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Calabi-Yau manifolds realizing symplectically rigid monodromy tuples

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We define an iterative construction that produces a family of elliptically fibered Calabi-Yau n-folds with section from a family of elliptic Calabi-Yau varieties of one dimension lower. Parallel to the geometric construction, we iteratively obtain for each family with a point of maximal unipotent monodromy, normalized to be at $t = 0$, its Picard-Fuchs operator and a closed-form expression for the period holomorphic at $t = 0$, through a generalization of the classical Euler transform for hypergeometric functions. In particular, our construction yields one-parameter families of elliptically fibered Calabi-Yau manifolds with section whose Picard-Fuchs operators realize all symplectically rigid Calabi-Yau differential operators with three regular singular points classified by Bogner and Reiter, but also non-rigid operators with four singular points.

1. Introduction

The study of Calabi-Yau manifolds, i.e., compact Kähler manifolds with trivial canonical bundle, has been an active field in algebraic geometry and mathematical physics ever since their christening by Candelas et al. [15] in 1985. For every positive integer n, the vanishing set of a non-singular homogeneous polynomial of degree $n+2$ in the complex projective space $\mathbb{P}^{n+1}$ is a compact Calabi-Yau manifold of n complex dimensions, or Calabi-Yau n-fold for short. The construction yields for $n=1$ an elliptic curve, while for $n=2$ one obtains a K3 surface. In two complex dimensions K3 surfaces are the only simply connected Calabi-Yau manifolds. The classification of Calabi-Yau threefolds remains an open problem. Results of tremendous ongoing activity in physics that included systematic computer searches have given us a better idea of the landscape of Calabi-Yau threefolds [50, 75]. For example, the work has impressively demonstrated the near omnipresence of elliptic fibrations on Calabi-Yau threefolds. Unfortunately, it has also been revealed that it is generally quite difficult to construct examples of families of Calabi-Yau threefolds with small Hodge number $h^{2,1}$ by specialization of multi-parameter families.

The mathematical study of mirror symmetry essentially began with the example of the quintic and mirror quintic family of Candelas, de la Ossa, Green and Parkes [14]. The quintic family is a generic quintic hypersurface $X \subset \mathbb{P}^5$, e.g., the Fermat quintic

$$X_0^5 + \cdots + X_4^5 = 0,$$

which is a Calabi-Yau threefold with Hodge numbers $h^{1,1} = 1$ and $h^{2,1} = 101$. The mirror family has a flipped Hodge diamond $h^{2,1} = 1$ and $h^{1,1} = 101$ and can be constructed via the Greene-Plesser orbifolding construction from the Dwork pencil

$$X_0^5 + \cdots + X_4^5 + 5\lambda\, X_0 X_1 \cdots X_4 = 0.$$

The mirror quintic family is a one-dimensional family of Calabi-Yau threefolds defined over the base $\mathbb{P}^1\backslash\{0, 1, \infty\}$ and has exactly three singular fibers: the large complex structure limit, the Gepner point, and the stacky point (the first two describe the discriminant locus of "bad" $N = (2, 2)$-SCFT). There are some further properties coming from the "special geometry" of the moduli space. The family has played a crucial role in the spectacular computations which suggested that mirror symmetry could be used to solve long-standing problems in enumerative geometry. These remarkable observations led to an enormous mathematical activity which both tried to explain the observed phenomena and to establish similar mirror recipes and results for other families of Calabi-Yau threefolds. For example, Batyrev described in [4] a way to construct the mirror of Calabi-Yau hypersurfaces in toric varieties via dual reflexive polytopes, and a proof of the so-called mirror theorem was given by Givental in [38] and Lian, Liu, and Yau in [52]. However, most of the standard recipes fail to consistently produce families with the desired properties. The goal of this paper is to present a construction which rectifies that.

The quintic-mirror family gives rise to a variation of Hodge structure and an associated Picard-Fuchs differential equation. In turn, the solutions to the differential equation, called periods, determine this variation of Hodge structure. Doran and Morgan [34] classified certain one-parameter variations of Hodge structure which arise from families of Calabi-Yau threefolds with one-dimensional rational deformation space, i.e., families of Calabi-Yau threefolds resembling the quintic-mirror family. It is worth pointing out that their result was achieved not by constructing families of Calabi-Yau threefolds, but by classifying all integral weight-three variations of Hodge structure which *can* underlie a family of Calabi-Yau threefolds over a thrice-punctured sphere $\mathbb{P}^1\backslash\{0, 1, \infty\}$ — subject to certain conditions on the monodromy coming from mirror symmetry. Calabi-Yau threefolds with large complex structure limit and $h^{2,1} = 1$ have since emerged in a pivotal role for both mathematics and physics, where classical geometric, toric, and analytical methods are used in their investigation. The Picard-Fuchs equations for other families of Calabi-Yau threefolds were constructed by Batyrev, van Straten and others in [6, 7] which lead to a general definition of a Calabi-Yau differential operator by Almkvist, van Enckevort, van Straten and Zudilin, rooted purely in the theory of differential operators; see [2, 77]. A large number of these are known today (currently over 500!) but they are mostly found via computer searches. The reason for the name is that they are conjectured to come from families of Calabi-Yau threefolds having a large complex structure limit and $h^{2,1} = 1$.

This paper addresses part of the conjecture. We devise an iterative geometric construction that finds explicit families of elliptically fibered Calabi-Yau manifolds whose Picard-Fuchs operators realize a big class of Calabi-Yau differential operators, including the so-called symplectically rigid operators, but also non-rigid operators.

2. Summary of results

The main result of this article is an *iterative twist construction* that produces projective families $\pi : X \to B = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ of elliptically fibered Calabi-Yau n-folds $X_t = \pi^{-1}(t)$ with $t \in B = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ with section from families of elliptic Calabi-Yau varieties with the same properties in one dimension lower, for $n = 1, 2, 3, 4$. In this paper we will always restrict ourselves to elliptic fibrations with sections, so-called Jacobian elliptic fibrations. These families are then presented as Weierstrass models which are ubiquitous in the description of families of elliptic curves and K3 surfaces. In fact, Gross proved [46] that there are only a finite number of distinct topological types of elliptically fibered Calabi-Yau threefolds up to birational equivalence. For an elliptically fibered Calabi-Yau threefold, the existence of a global section makes it then possible to find an explicit presentation as a Weierstrass model [62]. For example, our iterative procedure constructs from extremal families of elliptic curves[1] with rational total space, families of Jacobian elliptic K3 surfaces of Picard rank 18 or 19, and in turn from these elliptically fibered Calabi-Yau threefolds with $h^{2,1} = 1$, and can be continued further on. Moreover, all families are iteratively constructed from a *single* geometric object, the mirror family of Fermat quadrics in $\mathbb{P}^1 \backslash \{1, \infty\}$ given by

$$X_0^2 + X_1^2 + 2\, t\, X_0\, X_1 = 0. \tag{2.1}$$

The broad range of families of Jacobian elliptic Calabi-Yau manifolds obtained by our iterative construction includes the following noteworthy families with generic fibers of dimension n:

[$n = 1$] the universal families of elliptic curves over the modular curves for $\Gamma_0(k)$ with $k = 1, 2, 3, 4, 5, 6, 8, 9$,

[1] An elliptic fibration $\pi : X \to \mathbb{P}^1$ is called extremal if and only if for the group of sections we have $\operatorname{rank} \mathrm{MW}(\pi) = 0$ and the associated elliptic surface has maximal Picard number.

