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T-Duality for Orientifolds and Twisted KR-Theory

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Abstract. D-brane charges in orientifold string theories are classified by the KR-theory of Atiyah. However, this is assuming that all O-planes have the same sign. When there are O-planes of different signs, physics demands a "KR-theory with a sign choice" which up until now has not been studied by mathematicians (with the unique exception of Moutuou, who did not have a specific application in mind). We give a definition of this theory and compute it for orientifold theories compactified on S^1 and T^2. We also explain how and why additional "twisting" is implemented. We show that our results satisfy all possible T-duality relationships for orientifold string theories on elliptic curves, which will be studied further in subsequent work.

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1. Introduction

The purpose of this paper is to describe the versions of K-theory needed to describe T-duality for orientifolds, and to compute and analyze them in a few simple but important cases. By *orientifolds* we mean spacetimes of the form $\mathbb{R}^k \times X$, where X is a smooth 10-k dimensional (oriented) manifold equipped with an involution, ι, which defines the orientifold structure.[1] *Orientifold string theories* are defined by sigma-models with target space an orientifold (X, ι), where the fundamental strings are *equivariant* maps $\varphi\colon\ \Sigma \to X$, so that $\iota \circ \varphi = \varphi \circ \Omega$. Here Σ is an oriented 2-manifold, possibly with boundary (the case of open strings), called the

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[1]Note that in some of the literature, the word "orientifold" is used to denote the quotient space X/ι, but it is really essential to keep track of the *pair* (X, ι) and not just the quotient.

string worldsheet, and Ω, called the *worldsheet parity operator*, is an orientation-reversing involution on Σ. We require Σ/Ω, though not necessarily Σ itself, to be connected. (Thus an allowable possibility is $\Sigma = \Sigma_0 \amalg \overline{\Sigma}_0$, where Σ_0 is a connected oriented surface, $\overline{\Sigma}_0$ is the same surface with orientation reversed, and Ω interchanges the two.) See for example [13]; there some extra twisting data, which we are ignoring for the moment, is also taken into account, and the notation is slightly different.

As described in [29,37], D-branes in string theories are classified by K-theory, where the relevant type of K-theory depends on the string theory being considered. Since the physics of T-dual theories is indistinguishable, the groups classifying stable D-branes in two T-dual theories must be isomorphic. This led Bouwknegt, Evslin, and Mathai, and later Bunke and Schick, to describe the T-duality between the type IIB theory on a spacetime X that is a circle bundle over base Z, with H-flux H, and the type IIA theory on a dual circle bundle $\widetilde{X}$ over Z, with dual H-flux $\widetilde{H}$, as an isomorphism of twisted K-theories:

$$K^*(X, H) \cong K^{*+1}(\widetilde{X}, \widetilde{H}). \tag{1}$$

In the above equation

$$c_1(X) = \widetilde{\pi}_*(\widetilde{H}) \text{ and } c_1(\widetilde{X}) = \pi_*(H), \tag{2}$$

where $c_1(X) \in H^2(X; \mathbb{Z})$ is the first Chern class and $\pi_*\colon H^k(X) \to H^{k-1}(Z)$ is the Gysin push-forward map which in terms of de Rham cohomology is defined by integration along the fiber [9,12]. This was later generalized to the case where X is a T^n-bundle in [11,28].

For orientifolds, D-brane charges are classified by KR-theory [21,22], [37, Section 5.2], which we will review in Section 2. One benefit of using KR-theory is that it can be viewed as a sort of *universal K-theory*. It is "universal" in the sense that the K-theories KU for the type II theories, KO for the type I theory, and KSC for the type $\widetilde{\mathrm{I}}$ theory can all be built out of KR. This shows that by keeping track of the appropriate involution ι, one does not need to make a choice of which type of K-theory to use, and it is already accounted for just by using KR-theory. However, KR-theory has some immediate limitations that prevent us from generalizing a topological description of T-duality like equation (1) to orientifolds.

The first problem is that it is not immediately clear how to twist KR-theory, or even what is meant by H-flux. Traditionally, $H \in H^3(X; \mathbb{Z})$, so there is no reason to expect H to be equivariant. Another related issue is that orientifold theories involve extra information, which is not just topological, relevant to the stable O-plane charges. This issue is already apparent when studying circle orientifolds, even though the dimension of a circle is too low to have to worry about more general twistings.

In Section 3 we will review T-duality between all possible circle orientifolds and the classification of stable D-branes in the different theories. The T-dual of the type I theory on a circle (which is a type IIB orientifold on the circle with trivial involution) is a type IIA orientifold on a circle with involution given by reflection (referred to as the type IA or type I′ theory). The T-dual to the type IIB theory on the circle with the antipodal map (sometimes referred to as the type $\widetilde{I}$ theory) is also a type IIA orientifold on the circle with involution given by reflection (often called the type $\widetilde{\text{IA}}$ theory). The compactification manifolds for both the type IA and $\widetilde{\text{IA}}$ theories are topologically equivalent, with the difference being the charges of the O-planes at the two fixed points. There are physical descriptions of the classification of D-branes in the two theories [7,33]; however, we are not aware of any mathematical description for the classification of D-branes in the type $\widetilde{\text{IA}}$ theory via KR-theory. In fact, a topological invariant such as KR-theory cannot pick up the difference between the type IA and $\widetilde{\text{IA}}$ compactifications since the distinction is non-topological. In Section 4 we propose a variant of KR-theory, which we call KR-theory with a sign choice, that can distinguish between the two cases, giving a mathematical description of the brane charges in the type $\widetilde{\text{IA}}$ theory. We then give all possible sign choices for KR-theories for orientifolds of 2-tori.

A word about our sign convention: we say that an O-plane has positive sign, or is an O^+-plane, if the Chan–Paton bundle on it has orthogonal type, and has negative sign, or is an O^--plane, if the Chan–Paton bundle on it has symplectic type. The sign decorations that we attach to KR-theory follow the same convention. Since a tensor product of an orthogonal bundle with a symplectic bundle is symplectic, while the tensor product of two symplectic bundles is orthogonal, signs multiply as one would expect. This convention is the same as the one made by Witten in [38], but is the *reverse* of the convention made by Gao and Hori in [19]. Both sign conventions are in general use, but we feel that the multiplication rule indicates that this one is preferable, even though it means (as Witten points out) that the tadpoles are of opposite sign.

When we move up in dimension to 2-tori, the sign choice is no longer enough to account for all possible orientifold theories. In particular, KR-theory with a sign choice cannot describe the type I theory without vector structure [38]. For this we need to include more general twists of KR-theory, which will be discussed in Section 5. Here we use physics to motivate which KR-theories should be isomorphic, and check the results via topology. The twist applied to KR-theory is related to the geometry of its T-dual theory and is described in [15]. The purpose of the current paper is to describe the relevant twisted KR-theories needed to give the geometric interpretation in [15].

One of our motivations for a detailed analysis of T-duality via orientifold plane charges in KR-theory was the special case of $c=3$ Gepner models as studied in [6]. The authors of that paper used simple current techniques in CFT to construct the charges and tensions of Calabi–Yau orientifold planes, though a K-theoretic interpretation was missing. Although the interpretation of brane charges in KR-theory is sensitive to regions of stability, this K-theoretic interpretation does not depend on the specific structure of $c=3$ Gepner models, nor even on a rational conformal field theoretic description. These results should be contrasted with the recent work [17] where a twisted equivariant K-theory description of the D-brane charge content for WZW models is provided (see also [10] for examples which make explicit the isomorphism with topological K-theory in the case of some Gepner models). Work in progress seeks to establish an isomorphism between a suitable (real) variant of twisted equivariant K-theory, sufficient to capture orientifold charge content, and our KR-theory with sign choices for Gepner models. As a side-effect, such an isomorphism will then permit computation of KR-theory for complicated Calabi–Yau manifolds through a simpler computation at the Gepner point.

After the first version of this paper was completed, we became aware of the work of Moutuou [30–32] on groupoid twisted K-theory, which includes our KR-theory with a sign choice as a special case. Indeed, Moutuou's classification of possible twists of KR coincides with ours, though his point of view and motivation were quite different.

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2. Review of Classical KR-Theory

Let X be a locally compact space (in most physical situations it will be a smooth manifold) with involution ι. A Real vector bundle on X (in the sense of Atiyah [3]) is a complex vector bundle $p\colon\ E\to X$ together with a conjugate-linear vector bundle isomorphism $\varphi\colon\ E\to E$ such that $\varphi^2=1$ and φ is compatible with ι, in the sense that $p\circ\varphi=\iota\circ p$. $\mathrm{KR}(X)$ is the group of pairs of Real vector bundles (E,F) on X (with compact support) modulo the equivalence relation

$$(E,F)\sim(E\oplus H,F\oplus H), \tag{3}$$

for any Real vector bundle H. Note that $\mathrm{KR}(X)$ depends on the involution ι even though it is not explicitly stated. The compact support condition means that we can choose E and F to be trivialized off a sufficiently large ι-invariant compact set, with φ off this compact set being standard complex conjugation on a trivial bundle.

To define the higher KR-groups, $\mathrm{KR}^{-j}(X)$, we must first introduce some notation. Let $\mathbb{R}^{p,q}=\mathbb{R}^p+i\mathbb{R}^q$, where the involution is given by complex conjugation, and $S^{p,q}$ be the $p+q-1$ sphere in $\mathbb{R}^{p,q}$. *Caution*: In this notation, the roles of p and q are the reverse of those in the notation used by Atiyah in [3] but the same as the notation in [7,22,27,33]. Then we can define

$$\mathrm{KR}^{p,q}(X)=\mathrm{KR}(X\times\mathbb{R}^{p,q}).$$

This obeys the periodicity condition

$$\mathrm{KR}^{p,q}(X) \cong \mathrm{KR}^{p+1,q+1}(X),$$

so $\mathrm{KR}^{p,q}$ only depends on the difference $p-q$ and we can define

$$\mathrm{KR}^{q-p}(X) = \mathrm{KR}^{p,q}(X).$$

$\mathrm{KR}^{-j}(X)$ is periodic with period 8.

When ι is the trivial involution, the Reality condition is equivalent to E being the complexification of a real bundle. Thus KR gives a classification of real vector bundles and we find

$$\mathrm{KR}^{-j}(X) \cong \mathrm{KO}^{-j}(X), \tag{4}$$

when ι is trivial [3, p. 371]. Complex K-theory can also be obtained from KR-theory using

$$\mathrm{KR}^{-j}(X \times S^{0,1}) = \mathrm{KR}^{-j}(X \amalg X) \cong K^{-j}(X), \tag{5}$$

where the involution exchanges the 2 copies of X [3, Proposition 3.3]. And as shown by Atiyah [3, Proposition 3.5], $\mathrm{KR}^{-j}(X \times S^{0,2}) \cong \mathrm{KSC}^{-j}(X)$, the self-conjugate K-theory of Anderson [2] and Green [20], which is periodic with period 4.

In fact, when the involution ι has no fixed points, there is a spectral sequence ((11) below), whose E_2-term is 4-periodic, converging to $\mathrm{KR}^{-j}(X)$. This motivated Karoubi and Weibel [25, Proposition 1.8] to assert that $\mathrm{KR}^{-j}(X)$ is always 4-periodic when the involution is free, but in general this is not the case (unless one inverts the prime 2). The groups $\mathrm{KR}^{-j}(S^{0,4})$ provide a counterexample.

