrt{2} \int_{\theta=1}^{\infty} \left(\int_{r_-(\theta)}^{r_+(\theta)} \frac{dw}{\sqrt{wQ_\theta(w)}} \right) d\theta \ (> 0)$$

is one transcendental period, then using the automorphism

$$j : \mathrm{X} \to \mathrm{X}, \quad (w, y, \theta) \mapsto (-w, -iy, -\theta),$$

we have

$$\int_{j(\gamma)} \omega_0 = \int_\gamma j^* \omega_0 = i \int_\gamma \omega_0.$$

Normalizing ω_0 to $\omega := \omega_0 / (\int_\gamma i\omega_0)$, we find that $\Phi_{2,1}$ is described by

$$\mathrm{CH}^2(\mathrm{X}, 1) \to \mathbb{C}/\mathbb{Z}[i], \quad \mathcal{Z} \mapsto \int_\Gamma \omega,$$

which for our particular cycle is

$$\kappa := \int_\Gamma \omega = 2 \int_{\theta=1}^{\infty} \int_{w=r_+(\theta)}^{0} \omega = \frac{\int_1^\infty \int_{r_+(\theta)}^0 \frac{dw}{\sqrt{-wQ_\theta(w)}} d\theta}{\int_1^\infty \int_{r_-(\theta)}^{r_+(\theta)} \frac{dw}{\sqrt{wQ_\theta(w)}} d\theta} \in \mathbb{R}_+. \tag{7.1}$$

We have proved the following result.

Theorem 7.1. *Let $\mathcal{Z} \in \mathrm{CH}^2(\mathrm{X}_{1,0}, 1)$ be the image of a generator of $\mathrm{CH}^2_D(\mathrm{X}_{1,0}, 1)$, for D one of the two I_1 fibers in the alternate fibration. Then the transcendental regulator $\Phi_{2,1}(\mathcal{Z})_{\mathbb{Q}} \in \mathbb{C}/\mathbb{Q}[i]$ is non-zero if and only if $\kappa \notin \mathbb{Q}$.*

The situation is highly reminiscent of a computation by Harris [19] of the Abel–Jacobi map for the Ceresa cycle of the Fermat quartic curve. In that case, a computer computation suggested that the comparable invariant $\kappa' \in \mathbb{R}/\mathbb{Q}$ was non-trivial. This would have implied that the cycle was non-torsion modulo rational equivalence, a fact later proved by Bloch [3] using his ℓ-adic AJ map. Since the Fermat Jacobian is defined over $\bar{\mathbb{Q}}$, the Bloch–Beilinson conjecture predicts injectivity of the usual AJ map, and hence the irrationality of κ'. One might, in conclusion, speculate that a similar story unfolds here.

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