[$n = 2$] families of M_k-lattice polarized K3 surfaces over the modular curves for $\Gamma_0(k)^+$ with $k = 1, 2, 3, 4, 5, 6, 8, 9$, and families of M-lattice polarized K3 surfaces (and closely related lattices of Picard rank 18),

[$n = 3$] families of Calabi-Yau threefolds with $h^{2,1} = 1$ realizing all 14 one-parameter variations of Hodge structure classified by Doran and Morgan in [34],

[$n = 4$] families of Calabi-Yau fourfolds over a one-dimensional rational deformation space realizing all 14 hypergeometric one-parameter variations of Hodge structure of weight four and type $(1, 1, 1, 1, 1)$,

[$n \in \mathbb{N}$] mirror families of Dwork pencils in $\mathbb{P}^{n+1}$.

Katz discovered that linearly rigid monodromy tuples are obtained as tensor products and convolutions of rank-one local systems [51]; Doran and Morgan [34] classified all possible fourteen linearly rigid monodromy-tuples that could come from a B-side variation of Hodge structures. As it turns out, every single one of them admits geometric realization as hypersurfaces or complete intersections in a Gorenstein toric Fano variety or as Calabi-Yau threefolds fibered by high rank K3's by Clingher et al. [21]. Within the class of irreducible Fuchsian differential operators of Calabi-Yau type, the symplectically rigid differential operators with three regular singularities constitute an important subclass and were classified by Bogner and Reiter [12]. In addition to the 14 linearly rigid examples in [34], this class includes all operators whose associated monodromy representation is symplectically rigid. Following the results of Deligne [25] and Katz [51], there is a necessary and sufficient arithmetic criterion for the generalized rigidity of a monodromy tuple within any irreducible reductive algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$: choosing the subgroup to be $\mathrm{GL}(n, \mathbb{C})$ returns the aforementioned notion of linear rigidity, whereas choosing $\mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$ provides us with the more general notion of symplectic rigidity. We know that the elements of monodromy tuples induced by a rank-four Calabi-Yau operator must lie in $\mathrm{Sp}(4, \mathbb{C})$. Bogner and Reiter showed that all of the symplectically rigid monodromy tuples of quasi-unipotent elements admit a decomposition into a sequence of middle convolutions and tensor products of Kummer sheaves of rank one [12]. In particular, they are constructible using only tuples of rank-one. Among them, 60 tuples are associated with symplectically rigid Calabi-Yau operators having a maximal unipotent element; they are available in the database of Almkvist et al. [2], or AESZ database for short.

These results suggest that there should be a geometric explanation for the Bogner and Reiter result, and therefore some kind of "iterated fibration" construction, starting with the rank-one Picard-Fuchs operator of the family (2.1), realizing all symplectically rigid Calabi-Yau operators. The main result of this article is the following:

Theorem 2.1. *All symplectically rigid Calabi-Yau operators having a maximal unipotent element are the Picard-Fuchs operators of families of Jacobian elliptic Calabi-Yau varieties $\pi : X \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$. The families are obtained by the iterative twist construction applied to the quadric pencil (2.1). In particular, for each symplectically rigid differential operator L_t the iterative twist construction produces (1) a family of transcendental cycles $\Sigma(t)$, obtained iteratively from a lower dimensional cycle using a warped product, (2) a holomorphic top-form η_t on each fiber $X_t = \pi^{-1}(t)$, represented as a closed differential form, such that the period*

$$\omega(t) = \int_{\Sigma(t)} \eta_t$$

is holomorphic on the unit disk about the maximal unipotent monodromy point $t = 0$, and solves the Picard-Fuchs equation $L_t \omega(t) = 0$.

We point out that the restriction to symplectically rigid differential operators in Theorem 2.1 was chosen to simplify this exposition. For example, our iterative construction also provides a geometric realization of all rank-four, non-rigid Calabi-Yau operators with four regular singular points that were found in [13].

The outline of this paper is as follows: in Section 3 we recall crucial definitions and properties related to hypergeometric differential operators and Calabi-Yau operators, their behavior under the exterior square operation and the Hadamard product, as well as the notion of rigidity. The details of the proof of our main theorem are quite involved, but the basic idea is simple and already present in a series of examples that we present in Section 4. In Section 5, we describe the construction of twisted families with generalized functional invariant in full generality, including several modified variants needed later, and the computation of period integrals. In Section 6 we demonstrate that families of Calabi-Yau manifolds obtained by our iterative construction include universal families of elliptic curves over the modular curves for $\Gamma_0(k)$, families of M_k-lattice polarized K3 surfaces over the modular curves for $\Gamma_0(k)^+$, and other prominent families of lattice polarized K3 surfaces. In Section 7 we show that a sequence of generalized functional invariants captures all key features of the mirror families of the deformed Fermat pencils, including the existence of (new) elliptic fibrations and relations among their holomorphic periods. In Section 8 we apply linear and quadratic transformations to the rational parameter spaces of the twisted families of elliptic curves and K3 surfaces already obtained in previous sections. As we use our twist construction iteratively, applying base transformations between twists turns out to be a crucial step in order to construct a complete set of families realizing *all* symplectically rigid monodromy tuples. The proof of Theorem 2.1 will be completed in Section 9. In Section 10, we show that 30 non-rigid Calabi-Yau operators with four singular points are readily obtained by our twist construction as well.