When X is compact and X^ι is non-empty, the inclusion of an ι-fixed basepoint into X is equivariantly split, so the reduced KR-groups, $\widetilde{\mathrm{KR}}^{-j}(X)$, are defined such that

$$\mathrm{KR}^{-j}(X) \cong \widetilde{\mathrm{KR}}^{-j}(X) \oplus \mathrm{KR}^{-j}(\mathrm{pt}).$$

We will write simply KR^{-j} or KO^{-j} for $\mathrm{KR}^{-j}(\mathrm{pt})$. When $Y \subseteq X$ is closed and ι-invariant, we can define the relative KR-theory as

$$\mathrm{KR}^{-j}(X, Y) \cong \widetilde{\mathrm{KR}}^{-j}(X/Y).$$

As we will discuss in the following section, this is the relevant group for classifying D-brane charges.

3. Orientifolds on a Circle and T-Duality

In this section we will consider orientifold string theories with target space (S^1, ι), where we view S^1 as the unit circle in $\mathbb{R}^2$ and where the involution ι comes from a linear involution on $\mathbb{R}^2$. Since linear involutions are classified by the dimension of the (-1)-eigenspace, there are (up to isomorphism) exactly three possibilities for (S^1, ι): the trivial involution corresponding to $S^{2,0}$, reflection corresponding to $S^{1,1}$, and the antipodal map corresponding to $S^{0,2}$. $S^{2,0}$ and $S^{0,2}$ only support the type IIB theory since the involution is orientation preserving, while $S^{1,1}$ only supports the type IIA theory since the involution is orientation reversing.

The type IIB theory on $S^{2,0}$ is the type I theory compactified on a circle. It is known to be T-dual to the type IIA theory on $S^{1,1}$, sometimes referred to as the type IA or I′ theory [7,22,33]. The type IIB theory on $S^{0,2}$ is often called the type $\widetilde{\text{I}}$ theory and is T-dual to the type $\widetilde{\text{IA}}$ theory [38, Section 6.2], [19, Section 7.1].

In this section we will review these T-duality relations and their K-theoretic descriptions. The lack of a mathematical description for the K-theory description of the type $\widetilde{\text{IA}}$ theory will motivate the definition for a variant of KR-theory given in Section 4. Before describing the various T-dualities we will review how D-branes are classified by K-theory.

3.1. CLASSIFICATION OF D-BRANES BY KR-THEORY

D-branes on orientifolds (X, ι), where X is a smooth manifold and ι is an involution on X, are classified by pairs of vector bundles on X (the Chan–Paton bundles), each with conjugate-linear involutions compatible with ι, modulo creation and annihilation of charge zero D-brane systems (as in Equation (3)). So D-branes in orientifolds are classified by KR-theory [37, Section 5.2].

More generally, when we compactify string theory on an m-dimensional space M, so that the spacetime manifold is $\mathbb{R}^{10-m,0} \times M$, we are interested in the charges of D-branes in the non-compact dimensions. So we want to consider D-branes of codimension $9-m-p$ in $\mathbb{R}^{9-m,0}$. These can arise from both Dp-branes located at a particular point in M or higher dimensional D-branes that wrap non-trivial cycles in M. Furthermore, we only want to consider branes with finite energy, so we want to classify bundle pairs that are asymptotically equivalent to the vacuum in the transverse space $\mathbb{R}^{9-m-p,0}$. Mathematically this means we want to add a copy of M at infinity, i.e., take (the one-point compactification of $\mathbb{R}^{9-m-p,0}$) $\times M$, and consider bundles on $S^{10-m-p,0} \times M$ that are trivialized on the copy of M at infinity. Such bundles are classified by $\mathrm{KR}^{-i}(S^{10-m-p,0} \times M, M)$; the index i depends on the string theory and involution being considered. For purposes of calculation it is useful to relate this to the KR-theory of M.

PROPOSITION 1.

$$\mathrm{KR}^{-i}(S^{10-m-p,0} \times M, M) \cong \mathrm{KR}^{p+m-9-i}(M).$$

Proof. Note that by excision,

$$\mathrm{KR}^{-i}(S^{10-m-p,0} \times M, M) \cong \mathrm{KR}^{-i}((S^{10-m-p,0} \setminus \{\mathrm{pt}\}) \times M)$$

$$\cong \mathrm{KR}^{-i}(\mathbb{R}^{9-m-p,0} \times M)$$
$$\cong \mathrm{KR}^{p+m-9-i}(M).$$

□

Thus Dp-branes are classified by $\mathrm{KR}^{p+m-9-i}(M)$. It is important to keep track of the index. This point is often overlooked when studying D-branes in the (non-orientifold) type II theories, which are classified by KU-theory, since KU-theory has period 2 and only the parity of the index matters.

In what follows we will also have to study the charges of the O-*planes*, the components of the fixed set of the involution on spacetime.[2] The restriction of a Chan–Paton bundle to an O-plane must have either a real (positive) or symplectic (negative) structure. The classification of D-branes via KR-theory is only valid when all O-planes have positive charge, and breaks down when different O-planes have different charges. It is this breakdown that leads us to define KR-theory with a sign choice in Section 4.

3.2. THE TYPE I THEORY AND ITS T-DUAL

The type I theory compactified on a circle is formally identical to the type IIB orientifold theory compactified on $S^{2,0}$. Consider the bosonic fields in the type IIB theory compactified on $S^{2,0}$,

$$X = X_L + X_R.$$

The worldsheet parity operator reverses the orientation of the string, and so exchanges left-movers and right-movers. This leaves the bosonic fields invariant under Ω and therefore compatible with the trivial involution.

T-duality leaves the left-moving fields invariant, while reversing the sign of the right-moving fields, so the T-dual coordinates are

$$\tilde{X} = X_L - X_R.$$

Under the action of Ω, the T-dual coordinates transform as

$$\tilde{X} \mapsto -\tilde{X}.$$

This shows that the T-dual to the type I theory compactified on a circle must be the type IIA theory (since T-duality exchanges types IIA and IIB) mod the action of Ω combined with the spacetime involution that reflects the compact dimension. This is the type IIA theory compactified on $S^{1,1}$. In the literature it is often referred to as the type IA (or I′) theory. One could also show these two theories

[2]This terminology is unfortunate but standard; O-planes in general orientifold theories do not have to be planes. They can have more complicated topology.

Table I. D-brane charges in the type I theory compactified on a circle and the type IA theory

Dp-brane	$D8$	$D7$	$D6$	$D5$	$D4$	$D3$	$D2$	$D1$	$D0$	$D(-1)$	Type I on S^1	Type IIA on $S^{1,1}$
KO^{p-8}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$(p+1)$-brane wrapping $S^{2,0}$	Unwrapped p-brane
KO^{p-9}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	Unwrapped p-brane	$(p+1)$-brane wrapping $S^{1,1}$

are T-dual to one another by showing there is no momentum, but winding in the $S^{2,0}$ direction, while $S^{1,1}$ has momentum, but no winding.

The type I theory on S^1 has a space filling $O9^+$-plane wrapping the compact dimension. The T-dual type IA theory has $2O8^+$-planes located at the 2 fixed points of $S^{1,1}$. Recall that we use the plus sign to denote that the O-planes have negative D-brane charge and require the addition of D-branes to obtain a zero charge system.

Dp-brane charges in the type I theory compactified on a circle are classified by

$$\begin{aligned}\mathrm{KR}(S^{9-p,0}\times S^{2,0},S^{2,0}) &\cong \mathrm{KO}^{p-8}(S^1)\\ &\cong \mathrm{KO}^{p-8}\oplus\mathrm{KO}^{p-9}.\end{aligned}\tag{6}$$

The second factor in the last line of Equation (6) corresponds to Dp-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from branes wrapping S^1. The complete brane content is given in Table I.

Since the type IA theory is obtained from the type I theory compactified on a circle by a T-duality, the relevant KR-theory is shifted in index by 1. Therefore, Dp-brane charges in the type IA theory are classified by

$$\begin{aligned}\mathrm{KR}^{-1}(S^{9-p,0}\times S^{1,1},S^{1,1}) &\cong \mathrm{KR}^{p-9}(S^{1,1})\\ &\cong \mathrm{KO}^{p-9}\oplus\mathrm{KO}^{p-8},\end{aligned}\tag{7}$$

where the second factor on the right-hand side corresponds to Dp-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from wrapped branes. The complete brane content is given in Table I. The fact that T-duality exchanges wrapped and unwrapped branes is described by the exchanged roles for KO^{p-8} and KO^{p-9} in the two theories.

The non-BPS torsion-charged branes are not stable at all points of the moduli space. $D0$-brane charge in the type I theory receives an integral contribution from a wrapped BPS $D1$-brane and a $\mathbb{Z}_2$ contribution from an unwrapped non-BPS $D0$-brane. K-theory accurately predicts the entire brane charge spectrum everywhere, in and out of the region of stability for the non-BPS branes, but the sources of the charges may vary at different points of the moduli space. For more details, see [7].

3.3. THE TYPE $\widetilde{\mathrm{I}}$ AND $\widetilde{\mathrm{IA}}$ THEORIES

The type $\widetilde{\mathrm{I}}$ theory is the type IIB orientifold $(\mathbb{R}^9 \times S^1, \iota)$ where ι is the spacetime involution that rotates S^1 by π radians. In our notation, this is the type IIB theory on $\mathbb{R}^{9,0} \times S^{0,2}$. The T-dual of the type $\widetilde{\mathrm{I}}$ theory is the type $\widetilde{\mathrm{IA}}$ theory [7]. As we saw in the last section, the type IA theory contains $2O8^+$-planes. The type $\widetilde{\mathrm{IA}}$ theory is obtained from the type IA theory by replacing one of the $O8^+$-planes with an $O8^-$-plane. Here an O^--plane is an O-plane with symplectic Chan–Paton bundle and positive D-brane charge. (Note that if there were $O8^-$-planes at both fixed points, then a charge 0 system would require the addition of anti-branes and would not be supersymmetric.) We will refer to the compactification circle as $S^{1,1}_{(+,-)}$. It is topologically equivalent to a compactification on $S^{1,1}$, in that there are 2 fixed points. However, the net O-plane charge is zero.

Dp-brane charges in the type $\widetilde{\mathrm{I}}$ theory are classified by

$$\mathrm{KR}(S^{9-p,0} \times S^{0,2}, S^{0,2}) \cong \mathrm{KSC}^{p-8}. \tag{8}$$

KSC does not split into pieces from wrapped and unwrapped branes as in the previous case. The authors of [7] were still able to determine which charges come from wrapped and unwrapped branes using what we know about T-duality, the type IA theory and $O8^{\pm}$-planes.

Since the type $\widetilde{\mathrm{IA}}$ theory is T-dual to the type $\widetilde{\mathrm{I}}$ theory, Dp-brane charges in the type $\widetilde{\mathrm{IA}}$ theory must also be classified by KSC^{p-8}. It is important to note that there is no mathematical description for this that we are aware of. There is only the physical reasoning, which requires the assumption of T-duality. Since the underlying topological space for the type $\widetilde{\mathrm{IA}}$ theory is $S^{1,1}$, we should be able to classify D-brane charges by some *twisted* KR-theory of $S^{1,1}$. This idea motivates the definition given in the following section.

4. KR-Theory with a Sign Choice

The compactification manifolds for the type IA and $\widetilde{\mathrm{IA}}$ theories are topologically equivalent, even taking the involution ι into account. Therefore, KR-theory cannot differentiate between them. These two physical theories are differentiated by the signs of the O-planes located at their fixed sets, so we must enhance KR-theory with this information.

