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Lattice polarized K3 surfaces and Siegel modular forms

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Abstract

The goal of the present paper is two-fold. First, we present a classification of algebraic K3 surfaces polarized by the lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$. Key ingredients for this classification are as follows: a normal form for these lattice polarized K3 surfaces, a coarse moduli space and an explicit description of the inverse period map in terms of Siegel modular forms. Second, we give explicit formulas for a Hodge correspondence that relates these K3 surfaces to principally polarized abelian surfaces. The Hodge correspondence in question underlies a geometric two-isogeny of K3 surfaces, the details of which are described by the authors in Clingher and Doran (2011) [7].

Keywords: K3 surfaces; Kummer surfaces; Elliptic fibrations; Modular forms

1. Introduction

Let X be an algebraic K3 surface defined over the field of complex numbers. Denote by NS(X) the Néron–Severi lattice of X. This is an even lattice of signature $(1, \mathrm{p}_\mathrm{X} - 1)$ where p_X is the Picard rank. By definition (see [8]), a *lattice polarization* on the surface X is given by a primitive lattice embedding

$$i : \mathrm{N} \hookrightarrow \mathrm{NS(X)}$$

whose image contains a pseudo-ample class. Here N is a choice of even lattice of signature $(1, r)$ with $0 \le r \le 19$. Two N-polarized K3 surfaces (X, i) and (X', i') are said to be isomorphic if

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there exists an analytic isomorphism $\alpha : \mathrm{X} \to \mathrm{X}'$ such that $\alpha^* \circ i' = i$, where α^* is the appropriate cohomology morphism.

The present paper concerns the special class of K3 surfaces polarized by the even lattice of rank seventeen

$$\mathrm{N} = \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7.$$

Here H stands for the standard hyperbolic lattice of rank two and E_8, E_7 are negative definite lattices associated with the corresponding exceptional root systems. Surfaces in this class have Picard ranks taking four possible values: 17, 18, 19 or 20.

This special class of algebraic K3 surfaces is of interest because of a remarkable Hodge-theoretic feature. Any given N-polarized K3 surface (X, i) is associated uniquely with a well-defined principally polarized complex abelian surface (A, Π). This feature appears due to the fact that both types of surfaces mentioned above are classified, via appropriate versions of Torelli Theorem, by a Hodge structure of weight two on $\mathrm{T} \otimes \mathbb{Q}$ where T is the rank-five lattice $\mathrm{H} \oplus \mathrm{H} \oplus (-2)$. This fact determines a bijective map:

$$(\mathrm{X}, i) \leftrightarrow (\mathrm{A}, \Pi) \tag{1}$$

which is a Hodge correspondence. In fact, the map (1) can be regarded as a particular case of a more general Hodge-theoretic construction due to Kuga and Satake [23]. In particular, map (1) realizes an analytic identification between the moduli spaces of periods associated with the two types of surfaces, both of which could be seen as the classical Siegel modular threefold $\mathcal{F}_2 = \mathrm{Sp}_4(\mathbb{Z}) \setminus \mathbb{H}_2$.

The correspondence given by (1) can be further refined. The set of all isomorphism classes of N-polarized K3 surfaces divides naturally into two disjoint subclasses. The first subclass consists of those surfaces (X, i) for which the lattice polarization i extends canonically to a polarization by the unimodular rank-eighteen lattice $\mathrm{M} = \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. In terms of the Siegel modular threefold $\mathcal{F}_2$, this subclass is associated with the Humbert surface usually denoted by $\mathcal{H}_1$. Under (1), the principally polarized abelian surface (A, Π) associated to a M-polarized K3 surfaces (X, i) is of the form

$$\big(\mathrm{E}_1 \times \mathrm{E}_2, \mathcal{O}_{\mathrm{E}_1 \times \mathrm{E}_2}((\mathrm{E}_1 \times \{p_2\}) + (\{p_1\} \times \mathrm{E}_2))\big)$$

where (E_1, p_1) and (E_2, p_2) are complex elliptic curves, uniquely determined up to permutation.

The second subclass is given by those N-polarized K3 surfaces (X, i) for which the lattice polarization cannot be extended from N to M. These surfaces correspond in the Siegel threefold to the open region $\mathcal{F}_2 \setminus \mathcal{H}_1$. Their associated principally polarized abelian surfaces (A, Π) are of the form

$$\big(\mathrm{Jac}(\mathrm{C}),\ \mathcal{O}_{\mathrm{Jac}(\mathrm{C})}(\Theta)\big)$$

where C is a non-singular complex genus-two curve and Θ is the theta-divisor, the image of C under the Abel–Jacobi embedding. The genus-two curve C is uniquely determined by the pair (X, i) and (1) provides an analytic identification between $\mathcal{F}_2 \setminus \mathcal{H}_1$ and the moduli space $\mathcal{M}_2$ of complex genus-two curves.

The goal of the present paper is two-fold. First, we present a full classification theory for N-polarized K3 surfaces along the lines of the classical theory of elliptic curves defined over the field of complex numbers. Second, we give explicit formulas for the correspondence (1) in terms of Siegel modular forms.

A key ingredient for the results of this paper is the introduction of a *normal form* associated to K3 surfaces with N-polarizations. It will be instructive to first recall the classical Weierstrass normal form for complex elliptic curves and to trace our results in parallel with that case.

Theorem 1.1. *Let* (g_2, g_3) *be a pair of complex numbers. Denote by* $\mathrm{E}(g_2, g_3)$ *the curve in* $\mathbb{P}^2(x, y, z)$ *cut out by the degree-three homogeneous equation*

$$y^2z - 4x^3 + g_2xz^2 + g_3z^3 = 0. \tag{2}$$

(a) *If* $\triangle := g_2^3 - 27g_3^2$ *is nonzero, then* $\mathrm{E}(g_2, g_3)$ *is an elliptic curve.*
(b) *Given any elliptic curve* E*, there exists* $(g_2, g_3) \in \mathbb{C}^2$*, with* $\triangle \neq 0$*, such that the curves* E *and* $\mathrm{E}(g_2, g_3)$ *are isomorphic as elliptic curves.*

Our first result in this paper is analogous to the above.

Theorem 1.2. *Let* $(\alpha, \beta, \gamma, \delta)$ *be a quadruple of complex numbers. Denote by* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *the minimal resolution of the surface in* $\mathbb{P}^3(x, y, z, w)$ *cut out by the degree-four homogeneous equation*

$$y^2zw - 4x^3z + 3\alpha xzw^2 + \beta zw^3 + \gamma xz^2w - \frac{1}{2}(\delta z^2w^2 + w^4) = 0. \tag{3}$$

(a) *If* $\gamma \neq 0$ *or* $\delta \neq 0$*, then* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *is a* K3 *surface endowed with a canonical* N*-polarization.*
(b) *Given any* N*-polarized* K3 *surface* X*, there exists* $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$*, with* $\gamma \neq 0$ *or* $\delta \neq 0$*, such that surfaces* X *and* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *are isomorphic as* N*-polarized* K3 *surfaces.*

The quartic (3) extends a two-parameter family of K3 surfaces given by Inose in [18]. In the context of (3), the special case $\gamma = 0$ corresponds to the situation when the polarization extends to the lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$, whereas the N-polarizations of K3 surfaces $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ with $\gamma \neq 0$ cannot be extended to $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$.

As it turns out, the normal forms (3) are also ideal objects for establishing a moduli space for isomorphism classes of N-polarizations of K3 surfaces. Again let us first recall the classical case of Weierstrass elliptic curves.

Theorem 1.3. *Two curves* $\mathrm{E}(g_2, g_3)$ *and* $\mathrm{E}(g_2', g_3')$ *are isomorphic as elliptic curves if and only if there exists* $t \in \mathbb{C}^*$ *such that*

$$\left(g_2',\ g_3' \right) = \left(t^2g_2,\ t^3g_3 \right).$$

The open variety:

$$\mathcal{M}_{\mathrm{E}} = \left\{ [g_2, g_3] \in \mathbb{WP}^2(2, 3) \mid \triangle \neq 0 \right\}$$

forms a coarse moduli space for elliptic curves.

In the above context, the *j-invariant*

$$\mathrm{j}(\mathrm{E}) := \frac{g_2^3}{\triangle}$$

identifies $\mathcal{M}_{\mathrm{E}}$ and $\mathbb{C}$ (the "j-line"). The period map to the classifying space of Hodge structures is the isomorphism of quasi-projective varieties:

$$\mathrm{per}\colon \mathcal{M}_{\mathrm{E}} \to \mathcal{F}_1 = \mathrm{PSL}_2(\mathbb{Z}) \setminus \mathbb{H} \tag{4}$$

whose inverse is given by

$$\mathrm{per}^{-1} = [60\mathrm{E}_4,\ 140\mathrm{E}_6]$$

where $\mathrm{E}_4, \mathrm{E}_6 : \mathbb{H} \to \mathbb{C}$ are the classical Eisenstein series of weights four and six, respectively.

In our N-polarized K3 surface setting, we have following analogous result.

Theorem 1.4. *Two* K3 *surfaces* $\mathrm{X}(\alpha_1, \beta_1, \gamma_1, \delta_1)$ *and* $\mathrm{X}(\alpha_2, \beta_2, \gamma_2, \delta_2)$ *are isomorphic as* N*-polarized* K3 *surfaces if and only if there exists* $t \in \mathbb{C}^*$ *such that*

$$(\alpha_2,\ \beta_2,\ \gamma_2,\ \delta_2) = \left(t^2\alpha_1,\ t^3\beta_1,\ t^5\gamma_1,\ t^6\delta_1\right).$$

The open variety:

$$\mathcal{M}^{\mathrm{N}}_{\mathrm{K3}} = \left\{[\alpha, \beta, \gamma, \delta] \in \mathbb{WP}^3(2, 3, 5, 6) \mid \gamma \neq 0 \text{ or } \delta \neq 0\right\}$$

forms a coarse moduli space for N*-polarized* K3 *surfaces.*

In the context of Theorem 1.4, the period map to the associated classifying space of Hodge structures appears as a morphism of quasi-projective varieties:

$$\mathrm{per}\colon \mathcal{M}^{\mathrm{N}}_{\mathrm{K3}} \to \mathcal{F}_2 = \mathrm{Sp}_4(\mathbb{Z}) \setminus \mathbb{H}_2. \tag{5}$$

By the appropriate version of Global Torelli Theorem for lattice polarized K3 surfaces (see for instance [8]), one knows that (5) is in fact an isomorphism. We prove that the inverse of (5) has a simple description in terms of Siegel modular forms.

Theorem 1.5. *The inverse period map* $\mathrm{per}^{-1} : \mathcal{F}_2 \to \mathcal{M}^{\mathrm{N}}_{\mathrm{K3}}$ *is given by*

$$\mathrm{per}^{-1} = \left[\mathcal{E}_4,\ \mathcal{E}_6,\ 2^{12}3^5\mathcal{C}_{10},\ 2^{12}3^6\mathcal{C}_{12}\right]$$

where $\mathcal{E}_4, \mathcal{E}_6$ *are genus-two Eisenstein series of weight four and six, and* $\mathcal{C}_{10}$ *and* $\mathcal{C}_{12}$ *are Igusa's cusp forms of weight* 10 *and* 12*, respectively.*

Theorems 1.2, 1.4 and 1.5 allow one to give an explicit description of the dual principally polarized abelian surfaces associated by the Hodge-theoretic correspondence (1). In the case of K3 surfaces polarized by the lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$, explicit formulas were given previously by the authors [5] as well as Shioda [31].

Theorem 1.6. *Under the duality correspondence* (1)*, the principally polarized abelian surface* A *associated to* $\mathrm{X}(\alpha, \beta, 0, \delta)$ *is given by*

$$\left(\mathrm{E}_1 \times \mathrm{E}_2,\ \mathcal{O}_{\mathrm{E}_1 \times \mathrm{E}_2}(\mathrm{E}_1 + \mathrm{E}_2)\right)$$

where E_1 *and* E_2 *are complex elliptic curves with* j*-invariants satisfying*

$$\mathrm{j}(\mathrm{E}_1) + \mathrm{j}(\mathrm{E}_2) = \frac{\alpha^3 - \beta^2}{\delta} + 1, \qquad \mathrm{j}(\mathrm{E}_1) \cdot \mathrm{j}(\mathrm{E}_2) = \frac{\alpha^3}{\delta}.$$

In this paper, we use the formulas of Theorem 1.5 in order to explicitly identify the genus-two curves C corresponding to the remaining case, by computing the Igusa–Clebsch invariants $[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] \in \mathbb{WP}^3(2, 4, 6, 10)$ associated with these curves.

Theorem 1.7. *Assume $\gamma \neq 0$. Under the duality correspondence (1), the principally polarized abelian surface* A *associated to* $X(\alpha, \beta, \gamma, \delta)$ *is given by*

$$\big(\mathrm{Jac(C)},\ \mathcal{O}_{\mathrm{Jac(C)}}(\Theta)\big)$$

where C *is a smooth genus-two curve of Igusa–Clebsch invariants*

$$[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] = \Big[2^3 3\delta,\ 2^2 3^2 \alpha\gamma^2,\ 2^3 3^2 (4\alpha\delta + \beta\gamma)\gamma^2,\ 2^2\gamma^6\Big].$$

The present paper should be considered in connection with the companion paper [7]. This is because the proofs of the theorems mentioned above do not involve period computations. They rather rely on a very specific observation: the Hodge-theoretic correspondence (1) is a consequence of a purely geometric phenomenon — the existence of a pair of dual geometric two-isogenies of K3 surfaces between the N-polarized surface X and the Kummer surface Y associated to the abelian surface A corresponding to (X, i) under (1). The precise meaning of this isogeny concept is explained in [7]. In short, the observation consists of the existence of two special Nikulin involutions Φ_X and Φ_Y, acting on the surfaces X and Y, respectively, which lead to degree-two rational maps p_X and p_Y. The involutions Φ_X and Φ_Y are associated naturally with two particular elliptic fibrations φ_X and φ_Y on X and Y over a base rational curve B. The involutions are fiberwise two-isogenies in the sense that they correspond to translations by sections of order-two within the smooth fibers of the fibrations φ_X and φ_Y.

$$\begin{tikzcd} Y \arrow[loop left, "\Phi_Y"] \arrow[rr, dashed, bend right, "p_Y"'] \arrow[dr, "\varphi_Y"'] & & X \arrow[loop right, "\Phi_X"] \arrow[ll, dashed, bend right, "p_X"'] \arrow[dl, "\varphi_X"] \\ & B & \end{tikzcd} \tag{6}$$

The above geometric phenomenon allows one to make the duality map explicit, without involving an analysis of Hodge structures or period computations.

The present work focuses on the case of N-polarized K3 surfaces for which the lattice polarization does not extend to a polarization by the lattice $M = H \oplus E_8 \oplus E_8$. The case involving M-polarizations has been presented in [6], work on which the present paper builds.

Various partial ingredients pertaining to this construction have been discussed by the authors and others in earlier works. In his 1977 work [18], Inose presented a normal form for K3 surfaces and constructed the Nikulin involution Φ_Y on the Kummer surface associated with the product of two elliptic curves. The construction of Φ_Y in Inose's context uses a different elliptic fibration with respect to which the Nikulin involution is not a fiberwise isogeny. In paper [6], the authors constructed each piece of diagram (6) in the case of M-polarized K3 surfaces, including explicit equations for both elliptic fibrations φ_X and φ_Y. This case was also treated in [31] by Shioda. One particular sub-family of M-polarized K3 surfaces, with generic Picard lattice enhanced to $H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$ was considered by Van Geemen and Top in [36]. The Van Geemen–Top family corresponds, in terms of the duality (1), to pairs of two-isogenous elliptic curves.

An indication that the construction can be extended from M-polarized to N-polarized K3 surfaces was given by Dolgachev in his appendix to the paper [9] by Galluzzi and Lombardo, where, based on an analysis of Fourier–Mukai partners, they observe that K3 surfaces with Néron–Severi lattice exactly N are in correspondence with Jacobians of genus two curves.

The present paper has its origin in Dolgachev's observation. The authors extend the geometric arguments from [6] to the full N-polarized case by constructing in detail the two-isogenies between the N-polarized K3 surfaces and their dual Kummer surfaces of principally polarized abelian surfaces. An explicit computation based on parts of this construction was made by Kumar [24].

2. A four-parameter quartic family

Definition 2.1. Consider $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. Let $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ be the projective quartic surface in $\mathbb{P}^3(x, y, z, w)$ given by

$$y^2zw - 4x^3z + 3\alpha xzw^2 + \beta zw^3 + \gamma xz^2w - \frac{1}{2}(\delta z^2w^2 + w^4) = 0. \tag{7}$$

Denote by $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ the non-singular complex surface obtained as the minimal resolution of $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$.

The four-parameter quartic family $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ generalizes a special two-parameter family of K3 surfaces introduced by Inose in [18].

Theorem 2.2. *If $\gamma \neq 0$ or $\delta \neq 0$, then $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ is a K3 surface endowed with a canonical* N-*polarization.*

Proof. The hypothesis $\gamma \neq 0$ or $\delta \neq 0$ ensures that the singular locus of the quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ consists of a finite collection of rational double points. This fact implies, in turn, that $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ is a K3 surface.

Let us present the N-polarization on $\mathrm{X}(\alpha, \beta, \gamma, \delta)$. Note that $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ has two special singular points

$$\mathrm{P}_1 = [0, 1, 0, 0], \qquad \mathrm{P}_2 = [0, 0, 1, 0].$$

For a generic choice of quadruple $(\alpha, \beta, \gamma, \delta)$, the singular locus of $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ is precisely $\{\mathrm{P}_1, \mathrm{P}_2\}$. Under the condition $\gamma \neq 0$ or $\delta \neq 0$, both P_1 and P_2 are rational double point singularities. The point P_1 is always a rational double point of type A_{11}. The type of the rational double point P_2 is covered by two situations. If $\gamma \neq 0$ then P_2 has type A_5. When $\gamma = 0$, the singularity at P_2 is of type E_6.

The intersection locus of the quartic $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ with the plane of equation $w = 0$ consists of two distinct lines, L_1 and L_2, defined by $x = w = 0$ and $z = w = 0$, respectively. In addition, when $\gamma \neq 0$ one has an additional special curve on $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$ obtained as the intersection of the plane of equation $2\gamma x = \delta w$ with the cubic surface

$$2\gamma^3y^2z + (-\delta^3 + 3\alpha\gamma^2\delta + 2\beta\gamma^3)zw^2 - \gamma^3w^3 = 0. \tag{8}$$

This curve resolves to a rational curve in $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ which we denote by c.

After resolving the singularities at P_1 and P_2 one obtains a special configuration of rational curves on $\mathrm{X}(\alpha, \beta, \gamma, \delta)$. The dual diagram of this configuration, in the two cases in question, is presented in Figs. 1 and 2.

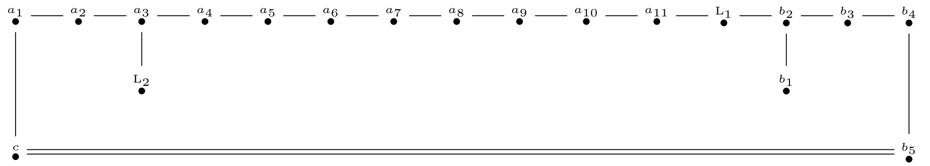


Fig. 1. Case $\gamma \neq 0$.