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3. Hypergeometric and Calabi-Yau type operators

3.1. Hypergeometric functions

The higher hypergeometric functions ${}_nF_{n-1}$ were introduced by Thomae [76] as series

$$ {}_nF_{n-1}\left(\begin{array}{c} \alpha_1, \quad \ldots \quad, \alpha_n \\ \beta_1, \ldots, \beta_{n-1} \end{array} \middle| \, t \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k} \frac{t^k}{k!}, \tag{3.1} $$

where $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$ is the Pochhammer symbol. When $n = 2$ this is the classical Gauss hypergeometric function. We always assume that we have rational parameters $\alpha_1, \ldots, \alpha_n \in (0, 1) \cap \mathbb{Q}$ and $\beta_1, \ldots, \beta_{n-1} \in (0, 1] \cap \mathbb{Q}$ such that $\alpha_i \neq \beta_j$. The rank-n and degree-one[2] differential equation satisfied by ${}_nF_{n-1}$ is given by

$$ \Big[\theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_{n-1} - 1) - t\,(\theta + \alpha_1) \cdots (\theta + \alpha_n)\Big]\, \omega(t) = 0, \tag{3.2} $$

[2] *degree* refers to the highest power in t

where $\theta = t\frac{d}{dt}$ and there are three regular singular points $t = 0, 1, \infty$. The differential operator has the Riemann symbol

$$\mathcal{P}\left(\begin{array}{ccc|c} 0 & 1 & \infty & \\ \hline 1-\beta_1 & 0 & \alpha_1 & \\ 1-\beta_2 & 1 & \alpha_2 & \\ \vdots & \vdots & \vdots & t \\ 1-\beta_{n-1} & n-2 & \alpha_{n-1} & \\ 0 & \sum_{j=1}^{n-1}\beta_j - \sum_{j=1}^{n}\alpha_j & \alpha_n & \end{array}\right). \tag{3.3}$$

From now on we shall denote the hypergeometric equation (3.2) by

$$L_t^{(n)}\Big((\alpha_1,\ldots,\alpha_n);(\beta_1,\ldots,\beta_{n-1})\Big)\,\omega(t) = 0. \tag{3.4}$$

In general, given a rank-n differential operator L_t with coefficients in $\mathbb{C}(t)$ and singular locus S with finite cardinality $|S| = r+1$, normalized to include $t = 0$ as a singular point, there is an induced rank-n local system $\mathbb{L}$ of solutions on $\mathbb{P}^1\backslash S$. If all singularities of L_t are regular, we call the operator a *Fuchsian* differential operator. We fix a base point $t_0 \in \mathbb{P}^1\backslash S$ to obtain the monodromy representation of the fundamental group given by

$$\pi_1(\mathbb{P}^1\backslash S, t_0) \to H \subset \mathrm{GL}(\mathbb{L}_{t_0}) \cong \mathrm{GL}(n,\mathbb{C}).$$

An operator is *irreducible* if the image of its monodromy representation is an irreducible subgroup. If we also fix an orientation and a set of based simple loops, i.e.,

$$\Big\{\gamma_s :\, (S^1,*) \to \big(\mathbb{P}^1\backslash S, t_0\big)\Big\}_{s\in S},$$

each circling a single point in the singular locus exactly once and an ordering of S, we obtain monodromy matrices $g_1,\ldots,g_{r+1}$ for the simple loops γ_s around the corresponding points together with the relation $g_1\cdots g_{r+1} = \mathbb{I}$. The latter follows because the product of all paths is a path encircling all of S, whence homotopic to the trivial path. The collection of monodromy matrices $T = (g_1,\ldots,g_{r+1})$ is called a *monodromy tuple* of rank n. Clearly, H is determined by the tuple of matrices $T = (g_1,\ldots,g_{r+1})$ with $g_i \in \mathrm{GL}(n,\mathbb{C})$ whose product is the identity, up to global conjugation, i.e., mapping $g_i \mapsto h\cdot g_i\cdot h^{-1}$ for a *single* $h \in \mathrm{GL}(n,\mathbb{C})$. We call a monodromy representation *linearly rigid* if the elements of the monodromy tuple are quasi-unipotent, generate an irreducible subgroup, and are completely determined by their individual conjugacy classes, i.e., Jordan forms.

For the generalized hypergeometric function this is the case if $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j; see [9]. Therefore, we have the following:

Theorem 3.1 ([9]). *The operator $L_t^{(n)}\big((\alpha_1, \ldots, \alpha_n); (1, \ldots, 1)\big)$ with $\alpha_i \in (0,1) \cap \mathbb{Q}$ and $\beta_j = 1$ in Equation (3.2) is a rank-n Fuchsian differential operator with the three regular singular points $t = 0, 1, \infty$ such that $t = 0$ is a point of maximally unipotent monodromy and the monodromy representation is linearly rigid.*

For rank $n = 1$, we have

$$ {}_1F_0(\alpha|\, t) = \big(1 - t\big)^{-\alpha}, \tag{3.5} $$

and the differential operator $L_t^{(1)}(1/2;)$ gives rise to a monodromy tuple of rank one, with monodromies $1, -1$, and -1 around the points $t = 0, 1, \infty$.

The monodromy representation for the differential operator (3.4) and the corresponding differential Galois group — which carries all information about algebraic relations between the solutions — were classified by Beukers and Heckman [9].

3.2. Convolution formulas

The *Hadamard product* of two power series $f(t) = \sum_{n\geq 0} f_n\, t^n$ and $g(t) = \sum_{n\geq 0} g_n\, t^n$ is defined by $(f \star g)(t) := \sum_{n\geq 0} f_n\, g_n\, t^n$. Using the Hadamard product, the following cancellation in the coefficients of convergent hypergeometric series is easily observed:

$$ \begin{aligned} &{}_nF_{n-1}\left(\begin{array}{c} \alpha_1, \quad \ldots \quad, \alpha_n \\ \rho_1, \ldots, \rho_l, \beta_1, \ldots, \beta_{n-l-1} \end{array} \middle|\, t \right) \star {}_mF_{m-1}\left(\begin{array}{c} \rho_1, \ldots, \rho_l, \alpha'_1, \ldots, \alpha'_{m-l} \\ \gamma_1, \ldots, \gamma_{m-1} \end{array} \middle|\, t \right) \\ &\qquad = {}_{m+n-l}F_{m+n-l-1}\left(\begin{array}{c} \alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_{m-l} \\ \beta_1, \ldots, \beta_{n-l-1}, \gamma_1, \ldots, \gamma_{m-1}, 1 \end{array} \middle|\, t \right) \end{aligned} \tag{3.6} $$

with $\alpha_i, \alpha'_{i'} \neq \beta_j, \gamma_{j'}$. The Hadamard product is used in explicit formulas for certain integral convolutions. We have the following:

Lemma 3.2. *For a function $\omega(t)$ which is holomorphic on the disc of radius 1 about $t = 0$ and has the absolutely convergent series $\omega(t) = \sum_{k\geq 0} f_k t^k$ for*

$|t| < 1$ *and* $\alpha \in (0,1) \cap \mathbb{Q}$, *we have the following convolution formulas:*

$$\int_0^1 \frac{dv}{v^{1-\alpha}\,(1-v)^{\alpha}}\,\omega(tv) = \pi \csc(\pi\alpha) \sum_{n\geq 0} \frac{f_n\,(\alpha)_n}{n!}\,t^n = \pi \csc(\pi\alpha)\; {}_1F_0(\alpha|\,t) \star \omega(t), \tag{3.7}$$

and

$$\int_{-1}^1 \frac{dv}{\sqrt{1-v^2}}\,\omega(tv) = \pi \sum_{n\geq 0} \frac{f_{2n}\,\left(\frac{1}{2}\right)_n}{n!}\,t^{2n} = \pi\; {}_1F_0\left(\frac{1}{2}\middle|\,t^2\right) \star \omega(t). \tag{3.8}$$