Along with the space X and the action of a group G (in our case $\mathbb{Z}_2$), we must also include a sign choice, α, on the components of the fixed set. Physically this sign choice determines the type of O-plane at the different components of the fixed set. In other words, it is a choice of orthogonal or symplectic Chan–Paton bundles on the different components. Recall our convention that a $+$ choice corresponds to an orthogonal Chan–Paton bundle, and a $-$ choice to a symplectic one. Note that the fixed sets for the type IA and $\widetilde{\mathrm{IA}}$ theories both have 2 components, each a point. The type IA theory is the sign choice $\alpha = (+, +)$, while the type $\widetilde{\mathrm{IA}}$ theory is the sign choice $\alpha = (+, -)$. We define an extension of KR-theory that contains this information and that fits into an exact sequence as in Theorem 2 below.

Intuitively, KR_α theory is defined in terms of a generalization of Real vector bundles, namely pairs (E, Φ), where E is a complex vector bundle over a real space (X, ι), and $\Phi\colon E \to E$ is a conjugate-linear vector bundle automorphism, equivariant with respect to ι, and with Φ^2 given by multiplication by $+1$ on components of the fixed set with a $+$ sign, -1 on components of the fixed set with a $-$ sign. Of course, if all components of the fixed set have a $+$ sign and $\Phi^2 \equiv 1$, then this is just Atiyah's definition of a Real vector bundle. If all components of the fixed set have a $-$ sign and $\Phi^2 \equiv -1$, then this is the corresponding notion in the symplectic case (used to define the theory often called KH—this is the name introduced in [21], but the theory already appeared much earlier in [16]). But for sign choices with both signs present, it is not clear how changing Φ^2 changes the notion of Real_α vector bundle, or how to get from this rough definition to a theory satisfying Bott periodicity. So for all these reasons (as in [39] and the literature on twisted K-theory, for example), a rigorous definition of KR_α requires non-commutative geometry.

Therefore what we really do is to define $\mathrm{KR}_\alpha(X)$ to be the topological K-theory of a certain non-commutative Banach algebra $\mathcal{A}_\alpha(X)$. In what follows, $\mathcal{K}$ and $\mathcal{K}_\mathbb{R}$ denote the algebras of compact operators on an infinite-dimensional separable complex Hilbert space and an infinite-dimensional separable real Hilbert space, respectively. Before we get to the rigorous definition of $\mathrm{KR}_\alpha(X)$, we first show that the requisite Banach algebras exist, and study their topological K-theory.

THEOREM 1. *Let (X, ι) be a Real locally compact space with an assignment of signs α to the components of the fixed set. Then an algebra $A_\alpha(X)$ exists satisfying the following properties:*

(1) *$A_\alpha(X)$ is a* **real** *continuous-trace C^*-algebra whose complexification $A_\alpha(X) \otimes_\mathbb{R} \mathbb{C}$ has spectrum X and trivial Dixmier–Douady invariant, and for which the induced action σ of $Gal(\mathbb{C}/\mathbb{R})$ on X is the given involution on X.*
(2) *The quotient of $A_\alpha(X)$ associated to any component Y^+ of X^G with positive sign choice is Morita equivalent (over Y^+) to $C_0^\mathbb{R}(Y^+)$.*
(3) *The quotient of $A_\alpha(X)$ associated to any component Y^- of X^G with negative sign choice is Morita equivalent (over Y^-) to $C_0^\mathbb{H}(Y^-)$, where $\mathbb{H}$ denotes the quaternions.*

Furthermore, there is a **canonical** *choice $\mathcal{A}_\alpha(X)$ of such an algebra $A_\alpha(X)$.*

Proof. Let Y^+ be the union of the components of the fixed set with $+$ sign choice, Y^- be the union of the components of the fixed set with $-$ sign choice, and $Z = X \smallsetminus (Y^+ \amalg Y^-)$, which is the open subset of X on which the involution ι acts freely. Let $\mathcal{A}(Z)$ denote the commutative real C^*-algebra $\mathcal{A}(Z) = \{f \in C_0(Z) \mid f(\iota(x)) = \overline{f(x)}\}$. (Recall that the K-theory of $\mathcal{A}(Z)$ is identical to $\mathrm{KR}^*(Z)$.). First we will show that there is a spectrum-fixing isomorphism of real C^*-algebras

$$\varphi\colon\ \mathcal{A}(Z) \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}} \xrightarrow{\cong} \mathcal{A}(Z) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}.$$

(The induced isomorphism on complexifications is equivariant for the involution σ.) In fact, there is a canonical choice for φ (up to homotopy). Then we can define $A_\alpha(X)$ by "clutching."

The algebra $\mathcal{A}(Z) \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$ is, as explained in [35, Section 3], the algebra of sections (vanishing at infinity on Z) of a bundle over $\overline{Z} = Z/\iota$ of real C^*-algebras with fibers $\mathcal{K}$ and structure group $P\mathcal{U}'$, the projective infinite-dimensional unitary/antiunitary group. This group is a semidirect product of $P\mathcal{U}$ by $\mathbb{Z}_2$ (acting by complex conjugation), and the bundle is induced from the $\mathbb{Z}_2$-bundle $Z \to \overline{Z}$ defined by the free involution ι. Now $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$, so $\mathcal{A}(Z) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$ is also the algebra of sections of a bundle over $\overline{Z}$ with fibers $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ and the same structure group, and since the bundle came from the original $P\mathcal{U}'$-bundle (via tensoring with $\mathbb{H}$) and induces the same covering map $Z \to \overline{Z}$, the bundles are isomorphic (as $P\mathcal{U}'$-bundles). This guarantees existence of the desired isomorphism φ. In fact, if we fix an isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \otimes \mathcal{K} \to \mathcal{K}$ (which is unique up to homotopy), we get a canonical choice of φ, also unique up to homotopy.

Now, let $X^+ = Z \cup Y^+$, $X^- = Z \cup Y^-$, which are both open subsets of X. $\mathcal{A}(Z)$ is an ideal in each of the commutative real C^*-algebras $\mathcal{A}(X^\pm) = \{f \in C_0(X^\pm) \mid f(\iota(x)) = \overline{f(x)}\}$. We can construct $A_\alpha(X)$ as the algebra of sections of a bundle of algebras obtained by clutching the stabilized bundles for $\mathcal{A}(X^+)$ and for $\mathcal{A}(X^-) \otimes \mathbb{H}$ together over $\overline{Z}$ via (the bundle isomorphism associated to) φ, i.e., we construct $A_\alpha(X)$ by gluing $\mathcal{A}(X^+) \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$ (which represents $\mathrm{KR}^*(X^+)$) to $\mathcal{A}(X^-) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$ (which represents $\mathrm{KR}^*_\alpha(X^-) \cong KSp^*(X^-)$) over Z using φ. It remains to show that we can choose φ so that the Dixmier–Douady invariant of $A_\alpha(X) \otimes_{\mathbb{R}} \mathbb{C}$ vanishes. This follows from the Mayer–Vietoris sequence for the diagram

$$\begin{array}{ccc} & X^+ & \\ \nearrow & & \searrow \\ Z & & X \\ \searrow & & \nearrow \\ & X^- & \end{array}$$

since the Dixmier–Douady invariant is trivial over X^+ and X^- (by construction) and thus the Dixmier–Douady invariant in $H^3(X)$ comes from the Phillips–Raeburn invariant of φ in $H^2(Z)$ via the Mayer–Vietoris boundary map. This invariant will be trivial for the canonical choice. (See the discussion in Remark 1 below for more details.) □

Remark 1. One has to be cautious; even though Theorem 1 guarantees *existence* of $A_\alpha(X)$, it does not guarantee *uniqueness*, since the isomorphism φ is only determined up to an automorphism of the $P\mathcal{U}'$-bundle over $\overline{Z}$. Such an automorphism, which (if $\overline{Z}$ is connected) we can assume is in the connected component of the identity in the automorphism group, is simply a section of the bundle of topological groups $\mathcal{B}_{P\mathcal{U}} = (Z \times_{\mathbb{Z}_2} P\mathcal{U}) \to \overline{Z}$, where the covering group $\mathbb{Z}_2$ acts on $\mathcal{U}$ and thus on $P\mathcal{U}$ by complex conjugation. The automorphism will not affect the K-groups if it is *inner*, i.e., comes from a section of $\mathcal{B}_{\mathcal{U}} = (Z \times_{\mathbb{Z}_2} \mathcal{U}) \to \overline{Z}$. From the exact sequence in sheaf cohomology for the exact sequence of sheaves of groups

$$1 \to \mathcal{B}_{\mathbb{T}} \to \mathcal{B}_{\mathcal{U}} \to \mathcal{B}_{P\mathcal{U}} \to 1,$$

where $\mathcal{B}_{\mathbb{T}} = (Z \times_{\mathbb{Z}_2} \mathbb{T}) \to \overline{Z}$ and we identify bundles of topological groups with their sheaves of sections, and from the fact that the sheaf $\mathcal{B}_{\mathcal{U}}$ is fine since $\mathcal{U}$ is contractible, we see that the obstruction to an automorphism being inner lies in $H^1(\overline{Z}, \mathcal{B}_{\mathbb{T}}) = H^2(\overline{Z}, \underset{\sim}{\mathbb{T}})$, where $\underset{\sim}{\mathbb{T}}$ is the sheaf of local sections of $\mathcal{B}_{\mathbb{T}}$. The obstruction group is via the exact sequence of sheaves

$$0 \to \underset{\sim}{\mathbb{Z}} \to \underset{\sim}{\mathbb{R}} \to \underset{\sim}{\mathbb{T}} \to 1$$

identifiable with $H^2(\overline{Z}, \underset{\sim}{\mathbb{Z}})$, where $\underset{\sim}{\mathbb{Z}}$ is the locally constant sheaf with stalks $\mathbb{Z}$ and twisting given by the covering map $Z \to \overline{Z}$. The obstruction is what we can call the *twisted Phillips–Raeburn invariant* (cf. [35, Section 1]). It vanishes when $H^2(\overline{Z}, \underset{\sim}{\mathbb{Z}}) = 0$, and in particular when $\dim Z = 1$, so in this case $A_\alpha(X)$ is unique up to spectrum-fixing Morita equivalence.

While we will always use the particular algebra $\mathcal{A}_\alpha(X)$ constructed in the proof of Theorem 1, any other algebra satisfying the properties in the Theorem gives the same K-groups up to extensions.

THEOREM 2. *For any $A_\alpha(X)$ with the properties of* Theorem 1, *the topological K-groups fit into the long exact sequence*

$$\cdots \to \mathrm{KR}^{-i}(Z) \to K_i(A_\alpha(X)) \to \mathrm{KO}^{-i}(Y^+) \oplus KSp^{-i}(Y^-) \to \mathrm{KR}^{-i+1}(Z) \to \cdots, \quad (9)$$

and the groups $K_*(A_\alpha(X))$ are uniquely determined at least up to extensions.

Proof. Let $A_\alpha(Z)$ be the ideal of $A_\alpha(X)$ associated to Z, and let $A_\alpha(Y^\pm)$ be the quotient associated to $Y^\pm$. The long exact sequence

$$\cdots \to K_i(A_\alpha(Z)) \to K_i(A_\alpha(X)) \to K_i(A_\alpha(X \setminus Z)) \to K_{i-1}(A_\alpha(Z)) \to \cdots. \quad (10)$$

follows from the long exact K-theory sequence of the extension of real C^*-algebras associated to the open inclusion $Z \subseteq X$ (see for example [35, equation (∗), p. 376]). Since the involution ι on Z is free, $K_i(A_\alpha(Z)) \cong \mathrm{KR}^{-i}(Z)$. Also

$$\begin{aligned} K_i(A_\alpha(X \setminus Z)) &\cong K_i(A_\alpha(Y^+ \amalg Y^-)) \\ &\cong K_i(A_\alpha(Y^+)) \oplus K_i(A_\alpha((Y^-)). \end{aligned}$$

But

$$K_i(R_\alpha(Y^+)) \cong \mathrm{KO}^{-i}(Y^+),$$

since Y^+ has trivial involution.