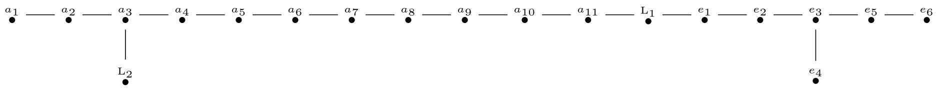
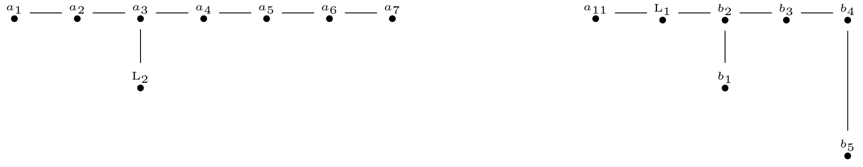
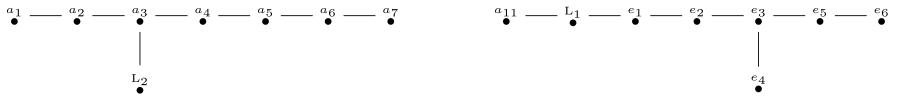


Fig. 2. Case $\gamma = 0$.

Note that when $\gamma \neq 0$ one has the following $E_8 \oplus E_7$ configuration.



When $\gamma = 0$, one has a similar $E_8 \oplus E_8$ configuration of curves.



The remaining orthogonal hyperbolic lattice H is spanned by the two classes associated to divisors a_9 and f where

$$\begin{aligned} f &= a_8 + 2a_7 + 3a_6 + 4a_5 + 5a_4 + 6a_3 + 3\mathrm{L}_2 + 4a_2 + 2a_1 \\ &= a_{10} + 2a_{11} + 3\mathrm{L}_1 + 4b_2 + 2b_1 + 3b_3 + 2b_2 + b_1 \end{aligned} \tag{9}$$

when $\gamma \neq 0$ and

$$\begin{aligned} f &= a_8 + 2a_7 + 3a_6 + 4a_5 + 5a_4 + 6a_3 + 3\mathrm{L}_2 + 4a_2 + 2a_1 \\ &= a_{10} + 2a_{11} + 3\mathrm{L}_1 + 4e_1 + 5e_2 + 6e_3 + 3e_4 + 2e_5 + e_6 \end{aligned} \tag{10}$$

when $\gamma = 0$. □

Remark 2.3. The surfaces $\mathrm{X}(\alpha, \beta, 0, 0)$ are rational surfaces. On the projective quartic surface $\mathrm{Q}(\alpha, \beta, 0, 0)$ the singularity at P_2 is no longer a rational double point, but an elliptic singularity.

Let us briefly discuss the discriminant locus of the family of quartics $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$. One particular discriminant component, given by the vanishing of

$$\mathcal{D}_1(\alpha, \beta, \gamma, \delta) = \gamma \tag{11}$$

was already mentioned during the proof of Theorem 2.2. This component corresponds (on its $\delta \neq 0$ region) to the situation when the N-polarization extends canonically to a lattice polarization by $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. These surfaces were discussed by the authors in [5]. In terms of the correspondence (1), surfaces $\mathrm{X}(\alpha, \beta, 0, \delta)$ correspond with principally polarized abelian surfaces (A, Π) which are products of two elliptic curves.

A second important discriminant component corresponds to the situation when, in addition to the singular points P_1 and P_2, extra singularities occur on the quartic surface $Q(\alpha, \beta, \gamma, \delta)$. One can check that this situation is determined by the vanishing of the polynomial

$$\begin{aligned}\mathcal{D}_4(\alpha, \beta, \gamma, \delta) = {} & -2^5 3^6 \alpha^6 \beta \gamma^3 + 2^6 3^6 \alpha^3 \beta^3 \gamma^3 - 2^5 3^6 \beta^5 \gamma^3 - 2^4 3^5 \alpha^5 \gamma^4 \\ & + 2^4 3^5 5^2 \alpha^2 \beta^2 \gamma^4 + 2 \cdot 3^3 5^4 \alpha \beta \gamma^5 + 5^5 \gamma^6 - 2^4 3^7 \alpha^7 \gamma^2 \delta \\ & + 2^5 3^7 \alpha^4 \beta^2 \gamma^2 \delta - 2^4 3^7 \alpha \beta^4 \gamma^2 \delta + 2^3 3^5 5 \cdot 19 \alpha^3 \beta \gamma^3 \delta \\ & + 2^3 3^5 5^2 \beta^3 \gamma^3 \delta + 3^3 5^3 11 \alpha^2 \gamma^4 \delta + 2^3 3^5 37 \alpha^4 \gamma^2 \delta^2 \\ & + 2^3 3^5 5 \cdot 7 \alpha \beta^2 \gamma^2 \delta^2 - 2^3 3^3 5^3 \beta \gamma^3 \delta^2 + 2^4 3^6 \alpha^6 \delta^3 - 2^5 3^6 \alpha^3 \beta^2 \delta^3 \\ & + 2^4 3^6 \beta^4 \delta^3 - 2^6 3^6 \alpha^2 \beta \gamma \delta^3 - 2^3 3^5 5^2 \alpha \gamma^2 \delta^3 \\ & - 2^5 3^6 \alpha^3 \delta^4 - 2^5 3^6 \beta^2 \delta^4 + 2^4 3^6 \delta^5. \end{aligned} \tag{12}$$

We shall see later (Remark 3.8), an interpretation of the polynomial (12) in terms of Segel modular forms. At this point, we note that this discriminant locus corresponds to the case in which the N-polarization extends canonically to a polarization by the lattice $H \oplus E_8 \oplus E_7 \oplus A_1$. In terms of the correspondence (1), surfaces $X(\alpha, \beta, 0, \delta)$ correspond with principally polarized abelian surfaces (A, Π) which admit an elliptic subgroup of degree two, or equivalently, A is two-isogenous with a product of two elliptic curves.

The overlap of the two discriminant components from above consists of the quartic surfaces $Q(\alpha, \beta, 0, \delta)$ with

$$\alpha^6 + \beta^4 + \delta^2 - 2\alpha^3 \beta^2 - 2\alpha^3 \delta - 2\beta^2 \delta = 0. \tag{13}$$

The K3 surfaces $X(\alpha, \beta, \gamma, \delta)$ associated with the above condition are precisely those for which the canonical N-polarization extends to a polarization by $H \oplus E_8 \oplus E_8 \oplus A_1$.

2.1. *Special features on* $X(\alpha, \beta, \gamma, \delta)$

Let us outline a few special properties of the four-parameter K3 family introduced above. These properties will play an important role in subsequent considerations.

Note that the isomorphism class of $X(\alpha, \beta, \gamma, \delta)$ does not change under a certain weighted scaling of the parameters $(\alpha, \beta, \gamma, \delta)$.

Proposition 2.4. *Let* $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ *with* $\gamma \neq 0$ *or* $\delta \neq 0$. *For any* $t \in \mathbb{C}^*$, *the two* N-*polarized* K3 *surfaces*

$$X(\alpha, \beta, \gamma, \delta) \quad \text{and} \quad X(t^2\alpha,\ t^3\beta,\ t^5\gamma,\ t^6\delta)$$

are isomorphic.

Proof. Let q be a square root of t. The proposition then follows from the fact that the projective automorphism

$$\Phi \colon \mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [x, y, z, w] \mapsto [q^8 x,\ q^9 y,\ z,\ q^6 w]$$

maps the quartic $Q(\alpha, \beta, \gamma, \delta)$ to $Q(t^2\alpha,\ t^3\beta,\ t^5\gamma,\ t^6\delta)$ while satisfying $\Phi(P_1) = P_1$, $\Phi(P_2) = P_2$. □

The K3 family $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ can therefore be regarded as being parametrized, up to an isomorphism, by the three-dimensional open analytic space

$$\mathcal{P}_{\mathrm{N}} = \left\{ [\alpha, \beta, \gamma, \delta] \in \mathbb{WP}^3(2, 3, 5, 6) \mid \gamma \neq 0 \text{ or } \delta \neq 0 \right\}. \tag{14}$$

One of the main results of this paper (to be justified by the subsequent sections) is that the space $\mathcal{P}_{\mathrm{N}}$ is a coarse moduli space for N-polarized K3 surfaces.

We also note that K3 surfaces $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ carry two special elliptic fibrations

$$\varphi^{\mathrm{s}}_{\mathrm{X}}, \varphi^{\mathrm{a}}_{\mathrm{X}} \colon \mathrm{X}(\alpha, \beta, \gamma, \delta) \to \mathbb{P}^1,$$

which we shall refer to as *standard* and *alternate*.[1] The two fibrations are associated with the pencils of planes in $\mathbb{P}^3$ containing the lines L_2 and L_1, respectively. In explicit coordinates, one can see $\varphi^{\mathrm{s}}_{\mathrm{X}}$ and $\varphi^{\mathrm{a}}_{\mathrm{X}}$ as induced, respectively, from the rational projections

$$\mathrm{pr}_1, \mathrm{pr}_2 \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1, \qquad \mathrm{pr}_1([x, y, z, w]) = [z, w], \qquad \mathrm{pr}_2([x, y, z, w]) = [x, w].$$

$$\begin{tikzcd}
 & & \mathbb{P}^1 \\
\mathrm{X}(\alpha,\beta,\gamma,\delta) \arrow[r] \arrow[urr, bend left, "\varphi^{\mathrm{s}}_{\mathrm{X}}"] \arrow[drr, bend right, "\varphi^{\mathrm{a}}_{\mathrm{X}}"'] & \mathbb{P}^3 \arrow[ur, dashed, "\mathrm{pr}_1"] \arrow[dr, dashed, "\mathrm{pr}_2"'] & \\
 & & \mathbb{P}^1
\end{tikzcd} \tag{15}$$

Using the above setting, one can easily write explicit Weierstrass forms for the elliptic fibrations $\varphi^{\mathrm{s}}_{\mathrm{X}}, \varphi^{\mathrm{a}}_{\mathrm{X}}$. For instance,

$$v^2 = u^2 + f_{\mathrm{s}}(\lambda)u + g_{\mathrm{s}}(\lambda)$$

with

$$f_{\mathrm{s}}(\lambda) = \lambda^4(\gamma\lambda + 3\alpha), \qquad g_{\mathrm{s}}(\lambda) = -\lambda^5(\delta\lambda^2 - 2\beta\lambda + 1)$$

describes the standard elliptic fibration $\varphi^{\mathrm{s}}_{\mathrm{X}}$ over the affine chart $\{[\lambda, 1] \mid \lambda \in \mathbb{C}\}$ of its base. A simple computation determines the discriminant of this elliptic curve family as

$$\begin{aligned} 4f_{\mathrm{s}}^3(\lambda) + 27g_{\mathrm{s}}^2(\lambda) = {} & \lambda^{10}(4\gamma^3\lambda^5 + 3(16\alpha\gamma^2 + 9\delta^2)\lambda^4 + 12(16\alpha^2\gamma - 9\beta\delta)\lambda^3 \\ & + 2(128\alpha^3 + 54\beta^2 + 27\delta)\lambda^2 - 108\beta\lambda + 27). \end{aligned}$$

For the alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$, one can describe the fibers over the affine chart $\{[\mu, 1] \mid \mu \in \mathbb{C}\}$ of the base as

$$v^2 = u^2 + f_{\mathrm{a}}(\mu)u + g_{\mathrm{a}}(\mu)$$

with

$$\begin{aligned} f_{\mathrm{a}}(\mu) &= \frac{1}{12}\left(-64\mu^6 + 96\alpha\mu^4 + 32\beta\mu^3 - 36\alpha^2\mu^2 - 6(4\alpha\beta + \gamma)\mu - 4\beta^2 + 3\delta\right) \\ g_{\mathrm{a}}(\mu) &= \frac{1}{108}(4\mu^3 - 3\alpha\mu - \beta)(128\mu^6 - 192\alpha\mu^4 - 64\beta\mu^3 + 72\alpha^2\mu^2 \\ &\quad + 6(8\alpha\beta + 3\gamma)\mu + 8\beta^2 - 9\delta). \end{aligned}$$

[1] The broader context of the elliptic fibrations $\varphi^{\mathrm{s}}_{\mathrm{X}}, \varphi^{\mathrm{a}}_{\mathrm{X}}$ is discussed in Section 4.1.

The discriminant of this family is

$$4f_{\mathrm{a}}^3(\mu)+27g_{\mathrm{a}}^2(\mu) = -\frac{1}{16}(2\gamma\mu-\delta)^2(16\mu^6-24\alpha\mu^4-8\beta\mu^3 \\ +9\alpha^2\mu^2+2(3\alpha\beta+\gamma)\mu+\beta^2-\delta).$$

An analysis based on Tate's algorithm [32], applied in the context of the above formulas, allows one to conclude the following.

Proposition 2.5. *Assume $\gamma \neq 0$ or $\delta \neq 0$. The standard elliptic fibration $\varphi_{\mathrm{X}}^{\mathrm{s}}\colon \mathrm{X}(\alpha,\beta,\gamma,\delta) \to \mathbb{P}^1$ carries a section, given by the curve a_9 from Figs. 1 or 2. In addition, there are two special singular fibers over the base points $[0,1]$ and $[1,0]$. The fiber $\varphi_{[0,1]}^{\mathrm{s}}$ has Kodaira type II^* and is represented by the divisor*

$$2a_1+4a_2+6a_3+3\mathrm{L}_2+5a_4+4a_5+3a_6+2a_7+a_8.$$

If $\gamma \neq 0$, then the fiber $\varphi_{[1,0]}^{\mathrm{s}}$ has type III^* and is represented by the divisor

$$b_5+2b_4+3b_3+4b_2+2b_1+3\mathrm{L}_1+2a_{11}+a_{10}$$

from Fig. 1. If $\gamma = 0$, then the fiber $\varphi_{[1,0]}^{\mathrm{s}}$ has type II^* and is represented by the divisor

$$2e_6+4e_5+6e_3+3e_4+5e_2+4e_1+3\mathrm{L}_1+2a_{11}+a_{10}$$

from Fig. 2.

Proposition 2.6. *Assume $\gamma \neq 0$ or $\delta \neq 0$. The alternate elliptic fibration $\varphi_{\mathrm{X}}^{\mathrm{a}}\colon \mathrm{X}(\alpha,\beta,\gamma,\delta) \to \mathbb{P}^1$ carries two disjoint sections, given by the pairs of curves a_1, b_4 or a_1, e_6 from Figs. 1 or 2, respectively. There is a special singular fiber over the base point $[1,0]$. If $\gamma \neq 0$, then the fiber $\varphi_{[1,0]}^{\mathrm{a}}$ has Kodaira type I_{10}^* and is represented by the divisor*

$$a_2+\mathrm{L}_2+2\,(a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+a_{11}+\mathrm{L}_1+b_2)+b_3+b_1$$

from Fig. 1. In such a case, one also has a singular fiber over $[\delta, 2\gamma]$ given by the divisor:

$$c+b_5$$

of Fig. 1. The fiber $\varphi_{[\delta,2\gamma]}^{\mathrm{a}}$ has type I_2 if

$$3\alpha\gamma^2\delta+2\beta\gamma^3-\delta^3 \neq 0$$

and type III *if*

$$3\alpha\gamma^2\delta+2\beta\gamma^3-\delta^3 = 0.$$

If $\gamma = 0$, then the fiber $\varphi_{[1,0]}^{\mathrm{a}}$ has type I_{12}^* and is represented by the divisor

$$a_2+\mathrm{L}_2+2\,(a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+a_{11} \\ +\mathrm{L}_1+e_1+e_2+e_3)+e_5+e_4$$

from Fig. 2.

Note that the standard fibration $\varphi_{\mathrm{X}}^{\mathrm{s}}$ offers an alternate way of defining the N-polarization on the K3 surface $\mathrm{X}(\alpha,\beta,\gamma,\delta)$. It is known (see [5,22,30]) that a pseudo-ample N-polarization on a K3 surface is equivalent geometrically with the existence of a jacobian elliptic fibration with two distinct special fibers of Kodaira types II^* and III^* (or higher), respectively.

However, it is the alternate elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$, that will play the major role in the consideration of this paper. Let us consider the case $\gamma \neq 0$. Then, the alternate fibration has two disjoint sections given by the curves a_1 and b_4 and a singular fiber of type I^*_{10} occurs over the base point $[1, 0]$ of $\varphi^{\mathrm{a}}_{\mathrm{X}}$. Consider the affine chart $[\mu, 1]$ as in Proposition 2.6. The elliptic fiber of $\varphi^{\mathrm{a}}_{\mathrm{X}}$ over $[\mu, 1]$ has then the cubic form

$$\left\{ y^2 z - \left(4\mu^3 - 3\alpha\mu - \beta\right) z w^2 + \gamma\mu z^2 w - \frac{1}{2}(\delta z^2 w + w^3) = 0 \right\} \subset \mathbb{P}^2(y, z, w), \tag{16}$$

with two special points $[1, 0, 0]$ and $[0, 1, 0]$ associated with the two sections. The affine version of the cubic equation in (16), in the base chart $w = 1$, is

$$y^2 z = z^2 \left(\frac{1}{2}\delta - \gamma\mu\right) + z\left(4\mu^3 - 3\alpha\mu - \beta\right) + \frac{1}{2}, \tag{17}$$

and one can easily verify that this affine cubic curve carries a special involution

$$(y, z) \mapsto \left(-y,\ \frac{1}{(\delta - 2\gamma\mu)z}\right). \tag{18}$$

The map (18) extends to an involution of (16) which exchanges the section points $[1, 0, 0]$ and $[0, 1, 0]$. For the smooth elliptic curves in (16), the point $[0, 1, 0]$ can be seen as a point of order two in the elliptic curve group with origin at $[1, 0, 0]$. The involution determined by (18) amounts then to a fiber-wise translation by $[0, 1, 0]$.

Note that, after multiplying (17) by $z\left(\frac{1}{2}\delta - \gamma\lambda\right)^2$, one gets

$$\begin{aligned}\left[yz\left(\frac{1}{2}\delta - \gamma\mu\right)\right]^2 &= \left[z\left(\frac{1}{2}\delta - \gamma\mu\right)\right]^3 + \left[z\left(\frac{1}{2}\delta - \gamma\mu\right)\right]^2 \left(4\mu^3 - 3\alpha\mu - \beta\right) \\ &\quad + \left[z\left(\frac{1}{2}\delta - \gamma\mu\right)\right] \frac{1}{2}\left(\frac{1}{2}\delta - \gamma\mu\right).\end{aligned}$$

With the coordinate change

$$y_1 = yz\left(\frac{1}{2}\delta - \gamma\mu\right), \qquad z_1 = z\left(\frac{1}{2}\delta - \gamma\mu\right),$$

one obtains

$$y_1^2 = z_1^3 + \mathcal{P}(\mu)\cdot z_1^2 + \mathcal{Q}(\mu)\cdot z_1, \tag{19}$$

where

$$\mathcal{P}(\mu) = 4\mu^3 - 3\alpha\mu - \beta, \qquad \mathcal{Q}(\mu) = \frac{1}{2}\left(\frac{1}{2}\delta - \gamma\mu\right).$$

One can recognize in (19) the classical equation for a jacobian elliptic fibration with a special section of order two (see, for instance, Section 4 of the work of Van Geemen and Sarti [35]). The involution of (18) can be described in this new coordinate context as

$$(z_1,\ y_1) \mapsto \left(\frac{\mathcal{Q}(\mu)}{z_1}, -\frac{\mathcal{Q}(\mu)\cdot y_1}{z_1^2}\right).$$

One obtains the following result.