Proof. The first identity easily follows from

$$\int_0^1 t^{a-1}(1-t)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where $\mathrm{Re}(a), \mathrm{Re}(b) > 0$, the formula $(1+z)^{-k} = \sum_{l\geq 0} \frac{\Gamma(l+k)}{\Gamma(k)\Gamma(l+1)}(-z)^l$, and the reflection formula $\Gamma(\alpha)\Gamma(1-\alpha) = \pi\csc(\pi\alpha)$. For the second equation we observe that

$$\int_{-1}^1 \frac{dv}{\sqrt{1-v^2}}\,\omega(tv) = \int_0^1 \frac{dw}{2\sqrt{w(1-w)}}\,\left(\omega(t\sqrt{w}) + \omega(-t\sqrt{w})\right). \tag{3.9}$$

□

The integral convolution in Equation (3.7) and (3.8) is also called Euler transform. That is, the Euler transform is an integral transform with parameter that relates (the holomorphic solution of) a Fuchsian differential equation of rank n with three regular singularities to a Fuchsian differential equation of rank $(n+1)$ with three regular singularities. We have the following:

Corollary 3.3. *In the situation above, we have*

$$\begin{aligned}
&{}_{n+1}F_n\left(\begin{matrix}\alpha_1 & \dots & \alpha_{n+1}\\ 1 & \dots & 1\end{matrix}\middle|\,t\right) = {}_1F_0(\alpha_1|\,t) \star {}_nF_{n-1}\left(\begin{matrix}\alpha_2 & \dots & \alpha_{n+1}\\ 1 & \dots & 1\end{matrix}\middle|\,t\right) && (3.10)\\
&= {}_1F_0(\alpha_1|\,t) \star \dots \star {}_1F_0(\alpha_{n+1}|\,t) && (3.11)\\
&= \left[\prod_{i=1}^n \frac{1}{\pi\csc(\pi\alpha_i)} \int_0^1 \frac{dz_i}{z_i^{1-\alpha_i}(1-z_i)^{\alpha_i}}\right](1 - t\,z_1\cdots z_{n-1})^{-\alpha_{n+1}}. && (3.12)
\end{aligned}$$

□

The Hadamard product in Equation (3.10) of the hypergeometric function ${}_nF_{n-1}$ with ${}_1F_0$ turns the holomorphic solution of

$$L_t^{(n)}((\alpha_2,\dots,\alpha_{n+1});(1,\dots,1))\,\omega(t)=0$$

into the holomorphic solution of $L_t^{(n+1)}((\alpha_1,\dots,\alpha_{n+1});(1,\dots,1))\,\tilde{\omega}(t)=0$. There is a corresponding notion of the Hadamard product (cf. [12, Def. 4.11]) for the differential operators involved: the Hadamard product of the differential operator $L_t^{(1)}(\alpha_1;)$ with the operator $L_t^{(n)}((\alpha_2,\dots,\alpha_{n+1});(1,\dots,1))$ yields the differential operator $L_t^{(n+1)}((\alpha_1,\dots,\alpha_{n+1});(1,\dots,1))$. In [12], this was denoted by $\mathcal{H}_{\alpha_1}(L_t^{(n)})=L_t^{(n+1)}$, and a corresponding operation, known as *middle Hadamard product*, was introduced for the monodromy tuple induced by a differential operator such that the monodromy tuple induced by $L_t^{(n+1)}$ becomes a sub-factor in the middle Hadamard product of the monodromy tuple induced by $L_t^{(n)}$. We make the following:

Definition 3.4. Differential operators are said to be of geometric origin if they are Picard-Fuchs operators annihilating the periods of a family of complex algebraic varieties. A monodromy tuple is of geometric origin, if it is induced by a differential operator of geometric origin.

Equation (3.11) decomposes the local system of

$$L_t^{(n+1)}((\alpha_1,\dots,\alpha_{n+1});(1,\dots,1))$$

into the convolution of $n+1$ local systems of rank one, each with a holomorphic solution of the type in Equation (3.5) with $\alpha=\alpha_i$. This is a special case of a general classification result by Katz [51] that applies to every linearly rigid local system; see also [12]. In fact, Katz proved that every linearly rigid local system is obtained as tensor products and convolutions of rank-one local systems associated with the holomorphic solution (3.5). More general, it is known that these operations on the level Fuchsian local systems and monodromy tuples preserve the geometric origin of an operator; see [26]. However, as we will prove in this article, such a decomposition into rank-one local systems is not necessarily meaningful in terms of geometry. For example, the classical rank-two local system for $L_t^{(2)}((\mu,1-\mu);(1))$ with $\mu\in\{\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$ is decomposed using Katz' procedure into two rank-one systems using the Hadamard product

$$(3.13)\qquad {}_2F_1\left(\begin{array}{c}\mu,1-\mu\\1\end{array}\middle|\,t\right)={}_1F_0(\,\mu|\,t)\star{}_1F_0(1-\mu|\,t)\,.$$

However, this decomposition is not the one to be used if one wants to relate a period of a zero-dimensional family of Calabi-Yau manifolds to a period of a family of elliptic curves. Instead one has to use the decomposition formula

$$ {}_2F_1\left(\begin{matrix} \mu, 1-\mu \\ 1 \end{matrix} \middle| t \right) = {}_2F_1\left(\begin{matrix} \mu, 1-\mu \\ \frac{1}{2} \end{matrix} \middle| t \right) \star {}_1F_0\left(\frac{1}{2} \middle| t \right). \tag{3.14} $$

Equation (3.14) then allows to build directly families of elliptic curves whose Picard-Fuchs operator is $L_t^{(2)}((\mu, 1-\mu); (1))$ for every $\mu \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ from a single geometric object, the family in Equation (6.1) with Picard-Fuchs operator $L_t^{(1)}(1/2;)$, using a generalized functional invariant which determines μ; see Lemma 6.1.

3.3. Rigid Calabi-Yau operators

Doran and Morgan classified in [34] all integral weight-three variations of Hodge structure which can underlie a family of Calabi-Yau threefolds over the thrice-punctured sphere $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ — subject to conditions on monodromy coming from mirror symmetry — through the irreducible monodromy representation generated by the local monodromies around the punctures of the base space. The monodromy representations turned out to be identical with the linearly rigid monodromy groups associated with the univariate generalized hypergeometric operators $L_t^{(4)}((\alpha_1, \ldots, \alpha_4); (1, \ldots, 1))$ with certain rational coefficients $\alpha_1, \ldots, \alpha_4$.