For a space M where all the components of M^G have $-$ sign choice, the only difference is that the quotient of $A_\alpha(M)$ associated to any component of M^G is Morita equivalent (over $\mathbb{R}$) to $C_0^{\mathbb{H}}(M^G)$ (instead of $C_0^{\mathbb{R}}(M^G)$). This defines a symplectic structure (instead of a real structure). In this case, KR-theory with a sign choice reduces to what is sometimes referred to as KSp- or KH-theory, which is just ordinary KO-theory with a shift in index by 4. Therefore,

$$K_i(A_\alpha(Y^-)) \cong \mathrm{KO}^{-i-4}(Y^-) \cong KSp^{-i}(Y^-).$$

Putting this all together gives the long exact sequence (9).

Since the connecting maps

$$\mathrm{KO}^{-i}(Y^+) \to \mathrm{KR}^{-i+1}(Z) \quad \text{and} \quad KSp^{-i}(Y^-) \to \mathrm{KR}^{-i+1}(Z)$$

in (9) are determined by the KR-theories of X^+ and X^-, respectively, we conclude that regardless of what choice one makes of $A_\alpha(X)$ satisfying the conditions of Theorem 1, the groups $K_*(A_\alpha(X))$ are uniquely determined at least up to extensions. □

DEFINITION 1. Let X be a Real locally compact space with an assignment of signs α to the components of the fixed set. Let $\mathcal{A}_\alpha(X)$ be the canonical real continuous-trace algebra constructed in Theorem 1. We define $\mathrm{KR}_\alpha^*(X)$ to be the topological K-theory of $\mathcal{A}_\alpha(X)$ (in the sense of [35, Section 3]). Note that these groups fit into the exact sequence given in Theorem 2.

COROLLARY 1. KR_α^{-j} *has periodicity with period* 8.

Proof. This is immediate from Bott periodicity for topological K-theory of real Banach algebras. □

Figure 1. Coefficient systems for the type IA and $\widetilde{\mathrm{IA}}$ theories.

It is now easy to see that KR-theory with a sign choice can be computed using a generalization of the equivariant Atiyah–Hirzebruch spectral sequence of [25, (A.2)]

$$E_2^{p,q} = H_G^p(X; \underset{\sim}{\mathrm{KR}}^q) \Rightarrow \mathrm{KR}^{p+q}(X), \tag{11}$$

where $\underset{\sim}{\mathrm{KR}}^*$ is the Bredon coefficient system for G associated to KR.

As described in [25], $\underset{\sim}{\mathrm{KR}}^*(G) = K^*$ and $\underset{\sim}{\mathrm{KR}}^*(\mathrm{pt}) = \mathrm{KO}^*$. We are now allowing for different components of the fixed set to have symplectic or orthogonal bundles, corresponding to the coefficient system being KO or KSp.

THEOREM 3. *There is a spectral sequence*

$$E_2^{p,q} = H_G^p(X; \underset{\sim}{\mathrm{KR}_\alpha}{}^q) \Rightarrow \mathrm{KR}_\alpha^{p+q}(X). \tag{12}$$

where $\underset{\sim}{\mathrm{KR}_\alpha}{}^*(G) \cong K^*$ *and*

$$\underset{\sim}{\mathrm{KR}_\alpha}{}^{-i}(pt_j) = \begin{cases} \mathrm{KO}^{-i}, & \textit{if } \alpha_j = +, \\ KSp^{-i}, & \textit{if } \alpha_j = -. \end{cases}$$

Proof. The proof is quite similar to the case handled in [25]. We filter $K_\alpha^*(X)$ using the equivariant skeletal filtration, but with fixed cells separated into two types. Then this is just the spectral sequence associated to this filtration. The picture of the coefficient system is as in Figure 1 (right side). □

If we remove the 2 fixed points from $S^{1,1}$ we are left with 2 copies of $\mathbb{R}$ that are exchanged by the involution. This gives $\mathrm{KR}_\alpha^*(S^{1,1} \setminus \text{fixed points}) \cong K^{*-1}$ by (5). The type IA theory has $O8^+$-planes (hence orthogonal bundles) at both fixed points, so has KO^* at both fixed points (see Figure 1) and matches with the spectral sequence as described in [25]. While motivated by physics, we are just decorating G–CW-complexes with some extra information on the equivariant cells of the form $(G/G) \times \mathrm{e}^n$ that we have called "sign." We show in [15] that the sign can be given a geometric interpretation in the T-dual theory, thus giving a completely mathematical description of T-duality.

Note that flipping the sign of every component of the fixed set exchanges KO and KSp and so just results in a shift of the index by 4. For example, type IIA theory on $S^{1,1}$ with 2 $O8^-$-planes does not make physical sense since it is not supersymmetric, and it is mathematically uninteresting since it is just the usual theory with an index shift.

We can now turn our attention to the only case of KR-theory of a circle with a non-trivial sign choice, corresponding to the type $\widetilde{\text{IA}}$ theory.

4.1. THE TYPE $\widetilde{\text{IA}}$ THEORY

As noted in Section 3, the compactification manifold for the $\widetilde{\text{IA}}$ theory is $S^{1,1}$, but with an $O8^-$-plane at one fixed point and an $O8^+$-plane at the other. Therefore, Dp-branes are classified by $\mathrm{KR}^{p-9}_{(+,-)}(S^{1,1})$. The index is determined as a shift by one from the T-dual theory $\mathrm{KR}^{p-8}(S^{0,2})$.

Recall that

$$S^{1,1} \setminus S^{1,0} \cong \mathbb{R}^{1,0} \times S^{0,1},$$

with an involution that exchanges the 2 copies of $\mathbb{R}$. Therefore

$$\begin{aligned}\mathrm{KR}^{-i}_{\alpha}(S^{1,1} \setminus S^{1,0}) &\cong \mathrm{KR}^{-i}(\mathbb{R}^{1,0} \times S^{0,1}) \\ &\cong K^{-i}(\mathbb{R}) \\ &\cong K^{-i-1},\end{aligned} \tag{13}$$

for all α. For $\alpha = (+,-)$,

$$\mathrm{KR}^{-i}_{\alpha}(S^{1,0}) \cong \mathrm{KO}^{-i} \oplus KSp^{-i}. \tag{14}$$

Plugging these into Equation (9) we get the long exact sequence

$$\cdots \longrightarrow K^{-i-1} \longrightarrow \mathrm{KR}^{-i}_{(+,-)}(S^{1,1}) \longrightarrow \mathrm{KO}^{-i} \oplus KSp^{-i} \xrightarrow{\delta} K^{-i} \longrightarrow \cdots. \tag{15}$$

The map δ is complexification on the first summand and doubling on the second summand, since symplectic bundles contain 2 complex bundles. Furthermore, the long exact sequence splits into 2 parts

$$0 \longrightarrow \mathrm{KR}^{0 \pmod 4}_{(+,-)}(S^{1,1}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \longrightarrow \mathrm{KR}^{-3 \pmod 4}_{(+,-)}(S^{1,1}) \longrightarrow 0. \tag{16}$$

$$0 \longrightarrow \mathrm{KR}^{-2 \pmod 4}_{(+,-)}(S^{1,1}) \longrightarrow \mathbb{Z}_2 \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\sigma} \mathrm{KR}^{-1 \pmod 4}_{(+,-)}(S^{1,1}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{17}$$

The map δ is $(m,n) \mapsto m + 2n$, which is surjective. This means $\mathrm{KR}^{0 \pmod 4}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$ and $\mathrm{KR}^{-3 \pmod 4}_{(+,-)}(S^{1,1}) \cong 0$. γ must be 0, showing $\mathrm{KR}^{-2 \pmod 4}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}_2$. This gives us an extension problem

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\sigma} \mathrm{KR}^{-1 \pmod 4}_{(+,-)}(S^{1,1}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{18}$$

However, since removing the fixed point with the $-$ sign from $S^{1,1}_{(+,-)}$ leaves $\mathbb{R}^{0,1}$, we also have an exact sequence

$$0 = KSp^{-2} \to \mathrm{KR}^{-1}(\mathbb{R}^{0,1}) \to \mathrm{KR}^{-1}_{(+,-)}(S^{1,1}) \to KSp^{-1} = 0,$$

and since $\mathrm{KR}^{-1}(\mathbb{R}^{0,1}) \cong \mathrm{KO}^0 = \mathbb{Z}$, we see that $\mathrm{KR}^{-1}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$. The corresponding argument where we remove the point with the $+$ sign instead shows that $\mathrm{KR}^{-5}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$. Putting this all together we find

$$\mathrm{KR}^{-i}_{(+,-)}(S^{1,1}) \cong \mathrm{KSC}^{-i+1}, \tag{19}$$

which has periodicity with period 4.

Now we can see T-duality between the type $\tilde{\mathrm{I}}$ and $\widetilde{\mathrm{IA}}$ theories as an isomorphism

$$\mathrm{KR}^{-i}(S^{0,2}) \cong \mathrm{KR}^{-i-1}_{(+,-)}(S^{1,1}). \tag{20}$$

The fact that we need to include a charge in the T-dual of the IIB theory on $S^{0,2}$ is contained in the geometry of $S^{0,2}$ in a way that is explored in [15]. Now let us turn to the different possibilities of sign choices for 2-torus orientifolds.

4.2. TORUS ORIENTIFOLDS WITH A SIGN CHOICE

For a 2-torus with an involution[3] the possible fixed point sets are empty, 1 copy of S^1, 2 disjoint copies of S^1, 4 isolated points, or the entire copy of T^2. Obviously, when the fixed set is empty there are no possible sign choices. Also, when the fixed set is a single copy of S^1 or the entire 2-torus, then the fixed set has only a single component. Therefore, there is only one possible sign choice giving either ordinary KR-theory ($+$ sign choice) or an index shift by 4 of ordinary KR-theory ($-$ sign choice). The only cases that do not immediately reduce to ordinary KR-theory are when the fixed point set is either 2 disjoint copies of S^1 or 4 isolated points.

Let us first consider the orientifold of the 2-torus with 4 fixed points, corresponding to when the involution is reflection. Topologically our orientifold is $S^{1,1} \times S^{1,1}$. There are 3 supersymmetric sign choices for the 4 fixed points: $\alpha = (+,+,+,+), (+,+,-,-)$, or $(+,+,+,-)$. (The non-supersymmetric cases $(-,-,-,-)$ and $(-,-,-,+)$ can be obtained from $(+,+,+,+)$ and $(+,+,+,-)$ by an index shift.)

The case of $(+,+,+,-)$ is considerably more subtle to compute than the other two, though as shown by Witten [38], the case of four O-planes, three with a $+$ charge and one with a $-$ charge, does indeed occur in physics.

To determine KR_α in each case, we first need the following result.

[3]Since this is what's needed for physics, we are assuming the torus can be identified with a complex smooth curve of genus 1, and the involution is either holomorphic or anti-holomorphic. This is explained further in [15].

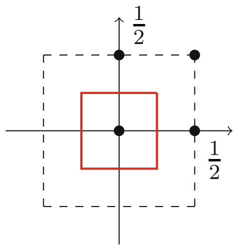

Figure 2. Fundamental domain of a 2-torus with the 4 fixed points shown.

PROPOSITION 2. *Let X be T^2 (realized as $\mathbb{R}^2/\mathbb{Z}^2$) with involution given by multiplication by -1. Let Y be the set of 4 points fixed by the involution. Then*

$$\mathrm{KR}_\alpha^{-i}(X \setminus Y) \cong \mathrm{KSC}^{-i-1} \oplus K^{-i-1} \oplus K^{-i-1}, \tag{21}$$

for all α.