Proposition 2.7. *Let* $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ *with* $\gamma \neq 0$ *or* $\delta \neq 0$. *The birational projective involution*

$$\begin{aligned} &\Psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3, \\ &\Psi([x, y, z, w]) = [xz(\delta w - 2\gamma x),\ -yz(\delta w - 2\gamma x),\ w^3,\ zw(\delta w - 2\gamma x)] \end{aligned} \tag{20}$$

restricts to a birational involution of the quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta)$. *Moreover* Ψ *lift to a non-trivial involution* Φ_{X} *of the* N*-polarized* K3 *surface* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$.

$$\begin{tikzcd} \mathrm{X}(\alpha,\beta,\gamma,\delta) \arrow[r, "\Phi_{\mathrm{X}}"] \arrow[d] & \mathrm{X}(\alpha,\beta,\gamma,\delta) \arrow[d] \\ \mathbb{P}^3 \arrow[r, dashed, "\Psi"] & \mathbb{P}^3 \end{tikzcd} \tag{21}$$

The involution Φ_{X} *exchanges the two disjoint sections of the alternate fibration* φ^{a} *and, on the smooth fibers of this fibration, amounts to a fiber-wise translation by a section of order two.*

Using the terminology of Definition 1.1 in [7], $\Phi_{\mathrm{X}} : \mathrm{X}(\alpha, \beta, \gamma, \delta) \to \mathrm{X}(\alpha, \beta, \gamma, \delta)$ is a Van Geemen–Sarti involution. In the context of the dual diagrams of Figs. 1 and 2, the involution Φ_{X} acts as a horizontal left–right flip.

3. Hodge theory and Siegel modular forms

A coarse moduli space for the isomorphism classes of N-polarized K3 surfaces can be constructed by gluing together spaces of local deformations. We refer the reader to [1,8] for a detailed description of the method. The moduli space $\mathcal{M}^{\mathrm{N}}_{\mathrm{K3}}$ so obtained is a quasi-projective analytic space of complex dimension three. Hodge theory, by the period map and the appropriate version of the Global Torelli Theorem provides one with an effective method of analyzing the structure of this space.

3.1. The period isomorphism

Recall that, up to an overall isometry, there exists a unique primitive embedding of N into the K3 lattice

$$\mathrm{L} = \mathrm{H} \oplus \mathrm{H} \oplus \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8.$$

Fix such a lattice embedding and denote by T the orthogonal complement of its image. The classical period domain associated to the lattice T is then

$$\Omega = \{\omega \in \mathbb{P}^1(\mathrm{T} \otimes \mathbb{C}) \mid (\omega, \omega) = 0, \quad (\omega, \bar{\omega}) > 0\}.$$

One also has the following group isomorphism.

$$\{\sigma \in \mathcal{O}(\mathrm{L}) \mid \sigma(\gamma) = \gamma \text{ for every } \gamma \in \mathrm{N}\} \xrightarrow{\simeq} \mathcal{O}(\mathrm{T}).$$

Via the classical Hodge decomposition, one associates to each N-polarized K3 surface (X, i) a well-defined point in the classifying space of N-polarized Hodge structures

$$\mathcal{O}(\mathrm{T}) \setminus \Omega.$$

Moreover, by the Global Torelli Theorem [8] for lattice polarized K3 surfaces, one has that the period map so constructed

$$\mathrm{per}\colon \mathcal{M}_{\mathrm{K3}}^{\mathrm{N}} \longrightarrow \mathcal{O}(\mathrm{T}) \setminus \Omega \tag{22}$$

is an isomorphism of analytic spaces.

Let us analyze in detail the period domain Ω. Note that the rank-five lattice T is naturally isomorphic to the orthogonal direct sum $\mathrm{H} \oplus \mathrm{H} \oplus (-2)$. We select an integral basis $\{p_1, p_2, q_1, q_2, r\}$ for T with intersection matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Since p_1, p_2, q_1, q_2 are all isotropic vectors, their intersection pairing with any given period line in Ω is non-zero. The Hodge–Riemann bilinear relations imply then that every period in Ω can be uniquely realized in this basis as

$$\omega(\tau, u, z) = (\tau,\ 1,\ u,\ z^2 - \tau u,\ z)$$

with $\tau, u, z \in \mathbb{C}$ satisfying $\tau_2 u_2 > z_2^2$. The 2-indices represent the fact that the imaginary part has been taken.

The period domain Ω has two connected components Ω_o and $\overline{\Omega}_o$ which get interchanged by the complex conjugation. Moreover, the map

$$\kappa = \begin{pmatrix} \tau & z \\ z & u \end{pmatrix} \to \omega(\tau, u, z) \tag{23}$$

provides an analytic identification between the classical Siegel upper-half space of degree two:

$$\mathbb{H}_2 = \left\{ \kappa = \begin{pmatrix} \tau & z \\ z & u \end{pmatrix} \mid \tau_2 u_2 > z_2^2,\ \tau_2 > 0 \right\} \tag{24}$$

and the connected component Ω_o. The action of the discrete group $\mathcal{O}(\mathrm{T})$ admits a nice reinterpretation under this identification. Note that the real orthogonal group $\mathcal{O}(\mathrm{T}, \mathbb{R})$ has four connected components and the kernel of its action on Ω is given by $\pm\mathrm{id}$. Let $\mathcal{O}^+(\mathrm{T}, \mathbb{R})$ be the (index-two) subgroup of $\mathcal{O}(\mathrm{T}, \mathbb{R})$ that fixes the connected component of Ω_o. This group can also be seen as

$$\mathcal{O}^+(\mathrm{T}, \mathbb{R}) = \{\pm\mathrm{id}\} \cdot \mathcal{SO}^+(\mathrm{T}, \mathbb{R})$$

where $\mathcal{SO}^+(\mathrm{T}, \mathbb{R})$ is the subgroup of $\mathcal{O}^+(\mathrm{T}, \mathbb{R})$ corresponding to isometries of positive spinor-norm. Finally, set $\mathcal{O}^+(\mathrm{T}) = \mathcal{O}^+(\mathrm{T}, \mathbb{R}) \cap \mathcal{O}(\mathrm{T})$. One has then the following isomorphism of groups.

$$\mathrm{Sp}_4(\mathbb{Z})/\{\pm \mathrm{I}_4\} \longrightarrow \mathcal{O}^+(\mathrm{T})/\{\pm\mathrm{id}\} \simeq \mathcal{SO}^+(\mathrm{T}). \tag{25}$$

The details of (25) are given in [11, see Lemma 1.1 therein]. Under (25) and in connection with the classical action of the group $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$ on $\mathbb{H}_2$, the identification (23) becomes equivariant. The following sequence of isomorphisms holds

$$\Gamma_2 \setminus \mathbb{H}_2 \simeq \mathcal{O}^+(\mathrm{T}) \setminus \Omega_o \simeq \mathcal{O}(\mathrm{T}) \setminus \Omega.$$

One obtains the following.

Proposition 3.1. *The period isomorphism (22) identifies the moduli space $\mathcal{M}^{\mathrm{N}}_{\mathrm{K3}}$ with the standard Siegel modular threefold*

$$\mathcal{F}_2 = \Gamma_2 \setminus \mathbb{H}_2. \tag{26}$$

As it is well-known (see, for instance, Chapter 8 of [2]), complex abelian surfaces (A, Π) endowed with principal polarizations are also classified by Hodge structures of weight two associated with the lattice T. Moreover, via an appropriate version of Global Torelli Theorem, one has that the corresponding period map establishes an analytic identification between the coarse moduli space $\mathcal{A}_2$ of isomorphism classes of principally polarized complex abelian surfaces and the Siegel modular threefold $\mathcal{F}_2$. In connection with the above considerations, one obtains then the following result.

Proposition 3.2. *There exists a Hodge theoretic correspondence:*

$$(\mathrm{A}, \Pi) \longleftrightarrow (\mathrm{X}, i) \tag{27}$$

associating bijectively to every N*-polarized* K3 *surface* (X, i) *a unique principally polarized abelian surface* (A, Π)*. The correspondence (27) underlies an analytic identification*

$$\mathcal{A}_2 \cong \mathcal{M}^{\mathrm{N}}_{\mathrm{K3}} \tag{28}$$

between the corresponding coarse moduli spaces.

One can further refine the correspondence (27). Recall (see, for instance, Chapter 4 of [10]) that a principal polarization Π on a complex abelian surface A can be of two types:

(i) $\Pi = \mathcal{O}_{\mathrm{A}}(\mathrm{E}_1 + \mathrm{E}_2)$ where E_1 and E_2 are smooth complex elliptic curves. In this case, the abelian surface A splits canonically as a cartesian product $\mathrm{E}_1 \times \mathrm{E}_2$.

(ii) $\Pi = \mathcal{O}_{\mathrm{A}}(\mathrm{C})$ where C is a smooth complex genus-two curve. In this case one can identify A canonically with the Jacobian variety Jac(C), with the divisor C being given by the image of the Abel–Jacobi map.

Case (i) corresponds with the situation when the abelian surface A admits an H-polarization. Under (27), one obtains then that the polarization N-polarization i of the corresponding K3 surface X can be extended to a polarization by the rank-eighteen lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. the case (ii) corresponds with the situation when the principal polarization given by Π cannot be extended to an H-polarization of A. Therefore, via (27) one obtains N-polarized K3 surfaces (X, i) for which the polarization i cannot be extended to an $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$-polarization.

The considerations of this section lead then to the following conclusion. The bijective correspondence (27) breaks into two parts. first, one has a bijective correspondence:

$$(\mathrm{E}_1,\ \mathrm{E}_2) \longleftrightarrow (\mathrm{X}, i) \tag{29}$$

between un-ordered pairs of complex elliptic curves and $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$-polarized K3 surfaces (X, i). Second, one has a bijective correspondence:

$$\mathrm{C} \longleftrightarrow (\mathrm{X}, i) \tag{30}$$

between smooth complex genus-two curves C and N-polarized K3 surfaces (X, i) with the property that polarization i does not extend to an $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ polarization.

The correspondence (29) was the central topic of the previous work [5] of the authors. The present paper gives an explicit description for (30).

3.2. Siegel modular forms in genus two

An effective way to understand the geometry of $\mathcal{F}_2$ is to use the Siegel modular forms of genus two. Let us enumerate here the main such forms that will be relevant to the present paper. For detailed references, we refer the reader to the classical papers of Igusa [15–17] and Hammond [12] as well as the more recent monographs of Van der Geer [34] and Klingen [20].

The simplest Siegel modular forms of genus two are those derived from Eisenstein series. These are modular forms of even weight and are defined through the classical series:

$$\mathcal{E}_{2t}(\kappa) = \sum_{(\mathrm{C},\mathrm{D})} \det(\mathrm{C}\kappa + \mathrm{D})^{-2t}, \quad t > 1. \tag{31}$$

The group $\Gamma_1 = \mathrm{SL}(2, \mathbb{Z})$ acts by simultaneous left-multiplication on the pairs (C, D) of symmetric 2×2 integral matrices, and the sum in (31) is taken over the orbits of this action. The Eisenstein forms $\mathcal{E}_{2t}$ are also integral, in the sense that their Fourier coefficients are integers.

A second special class of Siegel modular forms of degree two are the Siegel cusp forms, which lie in the kernel of the Siegel operator. The most important cusp forms are $\mathcal{C}_{10}$, $\mathcal{C}_{12}$ and $\mathcal{C}_{35}$, of weights 10, 12 and 35, respectively. One has[2]:

$$\mathcal{C}_{10} = -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} \, (\mathcal{E}_4\mathcal{E}_6 - \mathcal{E}_{10}) \tag{32}$$

$$\mathcal{C}_{12} = 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} \left(3^2 \cdot 7^2 \mathcal{E}_4^3 + 2 \cdot 5^3 \mathcal{E}_6^2 - 691 \mathcal{E}_{12}\right) \tag{33}$$

while $\mathcal{C}_{35}$ satisfies a polynomial equation $\mathcal{C}_{35}^2 = \mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12})$ where P is a specific polynomial with all monomials of weighted degree 70. The exact form of $\mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12})$ can be found in [16, p. 849].

The structure of the ring of Siegel modular forms of genus two is given by Igusa's Theorem:

Theorem 3.3. *(Igusa [17]) The graded ring* $\mathrm{A}(\Gamma_2, \mathbb{C})$ *of Siegel modular forms of degree two is generated by* $\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}$ *and* $\mathcal{C}_{35}$ *and is isomorphic to*

$$\mathbb{C}\,[\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}] \,/\, \left(\mathcal{C}_{35}^2 = \mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12})\right).$$

Note that, by Igusa's work [15], the cusp forms $\mathcal{C}_{10}$, $\mathcal{C}_{12}$ and $\mathcal{C}_{35}$ can also be introduced in terms of theta constants of even characteristics as follows:

$$\mathcal{C}_{10}(\kappa) = -2^{-14} \cdot \prod_{m \text{ even}} \theta_m(\kappa)^2 \tag{34}$$

$$\mathcal{C}_{12}(\kappa) = 2^{-17} \cdot 3^{-1} \cdot \sum_{(m_1 m_2 m_3 m_4 m_5 m_6)} \left(\theta_{m_1}(\kappa)\theta_{m_2}(\kappa)\theta_{m_3}(\kappa)\theta_{m_4}(\kappa)\theta_{m_5}(\kappa)\theta_{m_6}(\kappa)\right)^4 \tag{35}$$

$$\mathcal{C}_{35}(\kappa) = -i \cdot 2^{-39} \cdot 5^{-3} \cdot \left(\prod_{m \text{ even}} \theta_m(\kappa)\right) \cdot \left(\sum_{\substack{(m_1 m_2 m_3) \\ \text{asyzygous}}} \pm \left(\theta_{m_1}(\kappa)\theta_{m_2}(\kappa)\theta_{m_3}(\kappa)\right)^{20}\right). \tag{36}$$

[2] Note that in Igusa's original notation [15–17], the modular forms $\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}$ appear as ψ_4, ψ_6, χ_{10}, χ_{12} and χ_{35}.

The products in (34) and (36) are taken over the ten even characteristics. The sum in (35) is taken over the complements of the fifteen syzygous (Göpel) quadruples of even characteristics. The sum in (36) is taken over the sixty asyzygous triples of even characteristics. According to Igusa's terminology, a triple of even characteristics is called syzygous if the sum of the three characteristics is even. Otherwise, the triple is called asyzygous. A set of even characteristics is called syzygous (respectively asyzygous) if all triples of the set are syzygous (respectively asyzygous).

The factors

$$\mathcal{C}_5(\kappa) = 2^{-7} \cdot \prod_{m \text{ even}} \theta_m(\kappa) \tag{37}$$

$$\mathcal{C}_{30}(\kappa) = -i \cdot 2^{-32} \cdot 5^{-3} \cdot \left(\sum_{\substack{(m_1 m_2 m_3)\\ \text{asyzygous}}} \pm \left(\theta_{m_1}(\kappa)\theta_{m_2}(\kappa)\theta_{m_3}(\kappa)\right)^{20} \right). \tag{38}$$

are not Siegel modular forms in the traditional sense, as they carry non-trivial characters $\Gamma_2 \to \mathbb{Z}/2\mathbb{Z}$. The forms $\mathcal{C}_5, \mathcal{C}_{30}$ however satisfy the relations

$$\mathcal{C}_5^2 = -\mathcal{C}_{10}, \qquad \mathcal{C}_5\mathcal{C}_{30} = \mathcal{C}_{35}.$$

When computing with modular forms in practice, one can employ standard methods of [26,27,14] that reduce expressions involving the ten theta constants with even characteristics to four fundamental theta constants (as given in Section 5.2). Using Igusa's formulas in Section 4 of [14] and Section 3 of [17], one obtains explicit (and far from complicated) expressions

$$\begin{aligned} \mathcal{E}_4 &= 2^4 P_8 \\ \mathcal{E}_6 &= 2^6 P_{12} \\ \mathcal{C}_{10} &= -2^2 Q_{20} \\ \mathcal{C}_{12} &= 2^4 \cdot 3^{-1} Q_{24} \end{aligned} \tag{39}$$

where P_8, P_{12}, Q_{20} and Q_{24} are homogeneous polynomials in the fundamental theta constants a, b, c, d. The precise formulas for P_8, P_{12}, Q_{20} and Q_{24}, are given in Appendix A.1.

3.3. The Singular Locus of $\mathcal{F}_2$

The Siegel modular threefold $\mathcal{F}_2 = \Gamma_2 \setminus \mathbb{H}_2$ is non-compact and highly singular. The singular locus of $\mathcal{F}_2$ consists of the images under the projection

$$\mathbb{H}_2 \to \Gamma_2 \setminus \mathbb{H}_2 \tag{40}$$

of the points in $\mathbb{H}_2$ whose associated periods $\omega(\tau, u, z)$ are orthogonal to roots[3] of the rank-five lattice T. As T is isomorphic to the orthogonal direct sum $\mathrm{H} \oplus \mathrm{H} \oplus \mathrm{A}_1$, the set of roots of T forms two distinct orbits under the natural action of $\mathcal{O}(\mathrm{T})$. The two orbits are distinguished by the lattice type of the orthogonal complement $\{r\}^\perp \subset \mathrm{T}$ of a particular root r. For roots r in one orbit, the orthogonal complements $\{r\}^\perp$ are isomorphic to $\mathrm{H} \oplus \mathrm{H}$. For roots r belonging to the second orbit, $\{r\}^\perp$ are isomorphic to $\mathrm{H} \oplus (2) \oplus (-2)$. These facts can be shown either directly, or deduced from more general results such as the ones in [29].

[3] A root of T is an element $r \in \mathrm{T}$ such that $(r, r) = -2$.

The singular locus of $\mathcal{F}_2$ has therefore two connected components, which turn out to be the two Humbert surfaces $\mathcal{H}_1$ and $\mathcal{H}_4$. These surfaces are the images under the projection (40) of the two divisors in $\mathbb{H}_2$ associated to $z = 0$ and $\tau = u$, respectively. As analytic spaces, both these loci are Hilbert modular surfaces (see for instance Chapter IX of [33]). The Humbert surfaces $\mathcal{H}_1$, $\mathcal{H}_4$ are the vanishing locus of the cusp forms $\mathcal{C}_5$ and $\mathcal{C}_{30}$, respectively. The formal sum $\mathcal{H}_1 + \mathcal{H}_4$ is then the vanishing divisor of the Siegel cusp form $\mathcal{C}_{35}$.

We note that, under the period isomorphism of Proposition 3.1, the Humbert surface $\mathcal{H}_1$ corresponds to N-polarized K3 surfaces (X, i) for which the lattice polarization i extends to an $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$-polarization. The associated principally polarized abelian surfaces under (27) are products of two elliptic curves.

The complement of $\mathcal{H}_1$ in $\mathcal{F}_2$ corresponds to periods associated to smooth genus-two curves, which are nicely classified by the Igusa–Clebsch invariants.

Remark 3.4. Via the periods of the polarized Jacobian varieties Jac(C), one gets a natural identification between the open subset $\mathcal{F}_2 \setminus \mathcal{H}_1$ and the moduli space $\mathcal{M}_2$ of isomorphism classes of non-singular complex genus-two curves. The Igusa–Clebsch invariants [3,4,14]

$$[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] \ \in \mathbb{WP}(2, 4, 6, 10) \tag{41}$$

classify the isomorphism class of a genus-two curve, as well as realize explicit coordinates on $\mathcal{F}_2 \setminus \mathcal{H}_1$. The invariants can be defined [16], in terms of Siegel modular forms of genus two, as

$$[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] \ = \left[2^3 3 \frac{\mathcal{C}_{12}}{\mathcal{C}_{10}},\ 2^2 \mathcal{E}_4,\ 2^5 \frac{\mathcal{E}_4 \mathcal{C}_{12}}{\mathcal{C}_{10}} + 2^3 3^{-1} \mathcal{E}_6,\ 2^{14} \mathcal{C}_{10} \right]. \tag{42}$$

The above expression makes sense, as for period classes $[\kappa] \in \mathcal{F}_2 \setminus \mathcal{H}_1$, one has $\mathcal{C}_{10}(\kappa) \neq 0$.

In particular, the Igusa–Clebsch invariants realize an explicit identification between $\mathcal{F}_2 \setminus \mathcal{H}_1$, the moduli space $\mathcal{M}_2$ of genus-two curves and the open variety:

$$\{[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] \ \in \mathbb{WP}(2, 4, 6, 10) \mid \mathcal{D} \neq 0\}. \tag{43}$$

We also note that the periods in the Humbert surface $\mathcal{H}_4$ are given by N-polarized K3 surfaces (X, i) for which the lattice polarization i extends to an $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7 \oplus \mathrm{A}_1$-polarization. In terms of (27), the locus $\mathcal{H}_4$ corresponds to principally polarized abelian surfaces (A, Π) that are two-isogenous with a product of two elliptic curves.