Each case was realized as a family of, possibly singular, Calabi-Yau threefolds constructed as hypersurfaces or complete intersections in a Gorenstein toric Fano variety and Calabi-Yau threefolds fibered by high rank K3's by Clingher et al. [21] — where a non-generic geometric transition was needed in one of the cases. The construction of the 14 cases lead to the following definition of a Calabi-Yau type differential operator (or *Calabi-Yau operator* for short) by Almkvist, van Enckevort, van Straten and Zudilin:

Definition 3.5. A rank-n Calabi-Yau operator is an irreducible Fuchsian differential operator $L_t^{(n)}$ of rank n with coefficients in $\mathbb{C}(t)$ and singular locus S (with only regular singular points), normalized to include $t = 0$, such that (1) the monodromy at $t = 0$ is maximally unipotent, (2) $L_t^{(n)}$ is self-adjoint, i.e., there is a function $h(t) \neq 0$ algebraic over $\mathbb{Q}(t)$ such that $L_t^{(n)} h(t) = (-1)^n h(t) L_t^{(n)\dagger}$ where $L_t^{(n)\dagger}$ denotes the adjoint of $L_t^{(n)}$, and (3) $L_t^{(n)}\, \omega(t) = 0$ has an N-integral holomorphic solution $\omega(t) = \sum_{k \geq 0} f_k t^k$ at $t = 0$, i.e., there exists $N \in \mathbb{N}$ such that $f_k N^k \in \mathbb{N}$ for all k.

Remark 3.6. Condition (1) is equivalent to all exponents of the Riemann symbol for $L_t^{(n)}$ at $t=0$ being zero. For a general linear, rank-n, Fuchsian differential operator in the variable t, given by

$$L_t^{(n)} = \partial^n + \sum_{i=0}^{n-1} a_i(t)\,\partial^i,$$

with $\partial = \frac{d}{dt}$ and suitable rational coefficient functions $a_i(t)$ for $1 \le i \le n-1$, the formal adjoint operator is

$$L_t^{(n)\,\dagger} = \partial^n + \sum_{i=0}^{n-1}(-1)^{n+i}\,\partial^i a_i(t).$$

The condition of being self-adjoint implies, as a necessary condition, that the function $h(t)$ satisfies the differential equation

$$h'(t) = -\frac{2}{n}\,a_{n-1}(t)\,h(t).$$

Condition (2) implies that for n even the differential Galois group of $L_t^{(n)}$ is contained in $\mathrm{Sp}(n,\mathbb{C})$. Condition (3) implies that the monodromy matrices at each singular points are quasi-unipotent.

For a monodromy r-tuple $T=(g_1,\dots,g_r)$ with $g_i \in G$ where G is a reductive complex algebraic group, we define the *rigidity index of T in G* as

$$i_G(T) = \sum_{i=1}^{r} \operatorname{codim} C_G(g_i) - 2\,\dim G + 2\,\dim Z_G, \tag{3.15}$$

where $C_G(g_i)$ denotes the centralizer of g_i in G, and Z_G denotes the center of G. Deligne and Katz gave the following criterion:

Proposition 3.7 ([51]). *For $G = \mathrm{GL}(n,\mathbb{C})$ the monodromy r-tuple $T = (g_1,\dots,g_r)$ is linearly rigid if and only if $i_G(T)=0$.*

Therefore, one can extend the notion of rigidity from $\mathrm{GL}(n,\mathbb{C})$ to any reductive complex algebraic group G by considering the monodromy tuples with $i_G(T)=0$. In particular, since elements of monodromy tuples induced by a fourth order Calabi-Yau differential operator lie in $\mathrm{Sp}(4,\mathbb{C})$, we investigate those Calabi-Yau operators inducing an $\mathrm{Sp}(4,\mathbb{C})$-rigid monodromy representation, or *symplectically rigid* for short.

Bogner and Reiter [12] proved that all $\mathrm{Sp}(4,\mathbb{C})$-rigid monodromy tuples consisting of quasi-unipotent elements can be constructed using tensor products, rational pullbacks and the middle convolution [12, Thm. 3.1] of rank-one local systems associated with the holomorphic solution of the type in Equation (3.5). Simpson had already classified in [71] all irreducible, $\mathrm{Sp}(4,\mathbb{C})$-rigid monodromy tuples with quasi-unipotent elements *and* one maximally unipotent matrix. It turns out [71, Thm. 4] that these tuples necessarily have three matrices, analogous to Picard-Fuchs operators of families of Calabi-Yau threefolds over the thrice-punctured sphere $\mathbb{P}^1\backslash\{0,1\infty\}$ — subject to conditions on monodromy coming from mirror symmetry. Moreover, Simpson divided up these tuples into four families, called the *hypergeometric*, *odd*, *even*, and *extra* family. Bogner and Reiter found that these tuples are realized as monodromy tuples of Fuchsian differential operators. We have the following:

Corollary 3.8 ([12]). *There are* 60 *rank-four Calabi-Yau operators with three singularities whose associated symplectically rigid monodromy representations are generated by tuples* $T=(g_0,g_1,g_\infty)$ *with quasi-unipotent elements, having one maximally unipotent matrix, and satisfy* $i_G(T)=0$ *for* $G=\mathrm{Sp}(4,\mathbb{C})$.

Remark 3.9. The 60 symplectically rigid Calabi-Yau operators with three singularities are found in the AESZ database [2]. Among them, 14 Calabi-Yau operators are univariate generalized hypergeometric operators $L_t^{(4)}\big((\alpha_1,\dots,\alpha_4);(1,\dots,1)\big)$ with certain rational coefficients $\alpha_1,\dots,\alpha_4$ determined by Doran and Morgan [34].

Remark 3.10. As we demonstrate in Section 10, our geometric twist construction can also produce families of Calabi-Yau threefolds whose Picard-Fuchs operators realize all monodromy tuples of low degree with four quasi-unipotent elements and one maximally unipotent matrix. On the other hand, there are families of Calabi-Yau threefolds whose Picard-Fuchs operators have no point of maximally unipotent monodromy and do not underlie variations of Hodge structure of type $(1,1,1,1)$ [37].

The decomposition of the $\mathrm{Sp}(4,\mathbb{C})$-rigid monodromy tuples by Bogner and Reiter suffers from the same problem the decomposition of linearly rigid systems by Katz does — we demonstrated that with Equation (3.13) versus Equation (3.14): whereas it does imply that the symplectically rigid operators are of geometric origin, the decomposition is not constructive in the sense that it produces families of Calabi-Yau manifolds whose Picard-Fuchs operators realize them. In contrast, our iterative twist construction in Section 5 will build from a single geometric object, a family that we will present in Equation (6.1), the families of Calabi-Yau varieties using a generalized functional invariant such that their Picard-Fuchs operators realize all 60 cases in Corollary 3.8.

3.3.1. The Yifan-Yang pullback. As explained above, among the 60 rank-four Calabi-Yau operators with three singularities and symplectically rigid monodromy, 14 Calabi-Yau operators belong to the univariate generalized hypergeometric operators. Another 14 rank-four Calabi-Yau operators are uniquely determined by the fact that they are of degree two in t and their exterior squares are the univariate rank-five hypergeometric operators $L_t^{(5)}\big((\alpha_1, \alpha_2, 1/2, \alpha_3, \alpha_4); (1, \dots, 1)\big)$ for certain rational coefficients $\alpha_1, \dots, \alpha_4$.