Proof. A picture of the fundamental domain of the T^2 is shown in Figure 2. When we remove the four fixed points and the dashed lines along the boundary of the fundamental domain, what remains retracts onto the square with vertices at $(\pm\frac{1}{4}, \pm\frac{1}{4})$ shown in color. One can proceed to compute KR_α^* using this picture, but it will be faster to use the spectral sequence of Theorem 3. Since ι acts freely on $X \setminus Y$, the spectral sequence reduces to the one studied by Karoubi and Weibel [25, Example A.3]. Let $W = (X \setminus Y)/\iota$, which is diffeomorphic to $S^2 \setminus \{4 \text{ points}\}$. The map $(X \setminus Y) \to W$ is a 2-to-1 covering map. The spectral sequence has $E_2^{p,q} = 0$ for q odd, and reduces to $H_c^p(W, \mathbb{Z}(i)) \Rightarrow \mathrm{KR}_\alpha^{p+2i}(X \setminus Y)$, where $\mathbb{Z}(i) = \mathbb{Z}$ (the constant sheaf) for i even and $\mathbb{Z}(i)$ is the non-trivial local coefficient system determined by the covering map $(X \setminus Y) \to W$ for i odd. By Poincaré duality, $H_c^p(W, \mathbb{Z}) \cong H_{2-p}(W, \mathbb{Z})$, which is $\mathbb{Z}$ for $p = 2$, $\mathbb{Z}^3$ for $p = 1$, 0 for $p = 0$. The groups $H_c^p(W, \mathbb{Z}(1))$ are slightly harder to compute, but can be obtained, for example, from the exact sequence

$$\cdots \to H_c^p(\mathbb{R} \times S^1, \mathbb{Z}(1)) \to H_c^p(W, \mathbb{Z}(1)) \to H_c^p(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}(1)) \to \cdots,$$

coming from the fact that deleting two line segments from W, each one running between two of the branch points of the branched covering $T^2 \to S^2$, leaves an open subset diffeomorphic to $\mathbb{R} \times S^1$. Here $H_c^p(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}(1)) \cong H_c^p(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}) \cong \mathbb{Z}^2$ for $p = 1$, and 0 for other values of p, since each component of $\mathbb{R} \amalg \mathbb{R}$ is simply connected. The result is that $H_c^p(W, \mathbb{Z}(1))$ is isomorphic to $\mathbb{Z}^2$ for $p = 1$, $\mathbb{Z}_2$ for $p = 2$, and 0 for other values of p. The spectral sequence is shown in Figure 3. Note that there is no room for any non-trivial differentials or for any non-trivial extensions, and the Proposition follows. □

$q\backslash p$	0	1	2
0	0	$\mathbb{Z}^3$	$\mathbb{Z}$
−1	0	0	0
−2	0	$\mathbb{Z}^2$	$\mathbb{Z}_2$
−3	0	0	0
−4	0	$\mathbb{Z}^3$	$\mathbb{Z}$

Figure 3. E_2 of the spectral sequence for computing $\mathrm{KR}^*_\alpha(X \smallsetminus Y)$. The sequence repeats with vertical period 4.

For the set of fixed points, Y, the three options are

$$\mathrm{KR}^{-i}_\alpha(Y) = \begin{cases} 4\mathrm{KO}^{-i}, & \alpha = (+,+,+,+) \\ 2\mathrm{KO}^{-i} \oplus 2KSp^{-i}, & \alpha = (+,+,-,-) \\ 3\mathrm{KO}^{-i} \oplus KSp^{-i}, & \alpha = (+,+,+,-). \end{cases} \tag{22}$$

The case where $\alpha = (+,+,+,+)$ (in the notation of [33], this is $T^{1,2}$) just gives ordinary KR-theory, for which we get the calculation

$$\begin{aligned} \mathrm{KR}^{-i}(X) &\cong \mathrm{KR}^{-i}(S^{1,1}) \oplus \mathrm{KR}^{-i+1}(S^{1,1}) \\ &\cong \mathrm{KO}^{-i} \oplus KO^{-i+1} \oplus \mathrm{KO}^{-i+1} \oplus \mathrm{KO}^{-i+2}. \end{aligned} \tag{23}$$

The relevant long exact sequence for $\alpha = (+,+,+,-)$ is (via Proposition 2)

$$\begin{aligned} \cdots \to \mathrm{KSC}^{-i-1} \oplus 2K^{-i-1} \to \mathrm{KR}^{-i}_{(+,+,+,-)}(S^{1,1} \times S^{1,1}) \\ \to 3\mathrm{KO}^{-i} \oplus KSp^{-i} \to \cdots . \end{aligned} \tag{24}$$

This gives an extension problem in determining each of the KR-groups. Therefore, we need to look at some additional long exact sequences to determine $\mathrm{KR}^i_\alpha(S^{1,1} \times S^{1,1})$.

Let $Y_+ = S^{1,1}_{(+,+)} \vee S^{1,1}_{(+,+)}$ be the wedge of 2 circles going through the three fixed points with sign choice $+$. In terms of Figure 2, this is the image of the dotted lines. Then we get a long exact sequence

$$\cdots \longrightarrow \mathrm{KR}^{-i}(X \smallsetminus Y_+) \longrightarrow \mathrm{KR}^{-i}_\alpha(X) \longrightarrow \mathrm{KR}^{-i}_\alpha(Y_+) \longrightarrow \cdots . \tag{25}$$

Note that $X \smallsetminus Y_+ \cong \mathbb{R}^{0,2}_-$, where the fixed point of $\mathbb{R}^{0,2}$ is given the sign choice $-$. Therefore

$$\begin{aligned} \mathrm{KR}^{-i}_\alpha(X \smallsetminus Y_+) &\cong \mathrm{KR}^{-i}_-(\mathbb{R}^{0,2}) \\ &\cong KSp^{-i+2}. \end{aligned}$$

To determine $\mathrm{KR}^{-i}_{(+,+,+)}(Y_+)$, first note that this reduces to ordinary KR-theory since the sign choices are all positive. Now consider the split long exact sequence

$$\cdots \longrightarrow \mathrm{KR}^{-i}(Y_+ \setminus \{\mathrm{pt}\}) \longrightarrow \mathrm{KR}^{-i}(Y_+) \underset{\longleftarrow}{\longrightarrow} \mathrm{KR}^{-i}(\mathrm{pt}) \longrightarrow \cdots, \quad (26)$$

where the basepoint is the joining point of the two circles (a fixed point with sign +). Therefore, $Y_+ \setminus \{\mathrm{pt}\}$ is 2 copies of $\mathbb{R}^{0,1}$ and

$$\mathrm{KR}^{-i}(Y_+) \cong \mathrm{KO}^{-i+1} \oplus \mathrm{KO}^{-i+1} \oplus \mathrm{KO}^{-i}.$$

Plugging $\mathrm{KR}^{-i}_{\alpha}(Y_+)$ into the exact sequence

$$\cdots \to \mathrm{KR}^{-i}_{\alpha}(X \setminus Y_+) \cong KSp^{-i+2} \to \mathrm{KR}^{-i}_{\alpha}(X) \to \mathrm{KR}^{-i}_{\alpha}(Y_+) \to \cdots,$$

we find $\mathrm{KR}^{-i}_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$ is $\mathbb{Z}$ if $i = 4$ or $6, \mathbb{Z}^2$ for $i = 5$, and $\mathbb{Z}_2^2$ for $i = 3$. There are extension problems for the other 4 indices mod 8.

To solve the remaining extension problems, we can repeat the same process, but use the space Y_- which is the one point union of 2 circles joined at the fixed point with sign $-$ and going through 2 of the fixed points with sign choice +. This space is the image of the coordinate axes in Figure 2. Note that $X \setminus Y_- \cong \mathbb{R}^{0,2}$ (with a + sign at the fixed point). If we remove one circle, which we can identify with $S^{1,1}_{(+,-)}$, from Y_-, then what remains is $\mathbb{R}^{0,1}$ (with a + sign), so we get an exact sequence

$$\cdots \to \mathrm{KR}^{-i}(\mathbb{R}^{0,1}) \to \mathrm{KR}^{-i}_{\alpha}(Y_-) \to \mathrm{KR}^{-i}_{(+,-)}(S^{1,1}) \to \cdots,$$

or in other words,

$$\cdots \to \mathrm{KO}^{-i+1} \to \mathrm{KR}^{-i}_{\alpha}(Y_-) \to \mathrm{KSC}^{-i+1} \to \mathrm{KO}^{-i+2} \to \cdots. \quad (27)$$

In fact (27) splits, i.e., $\mathrm{KR}^{-i}_{\alpha}(Y_-) \cong \mathrm{KO}^{-i+1} \oplus \mathrm{KSC}^{-i+1}$, since the inclusion $S^{1,1}_{(+,-)} \hookrightarrow Y_-$ is split by the (sign-preserving) "fold map" sending both circles in Y_- onto $S^{1,1}_{(+,-)}$. Putting our result for Y_- into the exact sequence

$$\cdots \to \mathrm{KR}^{-i}(\mathbb{R}^{0,2}) \cong \mathrm{KO}^{-i+2} \to \mathrm{KR}^{-i}_{\alpha}(X) \to \mathrm{KR}^{-i}_{\alpha}(Y_-) \to \mathrm{KO}^{-i+3} \to \cdots$$

gives that $\mathrm{KR}^{-i}_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$ is $\mathbb{Z}$ for $i = 0, \mathbb{Z}^2$ for $i = 1$, $\mathbb{Z} \oplus (\mathbb{Z}_2)^2$ for $i = 2$ (for this case we must combine the information we get from (24), (26), and (27)), and 0 for $i = 7$. The results of the calculation are summarized in the last column of Table II in Section 5.1 below.

We could also use the spectral sequence in Theorem 3 for $S^{1,1} \times S^{1,1}$ with $\alpha = (+,+,+,-)$. To determine the E_2 term, we need to look at the groups $H^p_G(X; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q)$. These are most easily computed using the exact sequence

$$\cdots \to H^p_{G,c}(X \setminus X^\iota; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q) \to H^p_G(X; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q) \to H^p_G(X^\iota; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q) \to \cdots. \quad (28)$$

Here $H^p_G(X^\iota; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q)$ is nonzero only for $p = 0$, where it is $3\mathrm{KO}^q \oplus KSp^q$, and $H^p_{G,c}(X \setminus X^\iota; \underset{\sim}{\mathrm{KR}}_{\alpha}{}^q)$ was computed in the proof of Proposition 2. In (28) there

$q \backslash p$	0	1	2
0	$\mathbb{Z}$	0	$\mathbb{Z}$
−1	$(\mathbb{Z}_2)^3$	0	0
−2	$(\mathbb{Z}_2)^3$	$\mathbb{Z}^2$	$\mathbb{Z}_2$
−3	0	0	0
−4	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}$
−5	$\mathbb{Z}_2$	0	0
−6	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$
−7	0	0	0

Figure 4. E_2 of the spectral sequence for computing $\mathrm{KR}^*_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$. The sequence repeats with vertical period 8.

is one potentially nonzero connecting map, $\mathbb{Z}^4 \cong 3\mathrm{KO}^q \oplus KSp^q \to \mathbb{Z}^3$ when $q \equiv 0 \pmod 4$. This map can be computed by comparison with the corresponding sequences for the cases of $S^{1,1}_{+,+}$ and $S^{1,1}_{+,-}$, where the KR_α groups were computed from Equation (15) and the surrounding discussion. One finds that the connecting map has kernel $\mathbb{Z}$ in all cases, is surjective for $q \equiv 0 \pmod 8$, and has a cokernel of $\mathbb{Z}_2^2$ when $q \equiv 4 \pmod 8$. Thus the groups $H^p_G(X^\iota; \underset{\sim}{\mathrm{KR}_\alpha}{}^q)$ are as in Figure 4. This calculation is consistent with our computation of $\mathrm{KR}^*_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$, assuming that there are d_2 differentials that kill off the $\mathbb{Z}_2$'s in positions $(2,-2)$ and $(2,-6)$.