3.4. The main theorem

The main theorem of this paper asserts the following.

Theorem 3.5. *Let* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *be the four-parameter family of* N*-polarized* K3 *surfaces introduced in Section* 2*. For* $\gamma \neq 0$ *or* $\delta \neq 0$*, let* $\kappa \in \mathbb{H}_2$ *be a period point associated with* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$*. Then, one has the following identity involving weighted projective points in* $\mathbb{WP}(2, 3, 5, 6)$*:*

$$[\alpha,\ \beta,\ \gamma,\ \delta] = \left[\mathcal{E}_4,\ \mathcal{E}_6,\ 2^{12} 3^5 \mathcal{C}_{10},\ 2^{12} 3^6 \mathcal{C}_{12} \right]. \tag{44}$$

The computation leading to the above result is based on a special geometric two-isogeny of K3 surfaces, the details of which are presented in the companion paper [7]. An outline of this transformation is provided here in Section 4. The proof of Theorem 3.5 is given in Section 5.

In the light of Theorem 3.5 and based on the classical considerations of Igusa [14,16], one obtains that the period map

$$\mathrm{per}\colon \mathcal{P}_{\mathrm{N}} \to \mathcal{F}_2$$

is an isomorphism and (44) gives an explicit description of its inverse map. In particular, one obtains the following.

Corollary 3.6. *The open analytic space*

$$\mathcal{P}_{\mathrm{N}} = \Big\{[\alpha, \beta, \gamma, \delta] \in \mathbb{WP}^3(2, 3, 5, 6) \mid \gamma \neq 0 \text{ or } \delta \neq 0\Big\} \tag{45}$$

forms a coarse moduli space for isomorphism classes of N*-polarized* K3 *surfaces.*

We note that for $\gamma = 0$, case in which the K3 surface $\mathrm{X}(\alpha, \beta, 0, \delta)$ carries a canonical $\mathrm{H}\oplus\mathrm{E}_8\oplus\mathrm{E}_8$-polarization, an identity equivalent with (44) has been established by the authors in [5]. In this work we shall therefore focus on the $\gamma \neq 0$ case.

For $\gamma \neq 0$, Theorem 3.5 in connection with Remark 3.4, provides an explicit formula, in terms of Igusa–Clebsch invariants, for the geometric transformation underlying the Hodge theoretic correspondence (30).

Corollary 3.7. *Let* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *be a* N*-polarized* K3 *surface with* $\gamma \neq 0$*. Then, the genus-two curve* C *associated to* $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ *by the correspondence* (30) *has Igusa–Clebsch invariants given by*

$$[\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}] = \Big[2^3 3\delta,\ 2^2 3^2 \alpha\gamma^2,\ 2^3 3^2 (4\alpha\delta + \beta\gamma)\gamma^2,\ 2^2\gamma^6\Big].$$

The formula given by the above corollary can be seen to agree with the computation done by Kumar [24].

Remark 3.8. As a special remark, note that, under the formulas in (44), one obtains the expected period interpretation for the discriminants (11) and (12) of the quartic family $\mathrm{X}(\alpha, \beta, \gamma, \delta)$. Up to scaling by a constant, one has

$$\mathcal{D}_1(\alpha, \beta, \gamma, \delta) \cdot \mathcal{D}_4(\alpha, \beta, \gamma, \delta) \ = \mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}) = \mathcal{C}_{35}^2$$

where P is Igusa's weighted-degree 70 homogeneous polynomial (Theorem 3.3).

4. A geometric two-isogeny of K3 surfaces

This section outlines a purely geometric transformation upon which the main computation of this paper is based. For details regarding the transformation, as well as proofs, we refer the reader to the companion paper [7]. Various parts of the construction have also been discussed by Dolgachev (the Appendix Section of [9]) and Kumar [24].

4.1. Elliptic fibrations on N*-polarized K3 surfaces*

Let (X, i) be a N-polarized K3 surface. Assume also that the lattice polarization i cannot be extended to a polarization by the rank-eighteen lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. We are therefore in the case associated, under the Hodge theoretic correspondence (1), to principally polarized abelian surfaces obtained as Jacobians of genus-two curves.

By standard results on jacobian elliptic fibrations (see Kodaira's classical work [21], as well as [5,22,30]), the lattice polarization i determines a canonical elliptic fibration

$$\varphi^{\mathrm{s}}_{\mathrm{X}} : \mathrm{X} \to \mathbb{P}^1$$

with a section S^{s} and two singular fibers of Kodaira types II^* and III^*, respectively. We shall refer to $\varphi^{\mathrm{s}}_{\mathrm{X}}$ as the *standard* elliptic fibration of X. In the context of $\varphi^{\mathrm{s}}_{\mathrm{X}}$, one has the following dual configuration of rational curves on the K3 surface X.

$$\begin{tikzcd}[column sep=small]
\overset{a_1}{\bullet} \arrow[r, dash] & \overset{a_2}{\bullet} \arrow[r, dash] & \overset{a_3}{\bullet} \arrow[r, dash] \arrow[d, dash] & \overset{a_5}{\bullet} \arrow[r, dash] & \overset{a_6}{\bullet} \arrow[r, dash] & \overset{a_7}{\bullet} \arrow[r, dash] & \overset{a_8}{\bullet} \arrow[r, dash] & \overset{a_9}{\bullet} \arrow[r, dash] & \overset{S^s}{\bullet} \arrow[r, dash] & \overset{b_8}{\bullet} \arrow[r, dash] & \overset{b_7}{\bullet} \arrow[r, dash] & \overset{b_6}{\bullet} \arrow[r, dash] & \overset{b_4}{\bullet} \arrow[r, dash] \arrow[d, dash] & \overset{b_3}{\bullet} \arrow[r, dash] & \overset{b_2}{\bullet} \arrow[r, dash] & \overset{b_1}{\bullet} \\
 & & \overset{a_4}{\bullet} & & & & & & & & & & \overset{b_5}{\bullet} & & &
\end{tikzcd} \tag{46}$$

The fiber F^{s} of the elliptic fibration $\varphi^{\mathrm{s}}_{\mathrm{X}}$ is represented by the line bundle

$$\begin{aligned} &\mathcal{O}_{\mathrm{X}}\left(2a_1 + 4a_2 + 6a_3 + 3a_4 + 5a_5 + 4a_6 + 3a_7 + 2a_8 + a_9\right) \\ &\quad = \mathcal{O}_{\mathrm{X}}\left(b_1 + 2b_2 + 3b_3 + 4b_4 + 2b_5 + 3b_6 + 2b_7 + b_8\right). \end{aligned}$$

The N-polarization of X appears in this context as

$$\langle \mathrm{F}^{\mathrm{s}}, \mathrm{S}^{\mathrm{s}} \rangle \oplus \langle a_1, a_2, \ldots, a_8 \rangle \oplus \langle b_1, b_2, \ldots, b_7 \rangle.$$

A second *alternate* elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}} : \mathrm{X} \to \mathbb{P}^1$ is obtained via the same classical arguments. This alternate elliptic pencil is associated with the line bundle

$$\mathcal{O}_{\mathrm{X}}\left(a_2 + a_4 + 2(a_3 + a_5 + a_6 + a_7 + a_8 + \mathrm{S}^{\mathrm{s}} + b_8 + b_7 + b_6 + b_4) + b_3 + b_5\right).$$

The alternate elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ has two disjoint sections

$$\mathrm{S}^{\mathrm{a}}_1 = a_1, \qquad \mathrm{S}^{\mathrm{a}}_2 = b_2.$$

Moreover, the assumption that the polarization i does not extend to a lattice polarization by $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ implies that the rational curve b_1 is generically part of an I_2 type singular fiber. This implies the generic existence of an additional rational curve c, such that $b_1 + c$ belongs to the elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$. In the context of the explicit $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ surfaces, the curve c is described by Eq. (8). The diagram (46) completes to the following nineteen-curve diagram on X.

$$\begin{tikzcd}[column sep=small]
\overset{a_1}{\bullet} \arrow[r, dash] \arrow[dd, dash] & \overset{a_2}{\bullet} \arrow[r, dash] & \overset{a_3}{\bullet} \arrow[r, dash] \arrow[d, dash] & \overset{a_5}{\bullet} \arrow[r, dash] & \overset{a_6}{\bullet} \arrow[r, dash] & \overset{a_7}{\bullet} \arrow[r, dash] & \overset{a_8}{\bullet} \arrow[r, dash] & \overset{a_9}{\bullet} \arrow[r, dash] & \overset{S^s}{\bullet} \arrow[r, dash] & \overset{b_8}{\bullet} \arrow[r, dash] & \overset{b_7}{\bullet} \arrow[r, dash] & \overset{b_6}{\bullet} \arrow[r, dash] & \overset{b_4}{\bullet} \arrow[r, dash] \arrow[d, dash] & \overset{b_3}{\bullet} \arrow[r, dash] & \overset{b_2}{\bullet} \arrow[dd, dash] \\
 & & \overset{a_4}{\bullet} & & & & & & & & & & \overset{b_5}{\bullet} & & \\
\overset{c}{\bullet} \arrow[rrrrrrrrrrrrrr, dash, shift left=1pt] \arrow[rrrrrrrrrrrrrr, dash, shift right=1pt] & & & & & & & & & & & & & & \overset{b_1}{\bullet}
\end{tikzcd} \tag{47}$$

4.2. The Nikulin construction

As proved in [7], the section b_2 has order two, as a member of the Mordell–Weil group $\mathrm{MW}(\varphi^{\mathrm{a}}_{\mathrm{X}}, a_1)$. Translations by b_2 in the smooth fibers of the elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ extend then to a canonical Van Geemen–Sarti[4] involution

$$\Phi_{\mathrm{X}} : \mathrm{X} \to \mathrm{X}. \tag{48}$$

[4] For details regarding this concept, we refer the reader to Definition 1.1 of [7].

The involution Φ_{X} acts on the curves of diagram (47) as a horizontal left–right flip. In particular, Φ_{X} establishes a Shioda–Inose structure [19,25], as it exchanges the two E_8-configurations

$$\langle a_1, a_2, a_3, a_4.a_5, a_6, a_7, a_8\rangle, \qquad \langle b_2, b_3, b_4, b_5, b_6, b_7, \mathrm{S}^{\mathrm{s}}\rangle.$$

At this point one performs the Nikulin construction. Take the quotient of X by the involution Φ_{X} which produces a singular surface with eight rational double points of type A_1. Then take the minimal resolution of this quotient, hence obtaining a new K3 surface Y. The construction exhibits a rational two-to-one map

$$\mathrm{p}_{\Phi_{\mathrm{X}}} : \mathrm{X} \dashrightarrow \mathrm{Y}. \tag{49}$$

Moreover, as explained in [7], the surface Y inherits an elliptic fibration

$$\varphi_{\mathrm{Y}} : \mathrm{Y} \to \mathbb{P}^1 \tag{50}$$

which is induced from the alternate fibration on X. The elliptic fibration φ_{Y} carries a singular fiber of Kodaira type I_5^* and two disjoint sections $\widetilde{\mathrm{S}}_1, \widetilde{\mathrm{S}}_2$. As before, the section $\widetilde{\mathrm{S}}_2$ determines an element of order two in the Mordell–Weil group $\mathrm{MW}(\varphi_{\mathrm{Y}}, \widetilde{\mathrm{S}}_1)$ and fiber-wise translations by $\widetilde{\mathrm{S}}_2$ extend to a dual Van Geemen–Sarti involution

$$\Phi_{\mathrm{Y}} : \mathrm{Y} \to \mathrm{Y}. \tag{51}$$

The Nikulin construction associated to Φ_{Y} recovers back the K3 surface X as well as its alternate fibration. Hence, surfaces X and Y are naturally related by a geometric two-isogeny of K3 surfaces.

$$\begin{tikzcd}
\mathrm{Y} \arrow[loop left, "\Phi_{\mathrm{Y}}"] \arrow[rr, dashed, shift right, "\mathrm{p}_{\Phi_{\mathrm{Y}}}"'] \arrow[dr, "\varphi_{\mathrm{Y}}"'] & & \mathrm{X} \arrow[loop right, "\Phi_{\mathrm{X}}"] \arrow[ll, dashed, shift right, "\mathrm{p}_{\Phi_{\mathrm{X}}}"'] \arrow[dl, "\varphi_{\mathrm{X}}^{\mathrm{a}}"] \\
& \mathbb{P}^1 &
\end{tikzcd} \tag{52}$$

A key observation at this point is that the K3 surface Y carries a canonical Kummer structure. Let us summarize this fact. The Nikulin construction associated to the involution Φ_{X} induces a natural push-forward morphism at the cohomology level

$$(\mathrm{p}_{\Phi_{\mathrm{X}}})_* : \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \to \mathrm{H}^2(\mathrm{Y}, \mathbb{Z}). \tag{53}$$

Denote by U_i, with $1 \leq i \leq 8$, the exceptional rational curves on Y obtained from resolving the singularities associated with the eight fixed points of involution Φ_{X}. The curves $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_8$ form the even-eight configuration associated with the rational two-to-one map (49). The rank-eight lattice $\mathcal{N}$ defined as the minimal primitive sublattice of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ containing $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_8$ is a Nikulin lattice. One has

$$\big\langle (\mathrm{p}_{\Phi_{\mathrm{X}}})_*(x),\ y\big\rangle_{\mathrm{Y}} = 0,$$

for any $x \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ and $y \in \mathcal{N}$.

Set then $\mathcal{G}$ as the rank-seventeen sublattice of $\mathrm{NS}(\mathrm{Y})$ given by the orthogonal direct product

$$(\mathrm{p}_{\Phi_{\mathrm{X}}})_*(i(\mathrm{N})) \oplus \mathcal{N}.$$

Denote by $i(\mathrm{N})^{\perp}$ and $\mathcal{G}^{\perp}$ the orthogonal complements in $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ and $\mathrm{H}^2(\mathrm{Y}, \mathbb{Z})$, respectively. In this context, the Nikulin construction (see for instance Morrison's arguments in Section 3 of [25]), allows one to obtain the following lemma.

Lemma 4.1. (a) *The restriction of* (53) *induces a Hodge isometry*

$$(\mathrm{p}_{\Phi_{\mathrm{X}}})_* \colon i(\mathrm{N})^{\perp}(2) \xrightarrow{\simeq} \mathcal{G}^{\perp}. \tag{54}$$

(b) *Let $\mathcal{K}$ be the rank-sixteen Kummer lattice.*[5] *One has a canonical primitive lattice embedding*

$$\mathcal{K} \oplus (4) \hookrightarrow \mathcal{G}. \tag{55}$$

By the Nikulin criterion (see [28]), the lattice embedding (55) determines a canonical Kummer structure on Y, that is Y is a Kummer surface associated to a principally polarized abelian surface (A, Π) and the sixteen exceptional curves determining the Kummer structure are explicitly determined. Let $\pi : \mathrm{A} \dashrightarrow \mathrm{Y}$ be the rational two-to-one map associated to this Kummer structure. By restricting the map π_* to the orthogonal complement of the principal polarization Π in $\mathrm{H}^2(\mathrm{A}, \mathbb{Z})$, one obtains a classical Hodge isometry

$$\pi_* \colon \langle \Pi \rangle^{\perp}(2) \xrightarrow{\simeq} \mathcal{P}^{\perp}. \tag{56}$$

Connecting (54) and (56), one obtains an isometry of Hodge structures

$$(\pi_*)^{-1} \circ (\mathrm{p}_{\Phi_{\mathrm{X}}})_* \colon i(\mathrm{N})^{\perp} \xrightarrow{\simeq} \langle \Pi \rangle^{\perp}. \tag{57}$$

Both lattices $\langle \Pi \rangle^{\perp}$ and $i(\mathrm{N})^{\perp}$ are isometric to $\mathrm{H} \oplus \mathrm{H} \oplus (-2)$. Hence, via the considerations of Section 3, one obtains that (A, Π) is the abelian surface associated to (X, i) by the Hodge-theoretic correspondence (1). In particular (A, Π) is isomorphic, as principally polarized abelian surface, to

$$\big(\mathrm{Jac}(\mathrm{C}),\ \mathcal{O}_{\mathrm{Jac}(\mathrm{C})}(\Theta)\big)$$

with C a well-defined complex non-singular genus-two curve.

4.3. *Elliptic fibrations in the context of the Kummer structure*

As it turns out, the elliptic fibration φ_{Y}, as well as the Van Geemen–Sarti involution Φ_{Y} can be explicitly described from classical features of the Kummer surface $\mathrm{Y} = \mathrm{Km}(\mathrm{Jac}(\mathrm{C}))$. In order to present this description, we shall first need to establish some notations.

4.3.1. *Classical facts on* Km(C)

Let C be a complex non-singular genus-two curve. Assume a choice of labeling, $a_1, a_2, \ldots, a_6$, for the six ramification points of the canonical hyperelliptic structure. The Jacobian surface Jac(C) parametrizes the degree-zero line bundles on C. It comes equipped with a natural abelian group structure and contains sixteen two-torsion points that form a subgroup

$$\mathrm{Jac}(\mathrm{C})_2 \simeq (\mathbb{Z}/2\mathbb{Z})^4 .$$

The two-torsion points can be described as follows. Denote by $\mathrm{p}_{\varnothing}$ the neutral element of Jac(C), i.e. the point associated to the trivial line bundle of C. The fifteen points of order two are then given by p_{ij} representing the line bundles

$$\mathcal{O}_{\mathrm{C}}\left(a_i + a_j - 2a_1\right), \quad 1 \leq i < j \leq 6.$$

[5] As defined in [25] or [28].

The abelian group law on $\mathrm{Jac(C)}_2$ can be seen as

$$\mathrm{p_U} + \mathrm{p_V} = \mathrm{p_W}$$

where U, V and W are subsets of $\{1, 2, \ldots, 6\}$, containing either zero or two elements, and

$$\mathrm{W} = \begin{cases} \mathrm{U} & \text{if } \mathrm{V} = \varnothing \\ \mathrm{V} & \text{if } \mathrm{U} = \varnothing \\ \varnothing & \text{if } \mathrm{U} = \mathrm{V} \\ (\mathrm{U} \cup \mathrm{V}) \setminus (\mathrm{U} \cap \mathrm{V}) & \text{if } |\mathrm{U} \cap \mathrm{V}| = 1 \\ \{1, 2, \ldots, 6\} \setminus (\mathrm{U} \cup \mathrm{V}) & \text{if } \mathrm{U} \neq \varnothing,\ \mathrm{V} \neq \varnothing \text{ and } \mathrm{U} \cap \mathrm{V} = \varnothing \end{cases}. \tag{58}$$

The choice of labeling of the ramification points of C defines a level-two structure on Jac(C).

Consider the Abel–Jacobi embedding associated to the Weierstrass point a_0, i.e,

$$\mathrm{C} \hookrightarrow \mathrm{Jac(C)}, \qquad x \rightsquigarrow \mathcal{O}_{\mathrm{C}}(x - a_1)$$

and denote by $\Theta_\varnothing$ the image of C under this map. This is an irreducible curve on Jac(C), canonically isomorphic to C and containing the six two-torsion points $\mathrm{p}_\varnothing, \mathrm{p}_{12}, \mathrm{p}_{13}, \mathrm{p}_{14}, \mathrm{p}_{15}, \mathrm{p}_{16}$. Let then Θ_{ij} be the image of $\Theta_\varnothing$ under the translation by the order-two point p_{ij}. Each of the resulting sixteen Theta divisors $\Theta_\varnothing$, Θ_{ij} contains exactly six of the sixteen two-torsion points. For instance Θ_{1j}, for $2 \leq j \leq 6$, contains

$$\mathrm{p}_{1j},\ \mathrm{p}_{2j}, \ldots, \mathrm{p}_{j-1j},\ \mathrm{p}_\varnothing,\ \mathrm{p}_{jj+1}, \ldots, \mathrm{p}_{j6},$$

while Θ_{ij}, for $2 \leq i < j \leq 6$, contains

$$\mathrm{p}_{1i},\ \mathrm{p}_{1j},\ \mathrm{p}_{ij},\ \mathrm{p}_{kl},\ \mathrm{p}_{km},\ \mathrm{p}_{lm},$$

where $\{k, l, m\} = \{1, 2, \ldots, 6\} \setminus \{0, i, j\}$. Each two-torsion point lies on precisely six of the sixteen Theta divisors.