For a linear differential operator $L_t^{(n)}$ with linearly independent solutions $y_1(t), \dots, y_n(t)$, the *exterior square* is the linear differential operator of minimal rank with solutions $y_i(t)y_j'(t) - y_i'(t)y_j(t)$ for all $1 \leq i < j \leq n$. The exterior square of a general differential operator $L_t^{(4)}$ is a rank-six differential operator of the form $\partial^6 + \frac{1}{M(t)} \sum_{i=0}^{5} b_i(t)\, \partial^i$ where $M(t)$ is the right side of Equation (3.16). If $M(t) = 0$ for all t, then we say that the exterior square of $L_t^{(4)}$ is a rank-five operator.

For a general rank-four differential operator $L_t^{(4)} = \partial^4 + \sum_{i=0}^{3} a_i(t)\, \partial^i$ we have the following:

Lemma 3.11. *The following statements are equivalent:*

1) *The operator $L_t^{(4)}$ is self-adjoint.*
2) *The exterior square of $L_t^{(4)}$ is a rank-five operator.*
3) *The monodromy group of the operator $L_t^{(4)}$ is a discrete subgroup of* $\mathrm{Sp}(4, \mathbb{R})$.
4) *The following condition for the coefficients of $L_t^{(4)}$ holds*

$$\begin{aligned} 0 = 8\, a_1(t) - 8\frac{da_2(t)}{dt} + 4\frac{d^2 a_3(t)}{dt^2} \\ - 4\, a_2(t)\, a_3(t) + 6\, a_3(t)\, \frac{da_3(t)}{dt} + a_3(t)^3. \end{aligned} \tag{3.16}$$

Proof. The equivalence of (1), (2), (4) follows by an explicit computation. The equivalence with condition (3) was proved in [12]. □

Similarly, for a general rank-five differential operator

$$L_t^{(5)} = \partial^5 + \sum_{i=0}^{4} b_i(t)\,\partial^i$$

we have the following:

Lemma 3.12. *The following statements are equivalent:*

1) *The operator $L_t^{(5)}$ is self-adjoint.*
2) *The operator $L_t^{(5)}$ is the exterior square of a rank-four self-adjoint operator.*
3) *The monodromy group of the operator $L_t^{(5)}$ is a discrete subgroup of* $\mathrm{Sp}(4,\mathbb{R})$.
4) *The following two conditions for the coefficients of $L_t^{(5)}$ hold:*

$$\begin{aligned} b_2(t) = \frac{3}{2}\frac{db_3(t)}{dt} &+ \frac{3}{5}\,b_4(t)\,b_3(t) \\ &- \frac{d^2b_4(t)}{dt^2} - \frac{6}{5}b_4(t)\,\frac{db_4(t)}{dt} - \frac{4}{25}b_4(t)^3 \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} b_0(t) = {} & \frac{1}{5}\frac{d^4b_4(t)}{dt^4} - \frac{1}{4}\frac{d^3b_3(t)}{dt^3} + \frac{2}{5}b_4(t)\,\frac{d^3b_4(t)}{dt^3} - \frac{3}{10}b_4(t)\,\frac{d^2b_3(t)}{dt^2} \\ & + \left(\frac{8}{25}b_4(t)^2 + \frac{4}{5}\frac{db_4(t)}{dt} - \frac{1}{10}b_3(t)\right)\frac{d^2b_4(t)}{dt^2} + \frac{1}{2}\frac{db_1(t)}{dt} \\ & + \left(-\frac{3}{25}b_4(t)^2 - \frac{3}{10}\frac{db_4(t)}{dt}\right)\frac{db_3(t)}{dt} + \frac{12}{25}b_4(t)\,\left(\frac{db_4(t)}{dt}\right)^2 \\ & + \left(-\frac{3}{25}\,b_3(t)\,b_4(t) + \frac{16}{125}\,b_4(t)^3\right)\frac{db_4(t)}{dt} - \frac{2}{125}b_3(t)\,b_4(t)^3 \\ & + \frac{1}{5}\,b_1(t)\,b_4(t) + \frac{16}{3125}\,b_4(t)^5. \end{aligned} \tag{3.18}$$

Proof. The equivalence of (1), (2), (4) follows by an explicit computation. Yang and Zudilin [81] proved that $L_t^{(5)}$ has a projective monodromy group that is a discrete subgroup of $\mathrm{Sp}(4,\mathbb{R})$ if and only if $L_t^{(5)}$ satisfies conditions (3.17) and (3.18). In fact, we have the following identification with the polynomials p_1, p_2, and p_3 used in [81, Theorem 4]:

$$\begin{aligned}
\frac{p_1(t)}{t} &= \frac{1}{10} b_4(t) - \frac{1}{t},\\
\frac{p_2(t)}{t^2} &= \frac{1}{5} b_3(t) - \frac{1}{5}\frac{db_4(t)}{dt} - \frac{7}{100} b_4(t)^2,\\
\frac{p_3(t)}{t^4} &= \frac{1}{250} b_3(t)\, b_4(t)^2 - \frac{3}{10} b_4(t)\, \frac{db_3(t)}{dt} + \frac{1}{50} b_3(t)\, \frac{db_4(t)}{dt}\\
&\quad + \frac{17}{50}\left(\frac{db_4(t)}{dt}\right)^2 + \frac{2}{5}\frac{d^3 b_4(t)}{dt^3} + \frac{29}{125} b_4(t)^2 \frac{db_4(t)}{dt}\\
&\quad + \frac{14}{25} b_4(t)\, \frac{d^2 b_4(t)}{dt^2} + \frac{9}{1250} b_4(t)^4 - \frac{2}{25} b_3(t)^2\\
&\quad + \frac{1}{2} b_1(t) - \frac{9}{20}\frac{d^2 b_3(t)}{dt^2}.
\end{aligned} \tag{3.19}$$

□

We make the following:

Remark 3.13. The D-module associated with the differential operator $L_t^{(4)}$ underlies a variation of Hodge Structure $\mathcal{V}$ of rank four and weight three, corresponding to the standard representation of its Mumford-Tate group $\mathrm{Sp}(4,\mathbb{R})$. Since the exterior square of $L_t^{(4)}$ decomposes into a product of rank-five and rank-one operator, the exterior square of the D-module decomposes into a five-dimensional irreducible and a one-dimensional irreducible representation, and $\wedge^2\mathcal{V}$ decomposes accordingly. The five-dimensional sub-factor has weight six but level four, so the Tate twist $\wedge^2\mathcal{V}(1)$ has weight four and type $(1,1,1,1,1)$.