The case $\alpha = (+,+,-,-)$ can be obtained by the product of the type $\widetilde{\mathrm{IA}}$ theory with itself or the type IA theory. The equivariant decomposition

$$S^{1,1}_{(+,-)} \times S^{1,1} = (S^{1,1}_{(+,-)} \times \{\mathrm{pt}\}) \amalg (S^{1,1}_{(+,-)} \times \mathbb{R}^{0,1})$$

gives the calculation

$$\mathrm{KR}^i_{(+,+,-,-)}(S^{1,1} \times S^{1,1}) \cong \mathrm{KSC}^{i+2} \oplus \mathrm{KSC}^{i+1}. \tag{29}$$

The same case can also be obtained by looking at

$$S^{1,1}_{(+,-)} \times S^{1,1}_{(+,-)} = (S^{1,1}_{(+,-)} \times \{\mathrm{pt}\}) \amalg (S^{1,1}_{(+,-)} \times \mathbb{R}^{0,1}_{-}).$$

But crossing with $\mathbb{R}^{0,1}_{-}$ has the same effect as crossing with $\mathbb{R}^{4,1}$ or with $\mathbb{R}^{3,0}$, and since KSC^* is 4-periodic, we get the same result as in (29).

Now let us consider orientifolds of the 2-torus where the fixed set is 2 disjoint copies of S^1. Topologically, this is $S^{1,1} \times S^{2,0}$. There are 2 possible supersymmetric sign choices, $(+,+)$ and $(+,-)$. As usual, the non-supersymmetric case $(-,-)$ can be obtained from $(+,+)$ by an index shift. When both fixed circles have sign choice $+$, KR_α reduces to ordinary KR-theory,

$$\begin{aligned}\mathrm{KR}^{-i}(S^{1,1} \times S^{2,0}) &\cong \mathrm{KR}^{-i-1}(S^{1,1}) \oplus \mathrm{KR}^{-i}(S^{1,1})\\ &\cong \mathrm{KO}^{-i-1} \oplus \mathrm{KO}^{-i} \oplus \mathrm{KO}^{-i} \oplus \mathrm{KO}^{-i+1}.\end{aligned} \tag{30}$$

The case $\alpha = (+,-)$ is just the product of the type $\widetilde{\mathrm{IA}}$ theory, $S^{1,1}_{(+,-)}$, with a fixed circle, $S^{2,0}$, so we find

$$\begin{aligned}\mathrm{KR}^{-i}_{(+,-)}(S^{1,1} \times S^{2,0}) &\cong \mathrm{KR}^{-i-1}_{(+,-)}(S^{1,1}) \oplus \mathrm{KR}^{-i}_{(+,-)}(S^{1,1})\\ &\cong \mathrm{KSC}^{-i} \oplus \mathrm{KSC}^{-i+1}.\end{aligned} \tag{31}$$

To conclude this section, we explain how to compute KR-theory for a 2-torus orientifold where the involution ι is orientation reversing and has a fixed set that is topologically S^1. Unlike the cases above, this orientifold does *not* split as a product of two circle orientifolds, so a somewhat more complicated calculation is required.

THEOREM 4. *Let (X, ι) be a Real space where $X = T^2$ and ι is smooth, orientation reversing, and has a fixed set that is topologically S^1. The quotient space $M = X/\iota$ is topologically a closed Möbius strip.* (*Such a space arises from taking X to be the complex points of a smooth projective real curve of genus* 1 *when the real points have exactly one connected component, and taking ι to be the action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.) *Then* $\mathrm{KR}^j(X, \iota) \cong (\mathrm{KO}^j)^2 \oplus \mathrm{KU}^{j-1}$.

Proof. Step 1. Since X has a non-empty fixed set, $\mathrm{KR}^{-j}(T^2, \iota) \cong \widetilde{\mathrm{KR}}^{-j}(T^2, \iota) \oplus \mathrm{KO}^{-j}$, and we only need to compute $\widetilde{\mathrm{KR}}^{-j}(T^2, \iota)$. We begin by deducing two useful exact sequences. The first comes from observing that $X \smallsetminus X^\iota \cong S^{0,2} \times \mathbb{R}^{0,1}$ (as a Real space). Thus $\mathrm{KR}^{-j}(X \smallsetminus X^\iota) \cong \mathrm{KR}^{-j+1}(S^{0,2}) \cong \mathrm{KSC}^{-j+1}$. Since $\widetilde{\mathrm{KR}}^{-j}(X^\iota) \cong \mathrm{KR}^{-j}(\mathbb{R}^{1,0}) \cong \mathrm{KO}^{-j-1}$, we get the long exact sequence

$$\cdots \to \mathrm{KO}^{-j-2} \xrightarrow{\delta} \mathrm{KSC}^{-j+1} \to \widetilde{\mathrm{KR}}^{-j}(T^2, \iota) \to \mathrm{KO}^{-j-1} \xrightarrow{\delta} \mathrm{KSC}^{-j+2} \to \cdots, \tag{32}$$

where the connecting map δ will be determined later. However, note for now that δ vanishes after inverting 2, since $\mathrm{KO}^{-j-1}[\frac{1}{2}]$ is nonzero only for $j \equiv 3 \pmod 4$ and $\mathrm{KSC}^{-j+2} \cong \mathbb{Z}_2$ for these values of j. Thus the torsion-free part of $\widetilde{\mathrm{KR}}^{-j}(X, \iota)$ is the same as for $\mathrm{KSC}^{-j+1} \oplus \mathrm{KO}^{-j-1}$ and is thus $\mathbb{Z}$ for j odd, $\mathbb{Z}$ for $j \equiv 0 \pmod 4$, and 0 for $j \equiv 2 \pmod 4$.

To get the other exact sequence, choose an interval I in $M = X/\iota$ transverse to the central circle and meeting the boundary in two points. The inverse image of this interval in X is a copy of $S^{1,1}$, the unit circle in the complex plane with complex conjugation as the involution. Furthermore the complement of this copy of

$S^{1,1}$ is isomorphic (as a Real space) to $(0, 1) \times S^{1,1}$. Since $S^{1,1}$ with one fixed point removed is isomorphic (as a Real space) to $\mathbb{R}^{0,1}$, $\widetilde{\mathrm{KR}}^j(S^{1,1}) \cong \mathrm{KR}^j(\mathbb{R}^{0,1}) \cong \mathrm{KO}^{j+1}$ via [3, Theorem 2.3] and $\mathrm{KR}^j((0, 1) \times S^{1,1}) \cong \mathrm{KO}^{j-1} \oplus \mathrm{KO}^j$. So we get an exact sequence

$$\cdots \to \mathrm{KO}^j \xrightarrow{\rho} \mathrm{KO}^{j-1} \oplus \mathrm{KO}^j \to \widetilde{\mathrm{KR}}^j(X) \to \mathrm{KO}^{j+1} \xrightarrow{\rho} \mathrm{KO}^j \oplus \mathrm{KO}^{j+1} \to \cdots. \quad (33)$$

Step 2. Observe next that the connecting maps δ and ρ have to be compatible with cup products by the ground ring

$$\mathrm{KO}^* \cong \mathbb{Z}[b^{\pm}, \xi, \eta]/(2\eta, \eta^3, \xi\eta, \xi^2 - 4b).$$

Here the torsion-free generators are b in degree -8 and ξ in degree -4, and the torsion generator η is in degree -1. To prove this claim, simply replace T^2 by $T^2 \times \mathbb{R}^{p,q}$. Thus the connecting map $\delta: \mathrm{KO}^{j-1} \to \mathrm{KSC}^{j+2}$ has to be of the form $x \mapsto x \cdot y$, $x \in \mathrm{KO}^*$ and y some class in KSC^3, and the connecting map $\rho: \mathrm{KO}^j \xrightarrow{\rho} \mathrm{KO}^{j-1} \oplus \mathrm{KO}^j$ has to be of the form $x \mapsto (ax, bx)$, where $a \in \mathrm{KO}^{-1}$ and $b \in \mathrm{KO}^0$.

Step 3. The ground ring for KSC theory is

$$\mathrm{KSC}^* \cong \mathbb{Z}[\beta^{\pm}, \eta']/(2\eta', \eta'^2),$$

where the periodicity element β is in degree -4 and the torsion generator η' is in degree -1. There is a canonical ring homomorphism ε: $\mathrm{KO}^* \to \mathrm{KSC}^*$ (the map on KR induced by $S^{1,1} \to \mathrm{pt}$). Then $\varepsilon(\eta) = \eta'$, $\varepsilon(b) = \beta^2$, and $\varepsilon(\xi) = 2\beta$. These are standard facts which can be found in [8, Section 1], for example.

Step 4. We claim that δ is given by $x \mapsto \varepsilon(x) \cdot \beta^{-1}\eta'$ and that ρ is given by $x \mapsto (x \cdot \eta, 0)$. We get this by playing off the sequences (32) and (33) against each other. Start with $\delta(x) = \varepsilon(x) \cdot y$, $y \in \mathrm{KSC}^3$. If y were 0, we would have a short exact sequence

$$0 \to \mathrm{KSC}^{j+1} \to \widetilde{\mathrm{KR}}^j(T^2, \iota) \to \mathrm{KO}^{j-1} \to 0,$$

and this would imply for example that $\widetilde{\mathrm{KR}}^{-6}(T^2, \iota) \cong \mathrm{KSC}^{-5} \cong \mathbb{Z}_2$, which contradicts what we obtain from the other exact sequence (33) for $j = -6$. Thus $y = \beta^{-1}\eta'$ (the generator of KSC^3) and the claim follows.

Recall that ρ is of the form $x \mapsto (ax, bx)$ with $a \in \mathrm{KO}^{-1}$ and $b \in \mathrm{KO}^0 \cong \mathbb{Z}$. The number b must be 0; otherwise the torsion-free part of $\widetilde{\mathrm{KR}}^j(T^2, \iota)$ would contradict what we got in Step 1 from (32). And $a \in \mathrm{KO}^{-1}$ cannot vanish, because if it did, we would have a short exact sequence

$$0 \to \mathrm{KO}^{-2} \oplus \mathrm{KO}^{-1} \to \widetilde{\mathrm{KR}}^{-1}(T^2, \iota) \to \mathrm{KO}^0 \to 0,$$

giving $\widetilde{\mathrm{KR}}^{-1}(T^2, \iota) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}$, while (32) gives that $\widetilde{\mathrm{KR}}^{-1}(T^2, \iota)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. So this completes the calculation of the boundary maps δ and ρ.

Step 5. To conclude, we use a well-known fact in homotopy theory [1, p. 206], which is that if **KU** and **KO** are the complex and real topological K-theory spectra, then there is a fiber/cofiber sequence of spectra

$$\Sigma\mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \xrightarrow{c} \mathbf{KU}.$$

This corresponds to a famous long exact sequence [23, Theorem III.5.18] or [8, Definition 1.13(2)]:

$$\cdots \to \mathrm{KO}^{-n}(X) \xrightarrow{\eta} \mathrm{KO}^{-n-1}(X) \xrightarrow{c} \mathrm{KU}^{-n-1}(X) \xrightarrow{r\beta_U^{-1}} \mathrm{KO}^{-n+1}(X) \to \cdots.$$

Here c is complexification, r is realification, and β_U is the complex Bott element.