The sixteen two-torsion points together with the sixteen Theta divisors on Jac(C) yield, via the Kummer construction, a classical configuration of thirty-two smooth rational curves on Km(C) — the (16; 6) configuration. Sixteen of the curves, denoted $\mathrm{E}_\varnothing, \mathrm{E}_{ij}$ are the exceptional curves associated to the two-torsion points $\mathrm{p}_\varnothing$, p_{ij} of Jac(C), respectively. The remaining sixteen rational curves, denoted $\triangle_\varnothing$, $\triangle_{ij}$ are the proper transforms of the images of the Theta divisors $\Theta_\varnothing$, Θ_{ij}, respectively. Following the classical terminology, we shall refer to these latter sixteen curves as *tropes*.

On the Jacobian surface Jac(C), one has $h^0(\mathrm{Jac(C)}, 2\Theta_\varnothing) = 4$ and the linear system $|2\Theta_\varnothing|$ is base point free. The associated morphism

$$\varphi_{|2\Theta_\varnothing|} \colon \mathrm{Jac(C)} \to \mathbb{P}^3$$

is generically two-to-one and its image

$$\mathrm{S(C)} = \varphi_{|2\Theta_\varnothing|}(\mathrm{Jac(C)}) \subset \mathbb{P}^3$$

is a quartic surface. One has a canonical identification

$$\mathrm{S(C)} = \mathrm{Jac(C)}/\{\pm \mathrm{id}\}$$

and the images of the sixteen two-torsion points of Jac(C) are singularities on S(C): rational double points of type A_1. By convention, we shall also label these sixteen singularities as $\mathrm{p}_\varnothing$, p_{ij}.

The minimal resolution of S(C) is then isomorphic to the Kummer surfaces Km(C).

$$\begin{tikzcd} \mathrm{Jac(C)} \arrow[r, dashed] \arrow[dr, "\varphi_{|2\Theta_\varnothing|}"'] & \mathrm{Km(C)} \arrow[d, "\sigma"] \\ & \mathbb{P}^3 \end{tikzcd} \tag{59}$$

In this context, the sixteen curves $\mathrm{G}_\varnothing, \mathrm{G}_{ij}$ are resulting from resolving the sixteen singular points of S(C). The tropes $\triangle_\varnothing, \triangle_{ij}$ are conics resulting from intersecting the quartic surface S(C) with sixteen special planes of $\mathbb{P}^3$. The linear system of hyperplane sections associated to the morphism $\sigma : \mathrm{Km(C)} \to \mathbb{P}^3$ of diagram (59) is given by

$$\left| 2\triangle_\varnothing + \mathrm{G}_\varnothing + \sum_{2\le t\le 6} \mathrm{G}_{1t} \right| = \left| 2\triangle_{1j} + \mathrm{G}_\varnothing + \sum_{\substack{1\le t\le 6\\ t\neq j}} \mathrm{G}_{1t} \right|$$
$$= |2\triangle_{ij} + \mathrm{G}_{1i} + \mathrm{G}_{1j} + \mathrm{G}_{ij} + \mathrm{G}_{kl} + \mathrm{G}_{km} + \mathrm{G}_{lm}|.$$

Let $\mathrm{pr}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from the point $\mathrm{p}_\varnothing$. The images through this projection of the six planes associated with the tropes $\triangle_\varnothing, \triangle_{1j}, 2 \le j \le 6$ form a configuration of six distinct lines in $\mathbb{P}^2$:

$$\mathcal{L} = \{\mathrm{L}_1,\ \mathrm{L}_2,\ \mathrm{L}_3, \ldots \mathrm{L}_6\}. \tag{60}$$

The six lines are tangent to a common smooth conic and meet at fifteen distinct points $q_{ij} = \mathrm{pr}(p_{ij})$, $1 \le i < j \le 6$. After blowing up the points q_{ij}, one obtains a rational surface R with fifteen exceptional curves E_{ij}. Denote by L'_i with $1 \le i \le 6$, the rational curves on R obtained as proper transforms of the six lines L_i. Then, one has a double cover morphism

$$\pi : \mathrm{Km(C)} \to \mathrm{R} \tag{61}$$

with branched locus given by the six disjoint curves L'_i, $1 \le i \le 6$.

$$\begin{tikzcd} \mathrm{Jac(C)} \arrow[r, dashed] \arrow[dr, "\varphi_{|2\Theta_\varnothing|}"'] & \mathrm{Km(C)} \arrow[r, "\pi"] \arrow[d, "\sigma"] & \mathrm{R} \arrow[d, "\rho"] \\ & \mathbb{P}^3 \arrow[r, dashed, "\mathrm{pr}"] & \mathbb{P}^2 \end{tikzcd} \tag{62}$$

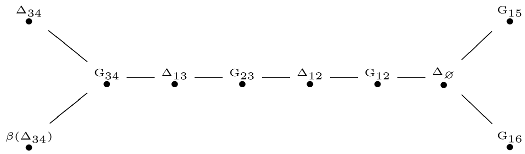
The deck transformation $\beta : \mathrm{Km(C)} \to \mathrm{Km(C)}$ associated with the double cover (61) is a non-symplectic involution with fixed locus given by the union of six curves $\triangle_\varnothing, \triangle_{1j}, 2 \le j \le 6$.

4.4. *Two elliptic fibrations on* Y

There are two elliptic fibrations on the Kummer surface $\mathrm{Y} = \mathrm{Km(C)}$ that play an important role in our discussion. The first one is the elliptic fibration $\varphi_\mathrm{Y} : \mathrm{Y} \to \mathbb{P}^1$ of (50). The geometric features of this fibration are discussed in detail in Chapter 3 of [7]. Let us outline here the main properties. The elliptic pencil φ_Y is associated with the line bundle

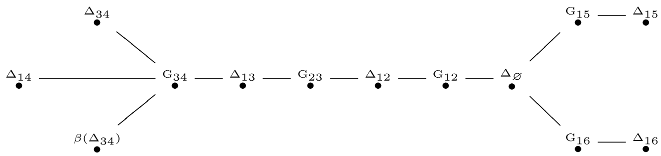
$$\mathcal{O}_\mathrm{Y}\left(\triangle_{34} + \beta(\triangle_{34}) + 2\left(\mathrm{G}_{34} + \triangle_{13} + \mathrm{G}_{23} + \triangle_{12} + \mathrm{G}_{12} + \triangle_\varnothing\right) + \mathrm{G}_{15} + \mathrm{G}_{16}\right). \tag{63}$$

The fibration carries therefore a singular fiber of Kodaira type I_5^*



(64)

In the generic situation, there are six additional singular fibers of type I_2 and one of type I_1. The tropes $\triangle_{15}$ and $\triangle_{16}$ are disjoint sections in φ_{Y}, whereas $\triangle_{14}$ is a bi-section.



(65)

As an element of the Mordell–Weil group $\mathrm{MW}(\varphi_{\mathrm{Y}}, \triangle_{15})$, the section $\triangle_{16}$ has order two. Hence, fiber-wise translations by $\triangle_{16}$ extend to define the Van Geemen–Sarti involution $\Phi_{\mathrm{Y}} : \mathrm{Y} \to \mathrm{Y}$ of (51).

A simple computation shows that, in the context of diagram (62), the I_5^* divisor in (63) is the pull-back under the double cover $\pi : \mathrm{Y} \to \mathrm{R}$ of

$$5\rho^*(\mathrm{h}) - 3\mathrm{E}_{13} - 2\left(\mathrm{E}_{14} + \mathrm{E}_{25} + \mathrm{E}_{26}\right) - \left(\mathrm{E}_{24} + \mathrm{E}_{35} + \mathrm{E}_{36} + \mathrm{E}_{56}\right) \tag{66}$$

where is the hyperplane class in $\mathbb{P}^2$. The fibers of φ_{Y} are therefore coming from a pencil of projective quintic curves in $\mathbb{P}^2$, with a triple point at q_{13}, three double points at q_{14}, q_{25}, q_{26} and also passing through the four points $q_{24}, q_{35}, q_{36}, q_{56}$. The divisor (66) determines a ruling

$$\varphi_{\mathrm{R}} : \mathrm{R} \to \mathbb{P}^1. \tag{67}$$

The generic fiber of this ruling is a rational curve with four distinct special points: the intersection with L_5', L_6' (sections) and L_4' (bi-section). The associated elliptic fiber of φ_{Y} is the double cover of this rational curve branched at the four special points. The elliptic fibration φ_{Y} factors through the ruling (67).

$$\varphi_{\mathrm{Y}} : \mathrm{Y} \xrightarrow{\pi} \mathrm{R} \xrightarrow{\varphi_{\mathrm{R}}} \mathbb{P}^1.$$

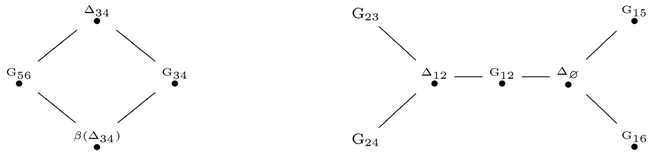
The second elliptic fibration on we consider on the K3 surface Y is associated, in a manner similar with the above description, with the pencil of conic curves in $\mathbb{P}^2$ passing through $q_{13}, q_{14}, q_{25}, q_{26}$. The line bundle

$$\mathcal{O}_{\mathrm{R}}\left(2\rho^*(\mathrm{h}) - \mathrm{E}_{13} - \mathrm{E}_{14} - \mathrm{E}_{25} - \mathrm{E}_{26}\right) \tag{68}$$

determines a ruling

$$\psi_{\mathrm{R}} : \mathrm{R} \to \mathbb{P}^1 \tag{69}$$

whose pull-back through the double cover $\pi : \mathrm{Y} \to \mathrm{R}$ gives an elliptic fibration $\psi_{\mathrm{Y}} : \mathrm{Y} \to \mathbb{P}^1$. The elliptic fibration ψ_{Y} carries two special singular fibers of Kodaira types I_3 and I_2^*.



(70)

In the generic situation, ψ_{Y} has six additional fibers of type I_2.

In the next section, we shall use the two elliptic fibrations φ_{Y} and ψ_{Y} in the context of the following property.

Proposition 4.2. *The product morphism $\varphi_{\mathrm{Y}} \times \psi_{\mathrm{Y}}$ factors through the double cover map*

$$\varphi_{\mathrm{Y}} \times \psi_{\mathrm{Y}} \colon \mathrm{Y} \xrightarrow{\pi} \mathrm{R} \xrightarrow{\varphi_{\mathrm{R}} \times \psi_{\mathrm{R}}} \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover $\varphi_{\mathrm{R}} \times \psi_{\mathrm{R}} \colon \mathrm{R} \to \mathbb{P}^1 \times \mathbb{P}^1$ is a birational morphism.

5. An explicit computation: Proof of Theorem 3.5

We shall prove the identity in Theorem 3.5 by explicitly describing the details of the geometric two-isogeny transformation outlined in Section 4. We give explicit formulas for the elliptic fibration φ_{Y} on the Kummer surface $\mathrm{Y} = \mathrm{Km}(\mathrm{C})$ from the points of view of the two contexts involved: the appearance of φ_{Y} from the four-parameter N-polarized K3 family $\mathrm{X}(\alpha, \beta, \gamma, \delta)$ (with $\gamma \neq 0$) via the Nikulin construction and the set-up of φ_{Y} in the context of the Kummer construction as described in Section 4.4. The first description will depend on the quadruple parameter $(\alpha, \beta, \gamma, \delta)$, while in the latter context, we give a formula for φ_{Y} in terms of Siegel modular forms. Identity (44) will follow from the matching of the explicit formulas on the two sides.

5.1. The fibration φ_{Y} via the Nikulin construction

Recall from Section 2.1 that, in the context of the K3 surface $\mathrm{X}(\alpha, \beta, \gamma, \delta)$, the alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ can be described by the affine equation

$$y_1^2 = z_1^3 + \mathcal{P}_{\mathrm{X}}(\mu) \cdot z_1^2 + \mathcal{Q}_{\mathrm{X}}(\mu) \cdot z_1, \tag{71}$$

where

$$\mathcal{P}_{\mathrm{X}}(\mu) = 4\mu^3 - 3\alpha\mu - \beta, \qquad \mathcal{Q}_{\mathrm{X}}(\mu) = \frac{1}{2}\left(\frac{1}{2}\delta - \gamma\mu\right).$$

The Van Geemen–Sarti involution Φ_{X} is described by Proposition 2.7 and, in the context of the affine coordinates (z_1, y_1) of (71), acts as

$$(z_1,\ y_1) \mapsto \left(\frac{\mathcal{Q}_{\mathrm{X}}(\mu)}{z_1},\ -\frac{\mathcal{Q}_{\mathrm{X}}(\mu) \cdot y_1}{z_1^2}\right).$$

Then, as explained, for instance, by Van Geemen and Sarti in Section 4 of [35], one can write an affine form for the elliptic fibration φ_{Y} as follows:

$$y_2^2 = z_2^3 + \mathcal{P}_{\mathrm{Y}}(\mu) \cdot z_2^2 + \mathcal{Q}_{\mathrm{Y}}(\mu) \cdot z_2, \tag{72}$$

where the affine coordinates (z_2, y_2) are

$$z_2 = \frac{y_1^2}{z_1^2}, \qquad y_2 = \frac{\left(\mathcal{Q}_{\mathrm{X}}(\mu) - z_1^2\right) y_1}{z_1^2}$$

and

$$\mathcal{P}_{\mathrm{Y}}(\mu) = -2\mathcal{P}_{\mathrm{X}}(\mu) = -8\mu^3 + 6\alpha\mu + 2\beta \tag{73}$$

$$\begin{aligned} \mathcal{Q}_{\mathrm{Y}}(\mu) \ = \ & \mathcal{P}_{\mathrm{X}}^2(\mu) - 4\mathcal{Q}_{\mathrm{X}}(\mu) = 16\mu^6 - 24\alpha\mu^4 - 8\beta\mu^3 \\ & + 9\alpha^2\mu^2 + 2(3\alpha\beta + \gamma)\mu + \beta^2 - \delta. \end{aligned} \tag{74}$$

5.2. *The fibration φ_{Y} via the Kummer construction*

The maps of diagram (62) can be described explicitly in terms of genus-two theta functions. Let $\kappa \in \mathbb{H}_2$ be a point of the Siegel upper half-space defined in (24). Furthermore, assume that κ is associated with a set of periods for the polarized Hodge structure of Jac(C). By classical results (see [26,27]), there are then sixteen theta functions

$$\theta_m(\kappa, \cdot)\colon \mathbb{C}^2 \to \mathbb{C},$$

with characteristics $m = (u, v)$, $u, v \in \{0, 1/2\} \times \{0, 1/2\}$. The theta functions $\theta_m(\kappa, \cdot)$ descend to sections in line bundles over the Jacobian surface Jac(C) determining the sixteen Theta divisors[6] $\Theta_\varnothing$, Θ_{ij}.

Among the possible sixteen characteristics $m = (u, v)$, ten are even and six are odd. The ten even theta functions are related by six independent Riemann theta relations. Our computation will be based on the following four *fundamental theta functions*

$$\theta_{m_1}(\kappa, \cdot), \qquad \theta_{m_2}(\kappa, \cdot), \qquad \theta_{m_3}(\kappa, \cdot), \qquad \theta_{m_4}(\kappa, \cdot) \tag{75}$$

with

$$\begin{aligned} m_1 &= ((0,0),(0,0))\,, \qquad m_2 = ((0,0),(1/2,1/2)) \\ m_3 &= ((0,0),(1/2,0))\,, \qquad m_4 = ((0,0),(0,1/2))\,. \end{aligned}$$

In this context, one can describe the morphism $\varphi_{|2\Theta_\varnothing|}$ of diagram (62) as

$$\begin{tikzcd} \mathbb{C}^2 \arrow[d] \arrow[dr, "\Xi"] & \\ \mathrm{Jac(C)} \arrow[r, "\varphi_{|2\Theta_\varnothing|}"'] & \mathbb{P}^3 \end{tikzcd} \tag{76}$$

where $\Xi\colon \mathbb{C}^2 \to \mathbb{P}^3$ is defined as

$$\Xi(Z) = \left[\theta_{m_1}(\kappa, 2\mathrm{Z}),\ \theta_{m_1}(\kappa, 2\mathrm{Z}),\ \theta_{m_3}(\kappa, 2\mathrm{Z}),\ \theta_{m_4}(\kappa, 2\mathrm{Z})\right].$$

Via Frobenius identities, one obtains then an explicit description for the quartic surface

$$\mathrm{S_C} = \varphi_{|2\Theta_\varnothing|}\,(\mathrm{Jac(C)}) \subset \mathbb{P}^3(x, y, z, w).$$

[6] One can arrange that $\theta_m(\kappa, \cdot) \in \mathrm{H}^0(\mathrm{Jac(C)}, \Theta_\varnothing)$ for $m = ((0,0),(0,0))$ and for the level-two structure induced by characteristics to match (58).

This is the classical equation of Hudson [13,10]:

$$x^4+y^4+z^4+w^4+2\mathrm{D}xyzw+\mathrm{A}(x^2w^2+y^2z^2)+\mathrm{B}(y^2w^2+x^2z^2)$$
$$+\mathrm{C}(x^2y^2+z^2w^2)=0. \tag{77}$$

The coefficients A, B, C, D of the Hudson quartic are rational functions in the four *fundamental theta constants*

$$a=\theta_{m_1}(\kappa,0),\qquad b=\theta_{m_2}(\kappa,0),\qquad c=\theta_{m_3}(\kappa,0),\qquad d=\theta_{m_4}(\kappa,0),$$

and appear as follows:

$$\mathrm{A}=\frac{b^4+c^4-a^4-d^4}{a^2d^2-b^2c^2},\qquad \mathrm{B}=\frac{c^4+a^4-b^4-d^4}{b^2d^2-c^2a^2},$$
$$\mathrm{C}=\frac{a^4+b^4-c^4-d^4}{c^2d^2-a^2b^2}, \tag{78}$$
$$\mathrm{D}=\frac{abcd(d^2+a^2-b^2-c^2)(d^2+b^2-c^2-a^2)(d^2+c^2-a^2-b^2)(a^2+b^2+c^2+d^2)}{(a^2d^2-b^2c^2)(b^2d^2-c^2a^2)(c^2d^2-a^2b^2)}.$$

Note that, as function of $\kappa\in\mathbb{H}_2$, the homogeneous polynomial

$$(ad-bc)(ad+bc)(ac-bd)(ac+bd)(ab-cd)(ab+cd)(a^2+d^2-b^2-c^2)$$
$$\times(a^2+c^2-b^2-d^2)(a^2+b^2-c^2-d^2)(a^2+b^2+c^2+d^2) \tag{79}$$

represents $\mathcal{C}_{10}$ scaled by a non-zero constant. The zero-divisor of $\mathcal{C}_{10}$ is the Humbert surface $\mathcal{H}_1$, and hence the denominators in (78) are all non-zero.