We obtain the following:

Corollary 3.14. *For a self-adjoint rank-five operator*

$$L_t^{(5)} = \partial^5 + \sum_{i=0}^{4} b_i(t)\, \partial^i,$$

a rank-four self-adjoint differential operator $L_t^{(4)}$ whose exterior square equals $L_t^{(5)}$ is given by

$$
\begin{aligned}
a_3(t) &= \frac{2}{5}\, b_4(t)\,,\\
a_2(t) &= -\frac{7\, b_4(t)^2}{50} - \frac{2}{5}\,\frac{d}{dt} b_4(t) + \frac{1}{2}\, b_3(t)\,,\\
a_1(t) &= -\frac{9\, b_4(t)^3}{250} - \frac{12\, b_4(t)\,\frac{d}{dt} b_4(t)}{25} + \frac{1}{10}\, b_4(t)\, b_3(t)\\
&\quad - \frac{3}{5}\,\frac{d^2}{dt^2} b_4(t) + \frac{1}{2}\,\frac{d}{dt} b_3(t)\,,\\
a_0(t) &= -\frac{2}{5}\,\frac{d^3}{dt^3} b_4(t) + \frac{3}{8}\,\frac{d^2}{dt^2} b_3(t) - \frac{23\, b_4(t)\,\frac{d^2}{dt^2} b_4(t)}{50}\\
&\quad + \frac{1}{5}\, b_4(t)\,\frac{d}{dt} b_3(t) - \frac{27\,\left(\frac{d}{dt} b_4(t)\right)^2}{100}\\
&\quad + \left(-\frac{18\, b_4(t)^2}{125} - \frac{1}{20}\, b_3(t)\right)\frac{d}{dt} b_4(t) - \frac{19\, b_4(t)^4}{10000}\\
&\quad - \frac{3\, b_4(t)^2\, b_3(t)}{200} + \frac{1}{16}\, b_3(t)^2 - \frac{1}{4}\, b_1(t)\,.
\end{aligned} \tag{3.20}
$$

Proof. The proof follows from an explicit computation using Lemmas 3.11 and 3.12. □

For a self-adjoint rank-five operator $L_t^{(5)}$ satisfying conditions (3.17) and (3.18), we denote the rank-four, self-adjoint, linear differential operator $L_t^{(4)}$ in Corollary 3.14 by $L_t^{(4)} = \vee_2 L_t^{(5)}$. Following [1], the latter is also called the *Yifan-Yang pullback* of $L_t^{(5)}$. The following is easy to check:

Lemma 3.15. *The rank-five operator $L_t^{(5)}$ satisfies conditions (3.17) and (3.18) if and only if for any algebraic function $g(t)$ the operator*

$$L_t^{(5),\,\langle 2g(t)\rangle} := e^{2\,g(t)}\, L_t^{(5)}\, e^{-2\,g(t)}$$

does. Moreover, the operator $\vee_2 L_t^{(5),\,\langle 2g(t)\rangle}$ coincides with $L_t^{(4),\,\langle g(t)\rangle} := e^{g(t)}\, L_t^{(4)}\, e^{-g(t)}$.

Proof. The proof follows by an explicit computation. □

We then have the following:

Proposition 3.16. *The hypergeometric operator*

$$L_t^{(5)}\big((\alpha_1,\ldots,\alpha_5);(1,\ldots,1)\big)$$

with $\alpha_1 = 1-\alpha_5$, $\alpha_2 = 1-\alpha_4$, $\alpha_3 = \frac{1}{2}$, *and* $\alpha_1 = p, \alpha_2 = q$, $p,q \in (0,1)\cap\mathbb{Q}$ *is a rank-five, self-adjoint differential operator whose Yifan-Yang pullback* $L_t^{(4),\,\langle g(t)\rangle}$ *is given by*

$$\begin{aligned}
\theta^4 - \frac{1}{4}\,t\,\Big(&8\,\theta^4 + 16\,\theta^3 - 2\,(p^2+q^2-p-q-9)\,\theta^2 \\
&- 2\,(p^2+q^2-p-q-5)\,\theta + 2 + p + q \\
&- pq - p^2 - q^2 + p^2q + p\,q^2 + p^2q^2\Big) \\
+ \frac{1}{16}\,t^2\, &(2\,\theta+2+p-q) \\
&\times (2\,\theta+1+p+q)\,(2\,\theta+2-p+q)\,(2\,\theta+3-p-q)\,,
\end{aligned} \tag{3.21}$$

with $\exp(-g(t)) = \sqrt[4]{t^2\,(t-1)^3}$. *In particular,* $\exp(-g(t)) = \sqrt[4]{t^2\,(t-1)^3}$ *is the unique non-trivial function (up to scaling) that minimizes the degree (in* t*) of* $L_t^{(4),\,\langle g(t)\rangle}$.

Proof. Using Lemma 3.12 and 3.15, the proof follows from an explicit computation. □

4. First examples from quadratic twists

The details of the proof of Theorem 2.1 are quite involved, but the basic idea is simple and present in the following series of examples for our iterative construction: One starts with a family of pairs of points and produces, by a quadratic twist, a family of elliptic curves whose total space is an extremal rational surface. One continues by constructing a family of Jacobian elliptic K3 surfaces of Picard rank 19, and in turn, a family of Calabi-Yau threefolds with $h^{2,1} = 1$ from the family of Jacobian elliptic K3 surfaces by two more quadratic twists. If one allows for the Picard rank of the K3 surfaces in the intermediate step to drop from 19 to 18, one can also construct a second, closely related family of Calabi-Yau threefolds with $h^{2,1} = 1$. The Picard-Fuchs operators for the two families of threefolds realize two simple rank-four and degree-one symplectically rigid Calabi-Yau operators in the AESZ database [2].

The construction of these examples was motivated by physics, in particular the embedding of gauge theory into F-theory [55–57]. An interpretation from the point of view of variations of Hodge structure might be provided by methods in [24, 40, 41]. The idea of using a quadratic twist to construct an isomorphism between different types of moduli problems also appeared in the work of Besser and Livné in [8].

4.1. A sequence of quadratic twists

We start with a pencil of ‘dimension zero’ Calabi-Yau manifolds which consists of the ramified family of pairs of points $\pm y_0$ given by

$$y_0^2 = 1 - t \tag{4.1}$$

for $t \in \mathbb{C}$. For this family, we define $\Sigma_0(t)$ to be the point t, take the branch cut branch cut along $\{\, t \,|\, 1 \leq t \leq \infty \}$, and consider the holomorphic 0-form $1/y_0$ and the period

$$\omega(t) = \int_{\Sigma_0(t)} \frac{1}{y_0} = \frac{2}{y_0} = 2 \; {}_1F_0\left(\left. \frac{1}{2} \right| t \right) = 2 \left(1-t\right)^{-\frac{1}{2}}, \tag{4.2}$$

which is a solution of the hypergeometric differential equation $L_t^{(1)}(\frac{1}{2};)\,\omega(t) = 0$.