Because of our calculation of the boundary map ρ, KO^j splits off as a direct summand in $\widetilde{\mathrm{KR}}^{-j}(T^2,\iota)$, and the complement can be identified with the cofiber of η with a degree shift. So this completes the proof. □

We conclude by noting that [25, Theorem 4.8] says that if X is a smooth projective variety defined over $\mathbb{R}$ (which in our case will be a curve of genus 1), identified with the Real space of its complex points with involution given by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, then the natural map $K_j(X;\mathbb{Z}_2) \to \mathrm{KR}^{-j}(X;\mathbb{Z}_2)$ sending algebraic to topological K-theory is an isomorphism for j sufficiently large (in our case $j \geq 1$ suffices). Here algebraic K-theory or KR-theory with $\mathbb{Z}_2$ coefficients is related to the integral theory by a universal coefficient sequence

$$0 \to \mathrm{KR}^{-j}(X)/2 \to \mathrm{KR}^{-j}(X;\mathbb{Z}_2) \to {}_2\mathrm{KR}^{-j+1}(X) \to 0, \tag{34}$$

where ${}_2\mathrm{KR}^{-j+1}(X;\mathbb{Z}_2)$ denotes the 2-torsion in $\mathrm{KR}^{-j+1}(X)$, and similarly for K_j. The torsion subgroup of $K_j(X)$ was computed in [34, Main Theorem 0.1] and agrees with our results under this isomorphism.[4]

5. More General Twists and Why They are Needed for Physics

5.1. TWISTED KO-THEORY

While twisted complex K-theory is by now well-known in both the mathematics literature (e.g., [4,5,14,24,35]) and the physics literature (e.g., [9,39]), its cousin, twisted real K-theory, is defined similarly but is less familiar. One way to define it is by using the K-theory of real continuous-trace algebras of real type (see [35, Section 3]). In the separable case, after stabilization, such an algebra is the algebra of sections vanishing at infinity of a bundle whose fibers are the compact operators $\mathcal{K}_\mathbb{R}$ on an infinite-dimensional separable real Hilbert space $\mathcal{H}_\mathbb{R}$. Since $O(\mathcal{H}_\mathbb{R})$ is contractible but the automorphism group of $\mathcal{K}_\mathbb{R}$ is the *projective* orthogonal group $PO(\mathcal{H}_\mathbb{R}) = O(\mathcal{H}_\mathbb{R})/\mathbb{Z}_2$, which is a $K(\mathbb{Z}_2,1)$ space, the relevant algebra bundles are classified by homotopy classes of maps from the space X to $BPO(\mathcal{H}_\mathbb{R})$,

[4]There is a small typo in the statement of [34, Main Theorem 0.1]. $K_2(X)_{\mathrm{tors}}$ should contain $\nu+1$ copies of $\mathbb{Z}_2$ (here ν is the species), not ν copies as written.

which is a $K(\mathbb{Z}_2, 2)$ space. Thus they are classified by a single characteristic class $\widetilde{w}_2 \in H^2(X, \mathbb{Z}_2)$, which one can identify with the characteristic class for Witten's type I string theory without vector structure in [38]. In other words, for each $\widetilde{w}_2 \in H^2(X, \mathbb{Z}_2)$, one gets an 8-periodic family of K-groups $\mathrm{KO}^*(X, \widetilde{w}_2)$, reducing to $\mathrm{KO}^*(X)$ when $\widetilde{w}_2 = 0$. This is analogous to twisting by H-flux for complex K-theory. Recall that the automorphism group of $\mathcal{K}$ (the compact operators on a *complex* infinite-dimensional separable Hilbert space $\mathcal{H}$) is the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/S^1$. In this case the relevant algebra bundles are classified by homotopy classes of maps from X to $BPU(\mathcal{H})$, which is a $K(\mathbb{Z}, 3)$ space. Therefore they are classified by the H-flux $H \in H^3(X; \mathbb{Z})$.

Just as in the complex case, the twisted real K-theory groups can be computed using an Atiyah–Hirzebruch spectral sequence (AHSS)

$$H_c^p(X, \mathrm{KO}^q) \Rightarrow \mathrm{KO}^{p+q}(X, \widetilde{w}_2),$$

where H_c^* is cohomology with compact supports and $\widetilde{w}_2$ appears in the differentials. We will primarily be interested in the case $X = T^2$, in which case the "compact supports" modifier can be dropped and there is only room for one differential,

$$d_2\colon\ H^0(T^2, \mathrm{KO}^q) = \mathrm{KO}^q \to \mathrm{KO}^{q-1} \cong H^2(T^2, \mathrm{KO}^{q-1}).$$

This differential is cup product with $\widetilde{w}_2$, viewed as an element of $H^2(T^2, \mathrm{KO}^{-1}) \cong \mathbb{Z}_2$. So if $\widetilde{w}_2$ is the non-trivial element of $H^2(T^2, \mathbb{Z}_2)$, the E_2 term of the spectral sequence with the nonzero d_2 differentials indicated is shown in Figure 5, and the $E_3 = E_\infty$ term is shown in Figure 6.

The groups $\mathrm{KO}^*(T^2, \widetilde{w}_2)$ are thus determined up to extensions by summing along the diagonals (where $p+q$ takes a constant value). We see that $\mathrm{KO}^0(T^2, \widetilde{w}_2)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}_2^2$, necessarily split, $\mathrm{KO}^{-2}(T^2, \widetilde{w}_2)$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}$, and the remaining groups $\mathrm{KO}^j(T^2, \widetilde{w}_2)$ are $\mathbb{Z}_2^2$ for $j = -1$, $\mathbb{Z}^2$ for $j = -3$, $\mathbb{Z}$ for $j = -4$, 0 for $j = -5$, $\mathbb{Z}$ for $j = -6$, $\mathbb{Z}^2$ for $j = -7$. The only case where we are left with an extension problem is $j = -2$. It turns out that $\mathrm{KO}^{-2}(T^2, \widetilde{w}_2) \cong \mathbb{Z}$, which we can see as follows. A map of degree one $T^2 \to S^2$ collapsing the 1-skeleton $S^1 \vee S^1$ to a point induces a map of spectral sequences which is an isomorphism on the columns with $p = 0$ and $p = 2$, hence shows that $\mathrm{KO}^{-2}(T^2, \widetilde{w}_2) \cong \mathrm{KO}^{-2}(S^2, \widetilde{w}_2)$ (with a non-trivial twist in $H^2(S^2, \mathbb{Z}_2) \cong \mathbb{Z}_2$), so we only need to compute this latter group and show that it is torsion-free. This will be done in Section 5.2 below.

There are many ways of seeing that this sort of $\widetilde{w}_2$ twisting of KO is needed for D-brane classification in the "no vector structure" theory of [38]. But the key feature is that Chan–Paton bundles are given not by $O(n)$ bundles but by $PO(n)$ bundles [38, Section 2.1], [19, Section 7.2], which is precisely how our twisting was defined.

The physics literature suggests that there should be a T-duality between the "type I with no vector structure" theory on T^2 and the type IIA orientifold on

$q \backslash p$	0	1	2
0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
-1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
-2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
-3	0	0	0
-4	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
-5	0	0	0
-6	0	0	0
-7	0	0	0

Figure 5. E_2 of the spectral sequence for computing $\mathrm{KO}^*(T^2, \widetilde{w}_2)$. The sequence repeats with vertical period 8.

$q \backslash p$	0	1	2
0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
-1	0	$\mathbb{Z}_2^2$	0
-2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	0
-3	0	0	0
-4	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
-5	0	0	0
-6	0	0	0
-7	0	0	0

Figure 6. E_∞ of the spectral sequence for computing $\mathrm{KO}^*(T^2, \widetilde{w}_2)$. The sequence repeats with vertical period 8.

an elliptic curve with anti-holomorphic involution of species 1 (i.e., a fixed set which is topologically just a single circle) [26]. The D-brane charges in this theory are described by the groups $\mathrm{KR}^j(T^2, \iota)$, where ι is an involution on T^2 with fixed set S^1. These groups were computed above in Theorem 4. Table II shows $\mathrm{KO}^j(T^2, \widetilde{w}_2)$ for $\widetilde{w}_2 \neq 0$, $\mathrm{KR}^j(T^2, \iota)$ for the species 1 anti-holomorphic involution

Table II. $\mathrm{KO}^j(T^2, \widetilde{w}_2)$, $\mathrm{KR}^j(T^2, \iota)$, and $\mathrm{KR}^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$

j mod 8	$\mathrm{KO}^j(T^2, \widetilde{w}_2)$	$\mathrm{KR}^j(T^2, \iota)$	$\mathrm{KR}^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$
0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2$	$\mathbb{Z}$
−1	$\mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2$
−2	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$
−3	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$
−4	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
−5	0	$\mathbb{Z}$	$\mathbb{Z}^2$
−6	$\mathbb{Z}$	0	$\mathbb{Z}$
−7	$\mathbb{Z}^2$	$\mathbb{Z}$	0

ι, and $\mathrm{KR}^j_{(+,+,+,-)}(S^{1,1} \times S^{1,1})$ from Section 4. The second column agrees precisely with the first column shifted down by 1, and the third column agrees with the second column shifted down by 1, as is predicted by T-duality. Note that the data of the B-field for the type IIB theory on $S^{1,1} \times S^{1,1}$ with $\alpha = (+,+,+,-)$ is encoded in the non-triviality of the d_2 differential for the spectral sequence in Figure 4. In [15] we will describe how the B-field is described by a sign choice under T-duality.

5.2. TWISTED KO-THEORY WITH AN H^1 TWIST

Twisting of $\mathrm{KO}^*(X)$ by $H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2)$ was already defined by Donovan and Karoubi in [14]. (The group of twists $HO(X)$ is actually a non-split abelian extension of $H^2(X, \mathbb{Z}_2)$ by $H^1(X, \mathbb{Z}_2)$.) For X compact and $\mathcal{A}$ a bundle over X whose fibers are $\mathbb{Z}_2$-graded simple $\mathbb{R}$-algebras, with $w(\mathcal{A}) = \alpha \in HO(X)$, $\mathrm{KO}^\alpha(X)$ is the Grothendieck group of graded real vector bundles X which are finitely generated projective modules for $\mathcal{A}$. Here $w(\mathcal{A}) = \big(w_1(\mathcal{A}), w_2(\mathcal{A})\big)$, where vanishing of $w_2(\mathcal{A}) \in H^2(X, \mathbb{Z}_2)$ is the condition for $\mathcal{A}$ to be the endomorphism bundle of a $\mathbb{Z}_2$-graded vector bundle, and $w_1(\mathcal{A}) = w_1(V)$ if $\mathcal{A}$ is the Clifford algebra bundle of a real vector bundle V for a negative definite metric [14, Lemma 7]. When $w_1 = 0$, we get back the twisted KO-groups of Section 5.1. The basic composition rule in $HO(X)$ is that

$$w_1(\mathcal{A} \hat{\otimes} \mathcal{B}) = w_1(\mathcal{A}) + w_1(\mathcal{B}), \qquad w_2(\mathcal{A} \hat{\otimes} \mathcal{B}) = w_2(\mathcal{A}) + w_2(\mathcal{B}) + w_1(\mathcal{A}) \cdot w_1(\mathcal{B}).$$

For general X, $HO(X)$ can have elements of order 4, but this will not happen if (as for S^1 or T^2) every element of $H^1(X, \mathbb{Z}_2)$ has square 0. Thus (assuming this condition) every element of $HO(X)$ is its own inverse, and by the Thom isomorphism theorem of [14, Section 6], if V is a real vector bundle over X,

$$\mathrm{KO}^j(V) \cong \mathrm{KO}^{j - \dim V}(X, w_1(V), w_2(V)). \tag{35}$$

Table III. $\mathrm{KO}^j(S^1, w_1)$ for the non-trivial twist

j	0	−1	−2	−3	−4	−5	−6	−7
$\mathrm{KO}^j(S^1, w_1)$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$

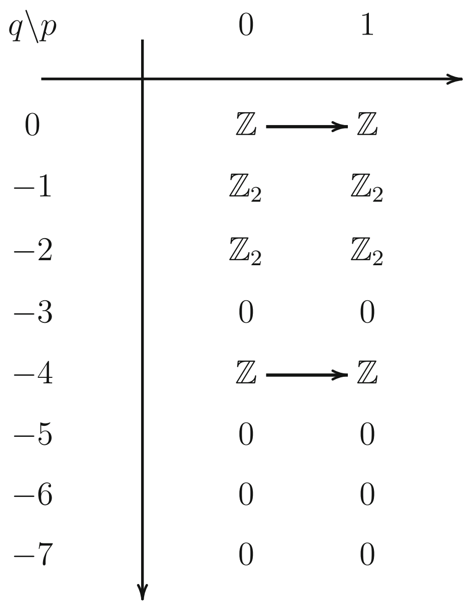

Figure 7. E_1 of the spectral sequence for computing $\mathrm{KO}^*(S^1, w_1)$. The sequence repeats with vertical period 8. *Arrows* represent multiplication by 2, so in $E_2 = E_\infty$, each $\mathbb{Z}$ in the $p=1$ column is replaced by a $\mathbb{Z}_2$, and each $\mathbb{Z}$ in the $p=0$ column dies.