In the Hudson quartic setting, the sixteen singularities $p_\varnothing$, p_{ij} of $\mathrm{S_C}$ are as follows:

- $p_\varnothing=[a,b,c,d]$
- $p_{12}=[c,d,a,b]$
- $p_{13}=[a,-b,-c,d]$
- $p_{14}=[-b,a,d,-c]$
- $p_{15}=[c,d,-a,-b]$
- $p_{16}=[-b,-a,d,c]$
- $p_{23}=[-c,d,a,-b]$
- $p_{24}=[d,-c,-b,a]$
- $p_{25}=[-a,-b,c,d]$
- $p_{26}=[d,c,-b,-a]$
- $p_{34}=[b,a,d,c]$
- $p_{35}=[-c,d,-a,b]$
- $p_{36}=[b,-a,d,-c]$
- $p_{45}=[d,-c,b,-a]$
- $p_{46}=[-a,b,-c,d]$
- $p_{56}=[d,c,b,a]$.

The sixteen tropes $\triangle_\varnothing$, $\triangle_{ij}$ correspond to the following sixteen hyperplanes

- $\triangle_\varnothing:\ dx-cy+bz-aw=0,$
- $\triangle_{12}:\ bx-ay+dz-cw=0,$
- $\triangle_{13}:\ dx+cy-bz+aw=0,$
- $\triangle_{14}:\ cx+dy-az-bw=0,$

- $\triangle_{15}: \ -bx + ay + dz - cw = 0$,
- $\triangle_{16}: \ -cx + dy + az - bw = 0$,
- $\triangle_{23}: \ -bx - ay + dz + cw = 0$,
- $\triangle_{24}: \ -ax - by + cz + dw = 0$,
- $\triangle_{25}: \ dx - cy - bz + aw = 0$,
- $\triangle_{26}: \ ax - by - cz + dw = 0$,
- $\triangle_{34}: \ -cx + dy - az + bw = 0$,
- $\triangle_{35}: \ bx + ay + dz + cw = 0$,
- $\triangle_{36}: \ cx + dy + az + bw = 0$,
- $\triangle_{45}: \ ax + by + cz + dw = 0$,
- $\triangle_{46}: \ dx + cy + bz + aw = 0$,
- $\triangle_{56}: \ -ax + by - cz + dw = 0$.

The rational projection $\mathrm{pr}\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ of diagram (62) has then the explicit form

$$\mathrm{pr}\left([x, y, z, w]\right) = [-bx + ay - dz + cw, \ cx - dy - az + bw, \ dx + cy - bz - aw] .$$

By a slight abuse of notation, we shall use homogeneous coordinates $[x, y, z]$ on the target space of the projection. In these coordinates, the six lines L_n, with $1 \leq n \leq 6$ forming the branch locus (60) can be described through the equations $\mathrm{L}_n(x, y, z) = 0$ where

- $\mathrm{L}_1(x, y, z) = 2(ac + bd)x + 2(ab - cd)y - (a^2 - b^2 - c^2 + d^2)z$
- $\mathrm{L}_2(x, y, z) = x$
- $\mathrm{L}_3(x, y, z) = z$
- $\mathrm{L}_4(x, y, z) = 2(ad - bc)x + (a^2 - b^2 + c^2 - d^2)y + 2(ab + cd)z$
- $\mathrm{L}_5(x, y, z) = (-a^2 - b^2 + c^2 + d^2)x + 2(ad + bc)y - 2(ac - bd)z$
- $\mathrm{L}_6(x, y, z) = y$.

The fifteen intersection points q_{ij} of the six-line configuration are

- $q_{12} = [0, \ -a^2 + b^2 + c^2 - d^2, \ -2ab + 2cd]$
- $q_{13} = [-2ab + 2cd, \ 2ac + 2bd, \ 0]$
- $q_{14} = [a^2 + b^2 - c^2 - d^2, \ -2bc - 2ad, \ 2ac - 2bd]$
- $q_{15} = [-2bc + 2ad, \ a^2 - b^2 + c^2 - d^2, \ 2ab + 2cd]$
- $q_{16} = [-a^2 + b^2 + c^2 - d^2, \ 0, \ -2ac - 2bd]$
- $q_{23} = [0, \ -a^2 - b^2 - c^2 - d^2, \ 0]$
- $q_{24} = [0, \ 2ab + 2cd, \ -a^2 + b^2 - c^2 + d^2]$
- $q_{25} = [0, \ -2ac + 2bd, \ -2bc - 2ad]$
- $q_{26} = [0, \ 0, \ a^2 + b^2 + c^2 + d^2]$
- $q_{34} = [a^2 - b^2 + c^2 - d^2, \ 2bc - 2ad, \ 0]$
- $q_{35} = [2bc + 2ad, \ a^2 + b^2 - c^2 - d^2, \ 0]$
- $q_{36} = [-a^2 - b^2 - c^2 - d^2, \ 0, \ 0]$
- $q_{45} = [-2ac - 2bd, \ -2ab + 2cd, \ a^2 - b^2 - c^2 + d^2]$
- $q_{46} = [2ab + 2cd, \ 0, \ 2bc - 2ad]$
- $q_{56} = [2ac - 2bd, \ 0, \ -a^2 - b^2 + c^2 + d^2]$.

5.3. The quintic pencil φ_{R}

As explained in Section 4.4, in order to describe explicitly the elliptic fibration $\varphi_{\mathrm{Y}} \colon \mathrm{Y} \to \mathbb{P}^1$ of (50), one needs to understand the ruling $\varphi_{\mathrm{R}} \colon \mathrm{R} \to \mathbb{P}^1$ of (67). This ruling is associated with the pencil of quintic curves in $\mathbb{P}^2$, with a triple point at q_{13}, three double points at q_{14}, q_{25}, q_{26} and passing through the four points $q_{24}, q_{35}, q_{36}, q_{56}$.

This pencil can be described explicitly. Note that a first such quintic curve is given by

$$\mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 + \mathrm{C} \tag{80}$$

where C is the unique conic passing through $q_{13}, q_{14}, q_{25}, q_{26}, q_{56}$. The pull-back of the divisor (80) determines the I_5^* fiber of the elliptic fibration φ_{Y}, as described in (64). The conic C is given by the following polynomial

$$\mathrm{C}(x, y, z) = c_{200}x^2 + c_{020}y^2 + c_{002}z^2 + c_{110}xy + c_{101}xz + c_{011}yz \tag{81}$$

with coefficients set as follows:

$$\begin{aligned}
c_{200} &= -2(ad - bc)(bc + ad)(ac + bd)(a^2 + b^2 - c^2 - d^2)\\
c_{020} &= -(bc + ad)(ab - cd)(a^2 + b^2 - c^2 - d^2)(a^2 - b^2 + c^2 - d^2)\\
c_{002} &= 0\\
c_{110} &= -(bc+ad)(a^2+b^2-c^2-d^2)(a^3c-3ab^2c+ac^3+3a^2bd-b^3d+3bc^2d-3acd^2-bd^3)\\
c_{101} &= -4(ad - bc)(bc + ad)(ac - bd)(ac + bd)\\
c_{011} &= (ac - bd)(ab - cd)(a^2 + b^2 - c^2 - d^2)(a^2 - b^2 + c^2 - d^2).
\end{aligned}$$

We have therefore a description for the divisor (80) as the zero-locus a special quintic

$$\mathrm{QIN}_1(x, y, z) = \mathrm{L}_1(x, y, z) \cdot \mathrm{L}_2(x, y, z) \cdot \mathrm{L}_3(x, y, z) \cdot \mathrm{C}(x, y, z). \tag{82}$$

In order to select a second quintic polynomial with the required properties, we choose to impose the extra condition that the quintic curve passes through q_{45}. In the generic situation, the pull-back of the strict transform of this quintic curve determines a singular fiber of Kodaira type I_2 on the elliptic fibration φ_{Y}. A polynomial describing this curve can be given as follows:

$$\begin{aligned}
\mathrm{QIN}_2(x, y, z) = {} & k_{500}x^5 + k_{050}y^5 + k_{005}z^5 + k_{410}x^4y + k_{401}x^4z + k_{140}xy^4 + k_{041}y^4z\\
& + k_{104}xz^4 + k_{014}yz^4 + k_{320}x^3y^2 + k_{302}x^3z^2 + k_{230}x^2y^3 + k_{032}y^3z^2\\
& + k_{203}x^2z^3 + k_{023}y^2z^3 + k_{311}x^3yz + k_{131}xy^3z\\
& + k_{113}xyz^3 + k_{122}xy^2z^2 + k_{212}x^2yz^2 + k_{221}x^2y^2z.
\end{aligned}$$

The coefficients k_{ijk} are homogeneous degree-sixteen polynomials the fundamental theta constants a, b, c, d. Their precise form is given in Appendix A.2.

The full pencil of quintic curves can be then described by

$$\mathrm{QIN}_{t_1,t_2}(x, y, z) = t_1 \cdot \mathrm{QIN}_1(x, y, z) + t_2 \cdot \mathrm{QIN}_2(x, y, z), \quad (t_1, t_2) \in \mathbb{C}^2. \tag{83}$$

5.4. The conic pencil ψ_{R}

As explained earlier, the quintic pencil in Section 5.3 determines a ruling φ_{R} on the rational surface R obtained by blowing up the fifteen points q_{ij} on $\mathbb{P}^2$. The proper transforms L_5', L_6' are sections in this ruling, while L_4' is a bi-section. On each smooth fiber of φ_{R}, these sections/

bi-sections determine four distinct points and the associated elliptic fiber of φ_{Y} is the double cover of the rational curve branched at these four special points.

Our strategy shall be to describe explicitly the location of the four branch points via a parametrization of the ruling. In order to accomplish this task, we shall use the second ruling $\psi_{\mathrm{R}}\colon \mathrm{R} \to \mathbb{P}^1$, the ruling associated to the pencil of projective conics passing through the four points $q_{13}, q_{14}, q_{25}, q_{26}$. This pencil can be written explicitly as

$$\mathrm{C}_{s_1,s_2}(x,y,z) = s_1 \cdot \mathrm{C}(x,y,z) + s_2 \cdot \mathrm{L}_1(x,y,z) \cdot \mathrm{L}_2(x,y,z), \quad (s_1,s_2) \in \mathbb{C}^2. \tag{84}$$

As explained in Section 4.4, the intersection between generic fibers of the rulings φ_{R} and ψ_{R}, respectively, consist of exactly one point and one obtains a birational morphism $\varphi_{\mathrm{R}} \times \psi_{\mathrm{R}}\colon \mathrm{R} \to \mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{tikzcd} \mathrm{R} \arrow[d, "\rho"'] \arrow[dr, "\varphi_{\mathrm{R}} \times \psi_{\mathrm{R}}"] & \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd} \tag{85}$$

5.5. *Explicit description of the elliptic fibration φ_{Y}*

Let $t \in \mathbb{C}$. Consider then the quintic curve

$$\mathrm{QIN}_{t,1}(x,y,z) = t(a^2+b^2+c^2+d^2) \cdot \mathrm{QIN}_1(x,y,z) + \mathrm{QIN}_2(x,y,z) = 0. \tag{86}$$

From the point of view of this work, one has four important points on the curve (86). These points are given by the residual intersections with the lines L_5, L_6, and L_4, respectively. The images of these four points through the rational map

$$\frac{\mathrm{C}(x,y,z)}{\mathrm{L}_1(x,y,z) \cdot \mathrm{L}_2(x,y,z)} \tag{87}$$

can be described as follows. The image through (87) of the intersection with L_5 is

$$A(t) = \frac{1}{4}(A_0 + A_1 t) \tag{88}$$

where

$$\begin{aligned} A_0 = {} & a^6 - 3a^4b^2 + 3a^4c^2 - 3a^4d^2 - 8a^3bcd + 3a^2b^4 + 2a^2b^2c^2 - 2a^2b^2d^2 \\ & + 3a^2c^4 + 2a^2c^2d^2 + 3a^2d^4 + 8ab^3cd - 8abc^3d + 8abcd^3 - b^6 + 3b^4c^2 \\ & - 3b^4d^2 - 3b^2c^4 - 2b^2c^2d^2 - 3b^2d^4 + c^6 - 3c^4d^2 + 3c^2d^4 - d^6, \\ A_1 = {} & a^2 - b^2 + c^2 - d^2. \end{aligned}$$

The image through (87) of the intersection with L_6 is

$$B(t) = \frac{1}{4}(B_0 + B_1 t) \tag{89}$$

where

$$\begin{aligned} B_0 &= -8(ad - bc)(ac + bd)(ab - cd), \\ B_1 &= a^2 - b^2 + c^2 - d^2. \end{aligned}$$

Finally, the two points of intersection with L_4, map under (87), to the two roots of the quadratic equation

$$C(t) \cdot u^2 + Du + E = 0, \tag{90}$$

where $C(t) = C_0 + C_1 t$ and

$$\begin{aligned}
C_1 &= a^2 - b^2 + c^2 - d^2, \\
C_0 &= a^6 + a^4b^2 - a^4c^2 + a^4d^2 - 8a^3bcd - a^2b^4 - 10a^2b^2c^2 + 10a^2b^2d^2 - a^2c^4 \\
&\quad - 10a^2c^2d^2 - a^2d^4 + 8ab^3cd - 8abc^3d + 8abcd^3 - b^6 - b^4c^2 + b^4d^2 + b^2c^4 \\
&\quad + 10b^2c^2d^2 + b^2d^4 + c^6 + c^4d^2 - c^2d^4 - d^6, \\
D &= -4(b^2c^2a^8 - b^2d^2a^8 + c^2d^2a^8 - b^2c^4a^6 - b^2d^4a^6 + c^2d^4a^6 + b^4c^2a^6 \\
&\quad - b^4d^2a^6 - c^4d^2a^6 + 6b^2c^2d^2a^6 - b^2c^6a^4 + b^2d^6a^4 - c^2d^6a^4 \\
&\quad - 6b^4c^4a^4 - 6b^4d^4a^4 - 6c^4d^4a^4 + 2b^2c^2d^4a^4 - b^6c^2a^4 + b^6d^2a^4 - c^6d^2a^4 \\
&\quad - 2b^2c^4d^2a^4 + 2b^4c^2d^2a^4 + b^2c^8a^2 + b^2d^8a^2 - c^2d^8a^2 + b^4c^6a^2 - b^4d^6a^2 \\
&\quad - c^4d^6a^2 + 6b^2c^2d^6a^2 - b^6c^4a^2 - b^6d^4a^2 + c^6d^4a^2 + 2b^2c^4d^4a^2 \\
&\quad - 2b^4c^2d^4a^2 - b^8c^2a^2 + b^8d^2a^2 + c^8d^2a^2 + 6b^2c^6d^2a^2 + 2b^4c^4d^2a^2 \\
&\quad + 6b^6c^2d^2a^2 + b^2c^2d^8 + b^2c^4d^6 - b^4c^2d^6 - b^2c^6d^4 - 6b^4c^4d^4 \\
&\quad - b^6c^2d^4 - b^2c^8d^2 - b^4c^6d^2 + b^6c^4d^2 + b^8c^2d^2), \\
E &= -4(ad - bc)(ad + bc)(ac - bd)(ac + bd)(ab - cd)(ab + cd) \\
&\quad \times (a^2 + b^2 - c^2 - d^2)(a^2 - b^2 - c^2 + d^2)(a^2 + b^2 + c^2 + d^2).
\end{aligned}$$

One obtains then an explicit affine expression for the elliptic fibration φ_{Y} as

$$v^2 = (u - A(t))\,(u - B(t))\left(C(t) \cdot u^2 + Du + E\right). \tag{91}$$

5.6. *Adjustments to formula (91)*

Next, we shall perform a series of transformations on the formula in expression (91) with the goal of making a comparison with (72). First, we operate a change in the affine coordinates (u, v) setting

$$\begin{aligned}
u_1 &= 16\left(C(t)A(t)^2 + DA(t) + E\right) \cdot \frac{u - B(t)}{u - A(t)} \\
v_1 &= 64\left(C(t)A(t)^2 + DA(t) + E\right) \cdot (A(t) - B(t)) \cdot \frac{v}{(u - A(t))^2}.
\end{aligned}$$

Intuitively, this operation amounts to sending $A(t)$ to infinity and $B(t)$ to zero. One obtains:

$$v_1^2 = u_1^3 + M(t)u_1^2 + N(t)u_1, \tag{92}$$

where

$$\begin{aligned}
M(t) &= -\left(2C(t)A(t)B(t) + DA(t) + DB(t) + 2E\right), \\
N(t) &= \left(C(t)A(t)^2 + DA(t) + E\right)\left(C(t)B(t)^2 + DB(t) + E\right).
\end{aligned}$$

An explicit evaluation of the above expressions gives

$$M(t) = -2\left(a^2-b^2+c^2-d^2\right)^3\left(t^3+M_2t^2+M_1t+M_0\right),$$
$$N(t) = \left(a^2-b^2+c^2-d^2\right)^6 t(t+N_1)(t+N_2)(t+N_3)(t+N_4)(t+N_5),$$

with coefficients as follows:

$$\begin{aligned}
M_0 &= -8(a^9bcd - 4a^6b^2c^2d^2 - 2a^5b^5cd - 2a^5bc^5d - 2a^5bcd^5 + 4a^4b^4c^4 \\
&\quad + 4a^4b^4d^4 + 4a^4c^4d^4 + 8a^3b^3c^3d^3 - 4a^2b^6c^2d^2 - 4a^2b^2c^6d^2 - 4a^2b^2c^2d^6 \\
&\quad + ab^9cd - 2ab^5c^5d - 2ab^5cd^5 + abc^9d - 2abc^5d^5 + abcd^9 + 4b^4c^4d^4) \\
M_1 &= a^8 - 32a^5bcd - 2a^4b^4 - 2a^4c^4 - 2a^4d^4 + 136a^2b^2c^2d^2 - 32ab^5cd \\
&\quad - 32abc^5d - 32abcd^5 + b^8 - 2b^4c^4 - 2b^4d^4 + c^8 - 2c^4d^4 + d^8
\end{aligned}$$

$$\begin{aligned}
M_2 &= 2(a^4+b^4+c^4+d^4-12abcd) \\
N_1 &= -16abcd \\
N_2 &= (a-b-c-d)(a+b+c-d)(a+b-c+d)(a-b+c+d) \\
N_3 &= (a^2-2ab+b^2+c^2-2cd+d^2)(a^2+2ab+b^2+c^2+2cd+d^2) \\
N_4 &= (a^2+b^2-2ac+c^2-2bd+d^2)(a^2+b^2+2ac+c^2+2bd+d^2) \\
N_5 &= (a^2+b^2-2bc+c^2-2ad+d^2)(a^2+b^2+2bc+c^2+2ad+d^2).
\end{aligned}$$

Next, we perform a change in affine coordinates, by setting $u_2 = q^2u_1, v_2 = q^3v_1$, where $q \in \mathbb{C}^*$ is chosen such $q^2 = 2^5 3^3 (a^2-b^2+c^2-d^2)^{-3}$. Note that $a^2-b^2+c^2-d^2 \neq 0$, as the expression is a factor in (79). We also choose to reparameterize the quintic pencil with a new parameter $\varepsilon = 6t + 2M_2$. In this context, formula (92) clears miraculously:

$$v_2^2 = u_2^3 + \widetilde{M}(\varepsilon)\cdot u_2^2 + \widetilde{N}(\varepsilon)\cdot u_2 \tag{93}$$

with

$$\widetilde{M}(\varepsilon) = -8\left(\varepsilon^3 - 122P_8\varepsilon - 16P_{12}\right)$$
$$\widetilde{N}(\varepsilon) = 16\left(\varepsilon^6 - 24P_8\varepsilon^4 - 32P_{12}\varepsilon^3 + 144P_8^2\varepsilon^2 + 384P_{20}\varepsilon + 256P_{24}\right),$$

where the terms $P_2, P_8, P_{12}, P_{20}, P_{24}$ are homogeneous polynomials in the fundamental theta constants a, b, c, d. The precise form of $P_2, P_8, P_{12}, P_{20}, P_{24}$ is given in Appendix A.1.