As an example of (35), we can compute $\mathrm{KO}^j(S^1, w_1)$ for the non-trivial element $w_1 \in H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Indeed, we have

$$\mathrm{KO}^j(S^1, w_1) \cong \mathrm{KO}^{j+1}(V) \cong \widetilde{\mathrm{KO}}^{j+1}(\mathbb{RP}^2),$$

where V is the non-trivial real line bundle over S^1, that is, the Möbius strip. And the KO-groups of $\mathbb{RP}^2$ were computed in [18, Theorem 1]. The result is given in Table III. Here the surprise is the existence of 4-torsion in $\widetilde{\mathrm{KO}}^0(\mathbb{RP}^2) \cong \mathrm{KO}^{-1}(S^1, w_1)$.

The groups in Table III can once again be explained by a twisted Atiyah–Hirzebruch spectral sequence with starting point $H^p(S^1, \mathrm{KO}^q)$, but this time the only differential is d_1, which is multiplication by 2 in the places indicated by the arrows in Figure 7.

The calculation of $\mathrm{KO}^*(S^1, w_1)$ also enables us to compute $\mathrm{KO}^*(T^2, w_1)$ for any choice of a twisting $w_1 \in H^1(T^2, \mathbb{Z}_2)$. The reason is that for any such $w_1 \neq 0$, we can choose a topological splitting $T^2 = S^1 \times S^1$ with respect to which w_1 lives only on the first factor, so that $(T^2, w_1) \cong (S^1, w_1) \times (S^1, 0)$. It follows that $\mathrm{KO}^j(T^2, w_1)$ splits as $\mathrm{KO}^j(S^1, w_1) \oplus \mathrm{KO}^{j-1}(S^1, w_1)$.

The calculation of $\mathrm{KO}^*(S^1, w_1)$ also enables us to compute $\mathrm{KO}^\alpha(T^2)$ in the sense of Donovan–Karoubi for a twist α with both $w_1(\alpha)$ and $w_2(\alpha)$ nonzero. Indeed, let V again be the nontrivial real line bundle over S^1, that is, the Möbius strip. Then $V \times V$ (the Cartesian product) is a rank-two real vector bundle over $S^1 \times S^1 = T^2$. If a and b are the elements of $H^1(T^2, \mathbb{Z}_2)$ dual to the two circles in the decomposition $T^2 = S^1 \times S^1$, then $V \times V$ can be identified with the Whitney sum $L_a \oplus L_b$, since the fiber of $V \times V$ over $(x, y) \in S^1 \times S^1$ is $L_a(x, y) \times L_b(x, y) = L_a(x, y) \oplus L_b(x, y)$. Note that $w_1(L_a \oplus L_b) = a + b$ and $w_2(L_a \oplus L_b) = ab$, a generator of $H^2(T^2, \mathbb{Z}_2)$. So $\mathrm{KO}^j(T^2, a+b, ab) = \mathrm{KO}^{j+2}(V \times V)$. The same holds for $\mathrm{KO}^j(T^2, w_1, w_2)$ for any nonzero w_1, w_2 since there is a self-homeomorphism of T^2 sending w_1 to $a+b$. Finally we can compute $\mathrm{KO}^j(T^2, w_1, w_2) \cong \mathrm{KO}^{j+2}(V \times V)$ using the fact that V has a closed subspace homeomorphic to $\mathbb{R}$, with $(V \smallsetminus \mathbb{R}) \cong \mathbb{R}^2$, so that we get from the pair $(V \times V, V \times \mathbb{R})$ an exact sequence

$$\cdots \to \mathrm{KO}^j(V) \to \mathrm{KO}^j(V) \to \mathrm{KO}^j(T^2, w_1, w_2) \to \mathrm{KO}^{j+1}(V) \to \cdots.$$

In particular, $\mathrm{KO}^j(T^2, w_1, w_2)$ is a 2-primary torsion group for all j.

Finally, we mention still another application of the Thom isomorphism (35), namely the completion of the calculation of $\mathrm{KO}^*(T^2, \widetilde{w}_2)$ when the twist is nonzero. Observe that the nonzero element $\widetilde{w}_2 \in H^2(T^2, \mathbb{Z}_2)$ is pulled back from the generator of $H^2(S^2, \mathbb{Z}_2)$ under a map $T^2 \to S^2$ of degree one, so to compute $\mathrm{KO}^*(T^2, \widetilde{w}_2)$, we can begin by computing $\mathrm{KO}^*(S^2, \widetilde{w}_2)$. The generator of $H^2(S^2, \mathbb{Z}_2)$ is $\widetilde{w}_2$ for the underlying real 2-plane bundle of the Hopf (complex) line bundle over $S^2 \cong \mathbb{CP}^1$, for which the total space is $\mathbb{CP}^2 \smallsetminus \{\mathrm{pt}\}$. So by (35), $\mathrm{KO}^{-j}(S^2, \widetilde{w}_2) \cong \widetilde{\mathrm{KO}}^{-j+2}(\mathbb{CP}^2)$, which is computed in [18, Theorem 2]. (The degree 0 part was computed earlier in [36, Section 3.6].) Rather surprisingly, $\widetilde{\mathrm{KO}}^*(\mathbb{CP}^2)$ is entirely torsion-free, with copies of $\mathbb{Z}$ in all even degrees and nothing in odd degrees. Thus $\mathrm{KO}^{-2}(S^2, \widetilde{w}_2) \cong \mathrm{KO}^{-2}(T^2, \widetilde{w}_2) \cong \widetilde{\mathrm{KO}}^0(\mathbb{CP}^2) \cong \mathbb{Z}$, not $\mathbb{Z} \oplus \mathbb{Z}_2$.

We should mention that even though we did not need it for studying D-brane charges in orientifold theories on 2-tori, in higher dimensional situations one might be forced to consider all the various kinds of twists of KR (sign choice, H^1, and H^2) simultaneously. The general framework for such twists is included in the work of Moutuou [30,32].

6. Conclusion

KR-theory with a sign choice (Definition 1) allows us to give a mathematical description of D-brane charges for all orientifolds including ones with both O^+- and O^--planes. The additional data of a sign choice is required to distinguish between topologically equivalent spaces with different O-plane content. As we saw, KR-theory with a sign choice gives a purely mathematical description of the D-

branes in the type $\widetilde{\mathrm{IA}}$ theory. This calculation provides further evidence for T-duality rather than requiring its assumption to determine the brane charges.

In addition to providing new tests of T-duality, KR-theory with a sign choice predicts the D-brane content in theories that could not be computed previously (which in turn can aid in the discovery of unknown dualities). We are not aware of the D-brane content for the type I theory without vector structure or either of its T-dual theories appearing anywhere in the literature. This extends the usefulness of K-theory as a first check for D-brane content to orientifold theories. As noted previously, the K-theoretic description cannot determine the sources for the D-brane charge, only that there is a stable charge. Determining the stable charges using KR-theory with a sign choice can greatly constrain what sources need to be tested for stability at different points in the moduli space using other methods (such as considering the boundary state description). Since boundary state descriptions can be quite difficult for orientifolds, any constraints are very useful, and as we show in [15], most of the sources can often be determined from the KR-theory using what we know about $O^{\pm}$-planes.

As noted in the introduction, one of our original motivations for a detailed analysis of T-duality via orientifold plane charges in KR-theory was the special case of $c=3$ Gepner models as studied in [6]. The authors of that paper used simple current techniques in CFT to construct the charges and tensions of Calabi–Yau orientifold planes. Using twisted KR-theory with a sign choice to classify the brane charges does not depend on the specific structure of $c=3$ Gepner models, nor even on a rational conformal field theoretic description. In [17] a twisted equivariant K-theory description of the D-brane charge content for WZW models is provided. Current work in progress attempts to generalize this work by establishing an isomorphism between a suitable (real) variant of twisted equivariant K-theory, sufficient to capture orientifold charge content, and our KR-theory with sign choices for Gepner models. Such an isomorphism would allow the computation of twisted KR-theory with a sign choice for complicated Calabi–Yau manifolds through a simpler computation at the Gepner point.

KR-theory with a sign choice provides a universal K-theory for classifying D-brane charges. In addition to being able to describe new orientifold cases it reduces to all other known classifications on smooth manifolds when using the correct involution. This unifies the K-theoretic classification of D-brane charges by not requiring one to change K-theories for different string theories. While its definition was motivated by a problem in physics, the last point exemplifies why KR-theory with a sign choice is also interesting mathematically.

KR-theory with a sign choice provides a framework for studying the underlying structure of K-theory. It was very surprising to see that KSC-theory (the KR-theory of $S^{0,2}$) can be described as a *twisting* of the KR-theory of $S^{1,1}$. While we have explicitly shown that twisted KR-theory with a sign choice satisfies all possible T-duality relationships for spaces where the compact dimensions are a circle or a 2-torus, in this paper we did not look at why there are isomorphisms between the twisted KR-theories of T-dual theories. The purpose of this paper was simply to set up the necessary topology to correctly classify brane charges. In [15] we explore why T-duality gives isomorphisms of twisted KR-theory with a sign choice. The extra data that we needed to include is contained in the geometry of T-dual theories.

We have already seen how considering the geometry is important. The complex structure constrains what involutions are possible on a 2-torus. Since the physical theory depends on the involution, the geometry of the torus constrains the allowable string theories. Another well known example that played a role in our analysis is the B-field, which is determined by the Kähler modulus. We were also compelled to explore more exotic twists in order to account for the T-duality of the type I theory without vector structure. Without the physical motivation we might not have considered looking at such additional mathematical structures. We have shown how such twistings must behave via an Atiyah–Hirzebruch spectral sequence. In [15] we give a more geometric reason for why such twistings must be included.

By exploring the underlying topology and geometry we were able to gain physical information and new evidence for hypothesized dualities. Additionally, this work shows how we can go in the opposite direction and use the additional structure of physics to gain insight into the underlying geometry and topology. This gives us a greater understanding of the interplay between the three structures: topology, geometry, and physics.

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