5.7. Matching of the two interpretations

Comparing (72) and (93), one obtains that the two affine forms describe isomorphic elliptic fibration if and only if the following identities hold, up to a common weighted scaling of type $(2, 3, 5, 6)$:

$$\begin{aligned}
\alpha &= 2^4P_8 \\
\beta &= 2^6P_{12} \\
\gamma &= 2^{10}\cdot 3\cdot(P_{20} - P_8P_{12}) \\
\delta &= 2^{12}\left(P_{12}^2 - P_{24}\right).
\end{aligned}$$

Via identities (96) and (97) the above provides the following matching of weighted points in $\mathrm{WP}(2,3,5,6)$:

$$[\alpha,\beta,\gamma,\delta]=\left[2^4P_8,\ 2^6P_{12},\ -2^{14}3^5Q_{20},\ 2^{16}3^5Q_{24}\right]. \tag{94}$$

After taking into account formulas (39), identity (94) becomes

$$[\alpha,\beta,\gamma,\delta]=\left[\mathcal{E}_4,\ \mathcal{E}_6,\ 2^{12}3^5\mathcal{C}_{10},2^{12}3^6\mathcal{C}_{12}\right]. \tag{95}$$

This completes the proof of Theorem 3.5.

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Appendix

During the computation presented in this paper, a few special polynomials played an important role. We include their precise form in this appendix section. The homogeneous polynomials of this section have as parameters the four fundamental theta constants a, b, c, d of Section 5.2. The polynomials presented below are available in electronic `Mathematica` format at http://www.arch.umsl.edu/~clingher/siegel-paper/Mathematica/.

A.1. Special polynomials : $P_2, P_8, P_{12}, P_{20}, P_{24}, Q_{20}, Q_{24}$

$$\begin{aligned}
P_2 &= a^2+b^2+c^2+d^2\\
P_8 &= a^8+14a^4b^4+14a^4c^4+14a^4d^4+168a^2b^2c^2d^2+b^8\\
&\quad +14b^4c^4+14b^4d^4+c^8+14c^4d^4+d^8\\
P_{12} &= a^{12}-33a^8b^4-33a^8c^4-33a^8d^4+792a^6b^2c^2d^2-33a^4b^8+330a^4b^4c^4\\
&\quad +330a^4b^4d^4-33a^4c^8+330a^4c^4d^4-33a^4d^8\\
&\quad +792a^2b^6c^2d^2+792a^2b^2c^6d^2+792a^2b^2c^2d^6+b^{12}\\
&\quad -33b^8c^4-33b^8d^4-33b^4c^8\\
&\quad +330b^4c^4d^4-33b^4d^8+c^{12}-33c^8d^4-33c^4d^8+d^{12}\\
P_{20} &= a^{20}-19b^4a^{16}-19c^4a^{16}-19d^4a^{16}-336b^2c^2d^2a^{14}-494b^8a^{12}\\
&\quad -494c^8a^{12}-494d^8a^{12}+716b^4c^4a^{12}+716b^4d^4a^{12}\\
&\quad +716c^4d^4a^{12}+7632b^2c^2d^6a^{10}+7632b^2c^6d^2a^{10}\\
&\quad +7632b^6c^2d^2a^{10}-494b^{12}a^8-494c^{12}a^8-494d^{12}a^8\\
&\quad +1038b^4c^8a^8+1038b^4d^8a^8
\end{aligned}$$

$$
\begin{aligned}
& + 1038c^4d^8a^8 + 1038b^8c^4a^8 + 1038b^8d^4a^8 \\
& + 1038c^8d^4a^8 + 129{,}012b^4c^4d^4a^8 + 7632b^2c^2d^{10}a^6 + 106{,}848b^2c^6d^6a^6 \\
& + 106{,}848b^6c^2d^6a^6 + 7632b^2c^{10}d^2a^6 + 106{,}848b^6c^6d^2a^6 \\
& + 7632b^{10}c^2d^2a^6 - 19b^{16}a^4 - 19c^{16}a^4 - 19d^{16}a^4 + 716b^4c^{12}a^4 \\
& + 716b^4d^{12}a^4 + 716c^4d^{12}a^4 + 1038b^8c^8a^4 + 1038b^8d^8a^4 \\
& + 1038c^8d^8a^4 + 129{,}012b^4c^4d^8a^4 \\
& + 716b^{12}c^4a^4 + 716b^{12}d^4a^4 + 716c^{12}d^4a^4 \\
& + 129{,}012b^4c^8d^4a^4 + 129{,}012b^8c^4d^4a^4 \\
& - 336b^2c^2d^{14}a^2 + 7632b^2c^6d^{10}a^2 \\
& + 7632b^6c^2d^{10}a^2 + 7632b^2c^{10}d^6a^2 \\
& + 106{,}848b^6c^6d^6a^2 + 7632b^{10}c^2d^6a^2 - 336b^2c^{14}d^2a^2 + 7632b^6c^{10}d^2a^2 \\
& + 7632b^{10}c^6d^2a^2 - 336b^{14}c^2d^2a^2 + b^{20} + c^{20} + d^{20} - 19b^4c^{16} \\
& - 19b^4d^{16} - 19c^4d^{16} - 494b^8c^{12} - 494b^8d^{12} \\
& - 494c^8d^{12} + 716b^4c^4d^{12} - 494b^{12}c^8 \\
& - 494b^{12}d^8 - 494c^{12}d^8 + 1038b^4c^8d^8 + 1038b^8c^4d^8 - 19b^{16}c^4 - 19b^{16}d^4 \\
& - 19c^{16}d^4 + 716b^4c^{12}d^4 + 1038b^8c^8d^4 + 716b^{12}c^4d^4
\end{aligned}
$$

$$
\begin{aligned}
Q_{20} = {} & (bc - ad)(ad + bc)(bd - ac)(ac + bd)(ab - cd) \\
& \times (ab + cd)(a^2 + b^2 - c^2 - d^2) \\
& \times (-a^2 + b^2 + c^2 - d^2)(-a^2 + b^2 - c^2 + d^2)(a^2 + b^2 + c^2 + d^2)
\end{aligned}
$$

$$
\begin{aligned}
P_{24} = {} & (a^4 - 12abcd + b^4 + c^4 + d^4)(a^4 + 12abcd + b^4 + c^4 + d^4) \\
& \times (a^4 - 6a^2b^2 - 6a^2c^2 - 6a^2d^2 + b^4 - 6b^2c^2 - 6b^2d^2 + c^4 - 6c^2d^2 + d^4) \\
& \times (a^4 - 6a^2b^2 + 6a^2c^2 + 6a^2d^2 + b^4 + 6b^2c^2 + 6b^2d^2 + c^4 - 6c^2d^2 + d^4) \\
& \times (a^4 + 6a^2b^2 - 6a^2c^2 + 6a^2d^2 + b^4 + 6b^2c^2 - 6b^2d^2 + c^4 + 6c^2d^2 + d^4) \\
& \times (a^4 + 6a^2b^2 + 6a^2c^2 - 6a^2d^2 + b^4 - 6b^2c^2 + 6b^2d^2 + c^4 + 6c^2d^2 + d^4)
\end{aligned}
$$

$$
\begin{aligned}
Q_{24} = {} & b^2c^2d^2a^{18} + 2b^4c^4a^{16} + 2b^4d^4a^{16} + 2c^4d^4a^{16} \\
& - 12b^2c^2d^6a^{14} - 12b^2c^6d^2a^{14} \\
& - 12b^6c^2d^2a^{14} - 2b^4c^8a^{12} - 2b^4d^8a^{12} - 2c^4d^8a^{12} - 2b^8c^4a^{12} - 2b^8d^4a^{12} \\
& - 2c^8d^4a^{12} + 76b^4c^4d^4a^{12} + 22b^2c^2d^{10}a^{10} \\
& - 52b^2c^6d^6a^{10} - 52b^6c^2d^6a^{10} + 22b^2c^{10}d^2a^{10} \\
& - 52b^6c^6d^2a^{10} + 22b^{10}c^2d^2a^{10} - 2b^4c^{12}a^8 - 2b^4d^{12}a^8 - 2c^4d^{12}a^8 \\
& + 36b^8c^8a^8 + 36b^8d^8a^8 + 36c^8d^8a^8 + 36b^4c^4d^8a^8 - 2b^{12}c^4a^8 \\
& - 2b^{12}d^4a^8 - 2c^{12}d^4a^8 + 36b^4c^8d^4a^8 \\
& + 36b^8c^4d^4a^8 - 12b^2c^2d^{14}a^6 - 52b^2c^6d^{10}a^6 \\
& - 52b^6c^2d^{10}a^6 - 52b^2c^{10}d^6a^6 - 8b^6c^6d^6a^6 \\
& - 52b^{10}c^2d^6a^6 - 12b^2c^{14}d^2a^6 \\
& - 52b^6c^{10}d^2a^6 - 52b^{10}c^6d^2a^6 - 12b^{14}c^2d^2a^6 + 2b^4c^{16}a^4 \\
& + 2b^4d^{16}a^4 + 2c^4d^{16}a^4 \\
& - 2b^8c^{12}a^4 - 2b^8d^{12}a^4 - 2c^8d^{12}a^4 + 76b^4c^4d^{12}a^4 - 2b^{12}c^8a^4 - 2b^{12}d^8a^4 \\
& - 2c^{12}d^8a^4 + 36b^4c^8d^8a^4 + 36b^8c^4d^8a^4 + 2b^{16}c^4a^4 + 2b^{16}d^4a^4 + 2c^{16}d^4a^4
\end{aligned}
$$

$$
\begin{aligned}
&+76b^4c^{12}d^4a^4+36b^8c^8d^4a^4+76b^{12}c^4d^4a^4+b^2c^2d^{18}a^2-12b^2c^6d^{14}a^2\\
&-12b^6c^2d^{14}a^2+22b^2c^{10}d^{10}a^2\\
&-52b^6c^6d^{10}a^2+22b^{10}c^2d^{10}a^2-12b^2c^{14}d^6a^2\\
&-52b^6c^{10}d^6a^2-52b^{10}c^6d^6a^2-12b^{14}c^2d^6a^2+b^2c^{18}d^2a^2-12b^6c^{14}d^2a^2\\
&+22b^{10}c^{10}d^2a^2-12b^{14}c^6d^2a^2+b^{18}c^2d^2a^2+2b^4c^4d^{16}\\
&-2b^4c^8d^{12}-2b^8c^4d^{12}-2b^4c^{12}d^8+36b^8c^8d^8-2b^{12}c^4d^8\\
&+2b^4c^{16}d^4-2b^8c^{12}d^4-2b^{12}c^8d^4+2b^{16}c^4d^4 \qquad .
\end{aligned}
$$

The above polynomials satisfy the following relations

$$P_{20}-P_8\cdot P_{12}=-2^4 3^4 Q_{20} \tag{96}$$

$$P_{12}^2-P_{24}=2^4\cdot 3^5\cdot Q_{24}. \tag{97}$$

A.2. *Coefficients of the quintic* $\mathrm{QIN}_2(x,y,z)$

$$
\begin{aligned}
&k_{500}==0\\
&k_{050}=-8(bc+ad)^2(ab-cd)^3(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)^2\\
&k_{005}=0\\
&k_{410}=4(bc+ad)(ac+bd)^3(a^2+b^2-c^2-d^2)^2(-a^2+b^2-c^2+d^2)^2\\
&k_{401}=16(bc-ad)^3(bc+ad)(ac+bd)^2(a^2+b^2-c^2-d^2)(a^2+b^2+c^2+d^2)\\
&k_{140}\ =\ 4(bc+ad)(ab-cd)^2(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)^2\\
&\qquad\quad \times(a^3b+ab^3-7abc^2-7a^2cd-7b^2cd+c^3d-7abd^2+cd^3)\\
&k_{041}\ =\ -16(bc+ad)(ab-cd)^3(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)\\
&\qquad\quad \times(a^3c-2ab^2c+ac^3-2a^2bd+b^3d-2bc^2d-2acd^2+bd^3)\\
&k_{104}=0\\
&k_{014}=0\\
&k_{320}\ =\ 4(bc+ad)(ac+bd)^2(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)^2\\
&\qquad\quad \times(3a^3b+3ab^3-5abc^2-5a^2cd-5b^2cd+3c^3d-5abd^2+3cd^3)\\
&k_{302}\ =\ -16(-bc+ad)^2(bc+ad)(ac+bd)^2(a^2+b^2+c^2+d^2)\\
&\qquad\quad \times(a^3b+ab^3-3abc^2+3a^2cd+3b^2cd-c^3d-3abd^2-cd^3)\\
&k_{230}\ =\ 12(bc+ad)(ac+bd)(ab-cd)(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)^2\\
&\qquad\quad \times(a^3b+ab^3-3abc^2-3a^2cd-3b^2cd+c^3d-3abd^2+cd^3)\\
&k_{032}\ =\ -8(-ac+bd)(ab-cd)^3(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)\\
&\qquad\quad \times(a^3c-5ab^2c+ac^3-5a^2bd+b^3d-5bc^2d-5acd^2+bd^3)\\
&k_{203}\\
&\quad =-32(-bc+ad)^2(bc+ad)(ac-bd)(ac+bd)^2(ab+cd)(a^2+b^2+c^2+d^2)\\
&k_{023}\\
&\quad =16(-ac+bd)^2(ab-cd)^3(ab+cd)(a^2+b^2-c^2-d^2)(-a^2+b^2-c^2+d^2)
\end{aligned}
$$

$$\begin{aligned}
k_{311} = {} & 4(ac+bd)(-2a^{11}bc^2+a^9b^3c^2+4a^7b^5c^2-2a^5b^7c^2-2a^3b^9c^2+ab^{11}c^2 \\
& +a^9bc^4-6a^7b^3c^4+12a^5b^5c^4+30a^3b^7c^4+11ab^9c^4+4a^7bc^6 \\
& +4a^5b^3c^6-24a^3b^5c^6-4ab^7c^6-2a^5bc^8 \\
& +14a^3b^3c^8-12ab^5c^8-2a^3bc^{10}+3ab^3c^{10} \\
& +abc^{12}-a^{12}cd+3a^8b^4cd-3a^4b^8cd+b^{12}cd+2a^{10}c^3d+3a^8b^2c^3d \\
& -12a^6b^4c^3d-22a^4b^6c^3d-6a^2b^8c^3d+3b^{10}c^3d+2a^8c^5d+14a^6b^2c^5d \\
& -14a^4b^4c^5d-38a^2b^6c^5d-12b^8c^5d-4a^6c^7d+4a^4b^2c^7d+32a^2b^4c^7d \\
& -4b^6c^7d-a^4c^9d-6a^2b^2c^9d+11b^4c^9d+2a^2c^{11}d+b^2c^{11}d-a^{11}bd^2 \\
& +2a^9b^3d^2+2a^7b^5d^2-4a^5b^7d^2-a^3b^9d^2+2ab^{11}d^2+6a^9bc^2d^2 \\
& -4a^7b^3c^2d^2+4a^3b^7c^2d^2-6ab^9c^2d^2-4a^7bc^4d^2+44a^5b^3c^4d^2 \\
& +20a^3b^5c^4d^2+32ab^7c^4d^2-14a^5bc^6d^2+4a^3b^3c^6d^2-38ab^5c^6d^2 \\
& -3a^3bc^8d^2-6ab^3c^8d^2-3a^{10}cd^3 \\
& +6a^8b^2cd^3+22a^6b^4cd^3+12a^4b^6cd^3-3a^2b^8cd^3 \\
& -2b^{10}cd^3-14a^8c^3d^3-4a^6b^2c^3d^3+4a^2b^6c^3d^3+14b^8c^3d^3-4a^6c^5d^3 \\
& -44a^4b^2c^5d^3+20a^2b^4c^5d^3-24b^6c^5d^3+6a^4c^7d^3 \\
& +4a^2b^2c^7d^3+30b^4c^7d^3-a^2c^9d^3-2b^2c^9d^3 \\
& -11a^9bd^4-30a^7b^3d^4-12a^5b^5d^4+6a^3b^7d^4 \\
& -ab^9d^4-32a^7bc^2d^4-20a^5b^3c^2d^4-44a^3b^5c^2d^4+4ab^7c^2d^4 \\
& +14a^5bc^4d^4-14ab^5c^4d^4+12a^3bc^6d^4-22ab^3c^6d^4 \\
& -3abc^8d^4+12a^8cd^5+38a^6b^2cd^5+14a^4b^4cd^5-14a^2b^6cd^5 \\
& -2b^8cd^5+24a^6c^3d^5-20a^4b^2c^3d^5+44a^2b^4c^3d^5+4b^6c^3d^5 \\
& -12a^4c^5d^5+12b^4c^5d^5-4a^2c^7d^5-2b^2c^7d^5+4a^7bd^6+24a^5b^3d^6 \\
& -4a^3b^5d^6-4ab^7d^6+38a^5bc^2d^6-4a^3b^3c^2d^6+14ab^5c^2d^6+22a^3bc^4d^6 \\
& -12ab^3c^4d^6+4a^6cd^7-32a^4b^2cd^7-4a^2b^4cd^7+4b^6cd^7-30a^4c^3d^7 \\
& -4a^2b^2c^3d^7-6b^4c^3d^7+2a^2c^5d^7+4b^2c^5d^7+12a^5bd^8-14a^3b^3d^8 \\
& +2ab^5d^8+6a^3bc^2d^8+3ab^3c^2d^8+3abc^4d^8-11a^4cd^9+6a^2b^2cd^9+b^4cd^9 \\
& +2a^2c^3d^9+b^2c^3d^9-3a^3bd^{10}+2ab^3d^{10}-a^2cd^{11}-2b^2cd^{11}-abd^{12}) \\
k_{131} = {} & 4(ab-cd)^2(-2a^{10}bc+a^8b^3c+4a^6b^5c-2a^4b^7c-2a^2b^9c+b^{11}c \\
& +9a^8bc^3-22a^6b^3c^3-4a^4b^5c^3 \\
& +30a^2b^7c^3+3b^9c^3+12a^6bc^5+20a^4b^3c^5-64a^2b^5c^5-4b^7c^5 \\
& -10a^4bc^7+46a^2b^3c^7-4b^5c^7-10a^2bc^9+3b^3c^9+bc^{11} \\
& -a^{11}d+2a^9b^2d+2a^7b^4d-4a^5b^6d-a^3b^8d+2ab^{10}d+10a^9c^2d \\
& -22a^7b^2c^2d-6a^5b^4c^2d+30a^3b^6c^2d+4ab^8c^2d \\
& +10a^7c^4d+18a^5b^2c^4d-66a^3b^4c^4d-6ab^6c^4d-12a^5c^6d \\
& +46a^3b^2c^6d-6ab^4c^6d-9a^3c^8d \\
& +4ab^2c^8d+2ac^{10}d-4a^8bcd^2-30a^6b^3cd^2+6a^4b^5cd^2+22a^2b^7cd^2 \\
& -10b^9cd^2-46a^6bc^3d^2+18a^4b^3c^3d^2 \\
& -26a^2b^5c^3d^2+46b^7c^3d^2-18a^4bc^5d^2 \\
& -26a^2b^3c^5d^2-64b^5c^5d^2+22a^2bc^7d^2+30b^3c^7d^2-2bc^9d^2-3a^9d^3
\end{aligned}$$

$$
\begin{aligned}
& -30a^7b^2d^3+4a^5b^4d^3+22a^3b^6d^3-9ab^8d^3-46a^7c^2d^3+26a^5b^2c^2d^3\\
& -18a^3b^4c^2d^3+46ab^6c^2d^3-20a^5c^4d^3\\
& -18a^3b^2c^4d^3-66ab^4c^4d^3+22a^3c^6d^3\\
& +30ab^2c^6d^3-ac^8d^3+6a^6bcd^4+66a^4b^3cd^4-18a^2b^5cd^4-10b^7cd^4\\
& +66a^4bc^3d^4+18a^2b^3c^3d^4+20b^5c^3d^4\\
& +6a^2bc^5d^4-4b^3c^5d^4-2bc^7d^4\\
& +4a^7d^5+64a^5b^2d^5-20a^3b^4d^5-12ab^6d^5+64a^5c^2d^5+26a^3b^2c^2d^5\\
& +18ab^4c^2d^5+4a^3c^4d^5-6ab^2c^4d^5-4ac^6d^5+6a^4bcd^6-46a^2b^3cd^6\\
& +12b^5cd^6-30a^2bc^3d^6-22b^3c^3d^6\\
& +4bc^5d^6+4a^5d^7-46a^3b^2d^7+10ab^4d^7\\
& -30a^3c^2d^7-22ab^2c^2d^7+2ac^4d^7-4a^2bcd^8+9b^3cd^8\\
& +bc^3d^8-3a^3d^9+10ab^2d^9+2ac^2d^9-2bcd^{10}-ad^{11})
\end{aligned}
$$

$$
\begin{aligned}
k_{113} ={} & (ac-bd)(ab-cd)(a^{12}-2a^{10}b^2-a^8b^4+4a^6b^6-a^4b^8-2a^2b^{10}+b^{12}\\
& -2a^{10}c^2+10a^8b^2c^2+12a^6b^4c^2\\
& -12a^4b^6c^2-10a^2b^8c^2+2b^{10}c^2-a^8c^4\\
& +12a^6b^2c^4+74a^4b^4c^4+12a^2b^6c^4-b^8c^4+4a^6c^6\\
& -12a^4b^2c^6+12a^2b^4c^6-4b^6c^6-a^4c^8-10a^2b^2c^8-b^4c^8\\
& -2a^2c^{10}+2b^2c^{10}+c^{12}-8a^9bcd\\
& +16a^5b^5cd-8ab^9cd+16a^5bc^5d+16ab^5c^5d\\
& -8abc^9d+2a^{10}d^2-10a^8b^2d^2\\
& -12a^6b^4d^2+12a^4b^6d^2+10a^2b^8d^2-2b^{10}d^2-10a^8c^2d^2-72a^6b^2c^2d^2\\
& -28a^4b^4c^2d^2-72a^2b^6c^2d^2-10b^8c^2d^2-12a^6c^4d^2-28a^4b^2c^4d^2\\
& +28a^2b^4c^4d^2+12b^6c^4d^2+12a^4c^6d^2\\
& -72a^2b^2c^6d^2+12b^4c^6d^2+10a^2c^8d^2\\
& -10b^2c^8d^2-2c^{10}d^2-64a^3b^3c^3d^3-a^8d^4+12a^6b^2d^4+74a^4b^4d^4\\
& +12a^2b^6d^4-b^8d^4+12a^6c^2d^4+28a^4b^2c^2d^4-28a^2b^4c^2d^4-12b^6c^2d^4\\
& +74a^4c^4d^4-28a^2b^2c^4d^4+74b^4c^4d^4\\
& +12a^2c^6d^4-12b^2c^6d^4-c^8d^4\\
& +16a^5bcd^5+16ab^5cd^5+16abc^5d^5-4a^6d^6+12a^4b^2d^6\\
& -12a^2b^4d^6+4b^6d^6+12a^4c^2d^6-72a^2b^2c^2d^6+12b^4c^2d^6-12a^2c^4d^6\\
& +12b^2c^4d^6+4c^6d^6-a^4d^8-10a^2b^2d^8\\
& -b^4d^8-10a^2c^2d^8+10b^2c^2d^8-c^4d^8-8abcd^9\\
& +2a^2d^{10}-2b^2d^{10}-2c^2d^{10}+d^{12})
\end{aligned}
$$

$$
\begin{aligned}
k_{221} ={} & 8(ab-cd)(2a^{11}bc^2-a^9b^3c^2-4a^7b^5c^2+2a^5b^7c^2+2a^3b^9c^2-ab^{11}c^2\\
& -3a^9bc^4+10a^7b^3c^4-4a^5b^5c^4-22a^3b^7c^4-5ab^9c^4-6a^7bc^6-8a^5b^3c^6\\
& +34a^3b^5c^6+4ab^7c^6+4a^5bc^8-22a^3b^3c^8+6ab^5c^8+4a^3bc^{10}-3ab^3c^{10}\\
& -abc^{12}+a^{12}cd-3a^8b^4cd+3a^4b^8cd-b^{12}cd-4a^{10}c^3d+a^8b^2c^3d\\
& +8a^6b^4c^3d-6a^4b^6c^3d-12a^2b^8c^3d-3b^{10}c^3d-4a^8c^5d-18a^6b^2c^5d\\
& +18a^4b^4c^5d+38a^2b^6c^5d+6b^8c^5d+6a^6c^7d-12a^4b^2c^7d-26a^2b^4c^7d
\end{aligned}
$$

$$
\begin{aligned}
& + 4b^6c^7d + 3a^4c^9d + 6a^2b^2c^9d - 5b^4c^9d - 2a^2c^{11}d - b^2c^{11}d + a^{11}bd^2 \\
& - 2a^9b^3d^2 - 2a^7b^5d^2 + 4a^5b^7d^2 + a^3b^9d^2 - 2ab^{11}d^2 - 6a^9bc^2d^2 \\
& + 16a^7b^3c^2d^2 - 16a^3b^7c^2d^2 + 6ab^9c^2d^2 \\
& + 12a^7bc^4d^2 - 34a^5b^3c^4d^2 + 24a^3b^5c^4d^2 \\
& - 26ab^7c^4d^2 + 18a^5bc^6d^2 - 8a^3b^3c^6d^2 + 38ab^5c^6d^2 - a^3bc^8d^2 - 12ab^3c^8d^2 \\
& + 3a^{10}cd^3 + 12a^8b^2cd^3 + 6a^6b^4cd^3 - 8a^4b^6cd^3 - a^2b^8cd^3 + 4b^{10}cd^3 \\
& + 22a^8c^3d^3 + 8a^6b^2c^3d^3 - 8a^2b^6c^3d^3 \\
& - 22b^8c^3d^3 + 8a^6c^5d^3 + 34a^4b^2c^5d^3 \\
& + 24a^2b^4c^5d^3 + 34b^6c^5d^3 - 10a^4c^7d^3 - 16a^2b^2c^7d^3 - 22b^4c^7d^3 + a^2c^9d^3 \\
& + 2b^2c^9d^3 + 5a^9bd^4 + 22a^7b^3d^4 + 4a^5b^5d^4 \\
& - 10a^3b^7d^4 + 3ab^9d^4 + 26a^7bc^2d^4 \\
& - 24a^5b^3c^2d^4 + 34a^3b^5c^2d^4 - 12ab^7c^2d^4 - 18a^5bc^4d^4 + 18ab^5c^4d^4 \\
& - 8a^3bc^6d^4 - 6ab^3c^6d^4 + 3abc^8d^4 - 6a^8cd^5 - 38a^6b^2cd^5 - 18a^4b^4cd^5 \\
& + 18a^2b^6cd^5 + 4b^8cd^5 - 34a^6c^3d^5 - 24a^4b^2c^3d^5 - 34a^2b^4c^3d^5 - 8b^6c^3d^5 \\
& + 4a^4c^5d^5 - 4b^4c^5d^5 + 4a^2c^7d^5 + 2b^2c^7d^5 - 4a^7bd^6 - 34a^5b^3d^6 + 8a^3b^5d^6 \\
& + 6ab^7d^6 - 38a^5bc^2d^6 + 8a^3b^3c^2d^6 - 18ab^5c^2d^6 + 6a^3bc^4d^6 + 8ab^3c^4d^6 \\
& - 4a^6cd^7 + 26a^4b^2cd^7 + 12a^2b^4cd^7 - 6b^6cd^7 + 22a^4c^3d^7 + 16a^2b^2c^3d^7 \\
& + 10b^4c^3d^7 - 2a^2c^5d^7 - 4b^2c^5d^7 - 6a^5bd^8 \\
& + 22a^3b^3d^8 - 4ab^5d^8 + 12a^3bc^2d^8 \\
& + ab^3c^2d^8 - 3abc^4d^8 + 5a^4cd^9 - 6a^2b^2cd^9 - 3b^4cd^9 - 2a^2c^3d^9 \\
& - b^2c^3d^9 + 3a^3bd^{10} - 4ab^3d^{10} + a^2cd^{11} + 2b^2cd^{11} + abd^{12})
\end{aligned}
$$

$$
\begin{aligned}
k_{212} = {} & (ac + bd)(5a^{12}bc - 6a^{10}b^3c - 9a^8b^5c + 12a^6b^7c \\
& + 3a^4b^9c - 6a^2b^{11}c + b^{13}c \\
& - 6a^{10}bc^3 + 18a^8b^3c^3 - 20a^6b^5c^3 - 84a^4b^7c^3 - 38a^2b^9c^3 + 2b^{11}c^3 \\
& - 9a^8bc^5 - 4a^6b^3c^5 + 178a^4b^5c^5 + 60a^2b^7c^5 - b^9c^5 + 12a^6bc^7 \\
& - 52a^4b^3c^7 + 76a^2b^5c^7 \\
& - 4b^7c^7 + 3a^4bc^9 - 22a^2b^3c^9 - b^5c^9 - 6a^2bc^{11} + 2b^3c^{11} + bc^{13} + a^{13}d \\
& - 6a^{11}b^2d + 3a^9b^4d + 12a^7b^6d - 9a^5b^8d - 6a^3b^{10}d + 5ab^{12}d - 6a^{11}c^2d \\
& - 6a^9b^2c^2d + 12a^7b^4c^2d + 60a^5b^6c^2d + 58a^3b^8c^2d + 10ab^{10}c^2d + 3a^9c^4d \\
& - 4a^7b^2c^4d + 10a^5b^4c^4d - 132a^3b^6c^4d - 37ab^8c^4d \\
& + 12a^7c^6d + 28a^5b^2c^6d - 148a^3b^4c^6d \\
& - 84ab^6c^6d - 9a^5c^8d + 42a^3b^2c^8d - 37ab^4c^8d - 6a^3c^{10}d \\
& + 10ab^2c^{10}d + 5ac^{12}d + 10a^{10}bcd^2 + 58a^8b^3cd^2 + 60a^6b^5cd^2 \\
& + 12a^4b^7cd^2 - 6a^2b^9cd^2 - 6b^{11}cd^2 + 42a^8bc^3d^2 - 56a^6b^3c^3d^2 \\
& + 60a^4b^5c^3d^2 - 88a^2b^7c^3d^2 - 22b^9c^3d^2 + 28a^6bc^5d^2 \\
& + 60a^4b^3c^5d^2 + 476a^2b^5c^5d^2 + 76b^7c^5d^2 - 4a^4bc^7d^2 \\
& - 88a^2b^3c^7d^2 + 60b^5c^7d^2 - 6a^2bc^9d^2 - 38b^3c^9d^2 - 6bc^{11}d^2 + 2a^{11}d^3 \\
& - 38a^9b^2d^3 - 84a^7b^4d^3 - 20a^5b^6d^3 + 18a^3b^8d^3 - 6ab^{10}d^3 - 22a^9c^2d^3 \\
& - 88a^7b^2c^2d^3 + 60a^5b^4c^2d^3 - 56a^3b^6c^2d^3 \\
& + 42ab^8c^2d^3 - 52a^7c^4d^3 + 60a^5b^2c^4d^3
\end{aligned}
$$

$$
\begin{aligned}
& -340a^3b^4c^4d^3 - 148ab^6c^4d^3 - 4a^5c^6d^3 \\
& -56a^3b^2c^6d^3 - 132ab^4c^6d^3 + 18a^3c^8d^3 \\
& +58ab^2c^8d^3 - 6ac^{10}d^3 - 37a^8bcd^4 - 132a^6b^3cd^4 \\
& +10a^4b^5cd^4 - 4a^2b^7cd^4 + 3b^9cd^4 - 148a^6bc^3d^4 \\
& -340a^4b^3c^3d^4 + 60a^2b^5c^3d^4 - 52b^7c^3d^4 \\
& +10a^4bc^5d^4 + 60a^2b^3c^5d^4 + 178b^5c^5d^4 + 12a^2bc^7d^4 - 84b^3c^7d^4 + 3bc^9d^4 \\
& -a^9d^5 + 60a^7b^2d^5 + 178a^5b^4d^5 - 4a^3b^6d^5 - 9ab^8d^5 + 76a^7c^2d^5 \\
& +476a^5b^2c^2d^5 + 60a^3b^4c^2d^5 + 28ab^6c^2d^5 + 178a^5c^4d^5 \\
& +60a^3b^2c^4d^5 + 10ab^4c^4d^5 - 20a^3c^6d^5 \\
& +60ab^2c^6d^5 - 9ac^8d^5 - 84a^6bcd^6 - 148a^4b^3cd^6 \\
& +28a^2b^5cd^6 + 12b^7cd^6 - 132a^4bc^3d^6 - 56a^2b^3c^3d^6 \\
& -4b^5c^3d^6 + 60a^2bc^5d^6 - 20b^3c^5d^6 + 12bc^7d^6 \\
& -4a^7d^7 + 76a^5b^2d^7 - 52a^3b^4d^7 + 12ab^6d^7 + 60a^5c^2d^7 \\
& -88a^3b^2c^2d^7 4ab^4c^2d^7 - 84a^3c^4d^7 + 12ab^2c^4d^7 \\
& +12ac^6d^7 - 37a^4bcd^8 + 42a^2b^3cd^8 - 9b^5cd^8 \\
& +58a^2bc^3d^8 + 18b^3c^3d^8 - 9bc^5d^8 - a^5d^9 - 22a^3b^2d^9 + 3ab^4d^9 \\
& -38a^3c^2d^9 - 6ab^2c^2d^9 + 3ac^4d^9 + 10a^2bcd^{10} \\
& -6b^3cd^{10} - 6bc^3d^{10} + 2a^3d^{11} - 6ab^2d^{11} \\
& -6ac^2d^{11} + 5bcd^{12} + ad^{13})
\end{aligned}
$$

$$
\begin{aligned}
k_{122} = {} & (ab - cd)(5a^{12}bc - 6a^{10}b^3c - 9a^8b^5c + 12a^6b^7c + 3a^4b^9c \\
& -6a^2b^{11}c + b^{13}c - 14a^{10}bc^3 + 58a^8b^3c^3 + 4a^6b^5c^3 - 92a^4b^7c^3 \\
& -22a^2b^9c^3 + 2b^{11}c^3 - 17a^8bc^5 - 20a^6b^3c^5 + 234a^4b^5c^5 + 28a^2b^7c^5 \\
& -b^9c^5 + 20a^6bc^7 - 108a^4b^3c^7 + 28a^2b^5c^7 \\
& -4b^7c^7 + 11a^4bc^9 - 22a^2b^3c^9 - b^5c^9 - 6a^2bc^{11} + 2b^3c^{11} + bc^{13} + a^{13}d \\
& -6a^{11}b^2d + 3a^9b^4d + 12a^7b^6d - 9a^5b^8d - 6a^3b^{10}d \\
& +5ab^{12}d - 6a^{11}c^2d + 34a^9b^2c^2d - 12a^7b^4c^2d - 12a^5b^6c^2d + 50a^3b^8c^2d \\
& +10ab^{10}c^2d + 11a^9c^4d \\
& -4a^7b^2c^4d + 146a^5b^4c^4d - 52a^3b^6c^4d - 5ab^8c^4d + 20a^7c^6d + 4a^5b^2c^6d \\
& -36a^3b^4c^6d - 20ab^6c^6d - 17a^5c^8d + 58a^3b^2c^8d - 5ab^4c^8d - 14a^3c^{10}d \\
& +10ab^2c^{10}d + 5ac^{12}d + 10a^{10}bcd^2 + 50a^8b^3cd^2 - 12a^6b^5cd^2 - 12a^4b^7cd^2 \\
& +34a^2b^9cd^2 - 6b^{11}cd^2 + 58a^8bc^3d^2 - 120a^6b^3c^3d^2 + 12a^4b^5c^3d^2 \\
& -248a^2b^7c^3d^2 - 22b^9c^3d^2 + 4a^6bc^5d^2 - 12a^4b^3c^5d^2 + 332a^2b^5c^5d^2 \\
& +28b^7c^5d^2 - 4a^4bc^7d^2 - 248a^2b^3c^7d^2 + 28b^5c^7d^2 + 34a^2bc^9d^2 \\
& -22b^3c^9d^2 - 6bc^{11}d^2 + 2a^{11}d^3 - 22a^9b^2d^3 - 92a^7b^4d^3 \\
& +4a^5b^6d^3 + 58a^3b^8d^3 - 14ab^{10}d^3 - 22a^9c^2d^3 \\
& -248a^7b^2c^2d^3 + 12a^5b^4c^2d^3 - 120a^3b^6c^2d^3 + 58ab^8c^2d^3 \\
& -108a^7c^4d^3 - 12a^5b^2c^4d^3 - 100a^3b^4c^4d^3 - 36ab^6c^4d^3 \\
& -20a^5c^6d^3 - 120a^3b^2c^6d^3 - 52ab^4c^6d^3 + 58a^3c^8d^3 \\
& +50ab^2c^8d^3 - 6ac^{10}d^3 - 5a^8bcd^4 \\
& -52a^6b^3cd^4 + 146a^4b^5cd^4 - 4a^2b^7cd^4 + 11b^9cd^4
\end{aligned}
$$

$$
\begin{aligned}
&- 36a^6bc^3d^4 - 100a^4b^3c^3d^4 - 12a^2b^5c^3d^4 - 108b^7c^3d^4 \\
&+ 146a^4bc^5d^4 + 12a^2b^3c^5d^4 + 234b^5c^5d^4 - 12a^2bc^7d^4 \\
&- 92b^3c^7d^4 + 3bc^9d^4 - a^9d^5 + 28a^7b^2d^5 + 234a^5b^4d^5 \\
&- 20a^3b^6d^5 - 17ab^8d^5 + 28a^7c^2d^5 + 332a^5b^2c^2d^5 - 12a^3b^4c^2d^5 \\
&+ 4ab^6c^2d^5 + 234a^5c^4d^5 + 12a^3b^2c^4d^5 + 146ab^4c^4d^5 + 4a^3c^6d^5 \\
&- 12ab^2c^6d^5 - 9ac^8d^5 - 20a^6bcd^6 - 36a^4b^3cd^6 + 4a^2b^5cd^6 + 20b^7cd^6 \\
&- 52a^4bc^3d^6 - 120a^2b^3c^3d^6 \\
&- 20b^5c^3d^6 - 12a^2bc^5d^6 + 4b^3c^5d^6 + 12bc^7d^6 - 4a^7d^7 + 28a^5b^2d^7 \\
&- 108a^3b^4d^7 + 20ab^6d^7 + 28a^5c^2d^7 - 248a^3b^2c^2d^7 - 4ab^4c^2d^7 \\
&- 92a^3c^4d^7 - 12ab^2c^4d^7 \\
&+ 12ac^6d^7 - 5a^4bcd^8 + 58a^2b^3cd^8 - 17b^5cd^8 + 50a^2bc^3d^8 + 58b^3c^3d^8 \\
&- 9bc^5d^8 - a^5d^9 - 22a^3b^2d^9 + 11ab^4d^9 - 22a^3c^2d^9 \\
&+ 34ab^2c^2d^9 + 3ac^4d^9 + 10a^2bcd^{10} - 14b^3cd^{10} - 6bc^3d^{10} + 2a^3d^{11} \\
&- 6ab^2d^{11} - 6ac^2d^{11} + 5bcd^{12} + ad^{13})
\end{aligned}
$$

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