To obtain from the family of points (4.1) a pencil of elliptic curves, one promotes the family parameter t to an additional complex variable x and carries out the quadratic twist $y_0^2 \mapsto -y_1^2/[x(x-t)]$ in Equation (4.1). This yields the classical Legendre pencil of elliptic curves given by

$$y_1^2 = x\,(x-1)\,(x-t), \tag{4.3}$$

where $t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\pi : X_t \to \mathbb{P}^1$ is the corresponding projection. The polarized Hodge filtration on $H_{\mathbb{Z}} = H^1(X_t, \mathbb{Z})$ of the elliptic curve X_t has two steps, F^0 and F^1 defining a pure Hodge structure of weight one and type $(1,1)$ in

$$\left\{ H_{\mathbb{Z}}, Q, F^1 \subset F^0 = H_{\mathbb{Z}} \otimes \mathbb{C} \right\}.$$

Here, F^0 is the entire cohomology group, and F^1 is $H^{1,0}(X_t)$, the one-dimensional space of holomorphic harmonic one-forms. The polarization Q is the natural non-degenerate, integer, bilinear form on $H_{\mathbb{Z}}$ derived from the cup product and varies holomorphically. The homology group of the elliptic curve is free of rank two, and the periods of dx/y_1 satisfy a second-order differential equation. In fact, Equation (4.3) defines a double covering of $\mathbb{P}^1$ branched at the four points $x = 0, 1, t, \infty$. We cut the Riemann sphere from

0 to 1 and from t to ∞. The two cuts are opened up into two ovals, and the two y-sheets are glued with opposite orientations to obtain an elliptic curve. The A-cycle $\Sigma_1(t)$ projects onto the closed cycle encircling the branching points at $x = 0$ and $x = t$. Then, flattening out the cycle, we obtain

$$\omega(t) = \oint_{\Sigma_1(t)} \frac{dx}{y_1} = 2 \int_0^t \frac{dx}{\sqrt{x\,(x-1)\,(x-t)}} \\ = 2 \int_0^1 \frac{dx}{\sqrt{x\,(1-x)\,(1-tx)}}, \tag{4.4}$$

that is, the Euler integral representation of $(2\pi)\,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1|t)$ and satisfies the differential equation $L_t^{(2)}((\frac{1}{2}, \frac{1}{2}); (1))\,\omega(t) = 0$. Alternatively, we can take a Pochhammer contour $C_{\{0,1\}}$ around $x = 0$ and $x = 1$ to obtain

$$\oint_{C_{\{0,1\}}} \frac{dx}{\sqrt{x\,(1-x)\,(1-tx)}} = \pi \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)_n t^n}{n!} \oint_{C_{\{0,1\}}} dx\, x^{n-\frac{1}{2}} (1-x)^{-\frac{1}{2}} \\ = 4\pi^2 {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t \right), \tag{4.5}$$

where we used Equation (4.13).

A fundamental observation is the following: the quadratic twist has turned the zero-dimensional family (4.1) into the family of elliptic curves (4.3); the holomorphic period for the family of elliptic curves is the Hadamard product of the function ${}_1F_0$ — which accounts for the quadratic twist — and the holomorphic period (3.5) of the zero-dimensional family. That is, for $|t| < 1$ we obtain

$$\oint_{\Sigma_1(t)} \frac{dx}{y_1} = 2\pi\, {}_1F_0\left(\left. \frac{1}{2} \right| t \right) \star {}_1F_0\left(\left. \frac{1}{2} \right| t \right) = 2\pi\, {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t \right), \tag{4.6}$$

where we used Equation (3.6).

To obtain from the family of elliptic curves in Equation (4.3) a family of K3 surfaces, one again promotes the parameter t to an additional complex variable u and carries out the quadratic twist $y_1^2 \mapsto y_2^2/[u(u-t)]$ in Equation (4.3). This yields the twisted Legendre pencil given by

$$y_2^2 = x\,(x-1)\,(x-u)\,u\,(u-t). \tag{4.7}$$

Equation (4.14) defines the Néron model for a family of elliptically fibered K3 surfaces X_t of Picard rank 19 with section over $\mathbb{P}^1 \ni [u : 1]$. Hoyt [48] and Endo [36] extended arguments of Shimura and Eichler [35] to show that on the parabolic cohomology group $H_{\mathbb{Z}} \cong \mathbb{Z}^3$ associated with $\pi : X_t \to \mathbb{P}^1$ there is a natural polarized Hodge filtration given by

$$\left\{ H_{\mathbb{Z}}, Q, F^2 \subset F^1 \subset H_{\mathbb{Z}} \otimes \mathbb{C} \right\}.$$

Here, $H_{\mathbb{Z}} \otimes \mathbb{C}$ consists of cohomology classes spanned by suitable meromorphic differentials of the second kind, and Q is a non degenerate $\mathbb{Q}$-valued bilinear form determined by period relations of the holomorphic two-form $du \wedge dx/y_2$ [49]. The period point lies on the cone $Q = 0$. In the situation of Equation (4.7), it follows $Q = 2z_1^2 + 2z_2^2 - 2z_3^2$, the local system $R^2\pi_*\mathbb{C}_X$ of middle cohomology is irreducible, and the cohomology group $H_{\mathbb{Z}}$ carries a pure Hodge structure of weight two and type $(1, 1, 1)$.

A basis of transcendental cycles is constructed from cycles in the elliptic fiber and carefully chosen curves in the base connecting the cusps $0, 1, t, \infty$ [48]. As in Shimura [69], for continuously varying families of closed one-cycles $\Sigma_1(u), \check{\Sigma}_1(u)$, that form bases of the first homology of the fiber, the expression

$$\int_t^u du \begin{pmatrix} \int_{\Sigma_1(u)} \frac{dx}{y_2} \\ \int_{\check{\Sigma}_1(u)} \frac{dx}{y_2} \end{pmatrix} \tag{4.8}$$

defines a vector-valued holomorphic function that converges as u approaches the cusps at $u = 0, 1, \infty$. It was shown by Cox and Zucker [23] that the components of (4.8) are $\mathbb{Q}$-linear combinations of periods of associated meromorphic two-forms on X that are of the second kind and holomorphic on singular fibers; they represent generalized cusp forms of weight three associated with Equation (4.7). In particular, periods of the holomorphic two-form $du \wedge dx/y_2$ satisfy the third-order differential equation $L_t^{(3)}((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}); (1, 1))\, \omega(t) = 0$. Since the singular fiber over $u = 0$ and $u = t$ is of Kodaira-type I_2^* and I_0^*, respectively — the latter having monodromy $-\mathbb{I}$ independent of the chosen homological invariant — there is a unique A-cycle that is transformed into itself as a path encircles the cusps at $u = 0$ or $u = t$. One obtains a transcendental two-cycle $\Sigma_2(t)$ on X_t by tracing out this A-cycle $\Sigma_1(u)$ in the fiber over the line segment between the cusps at $u = 0$ and $u = t$ in the base. Then, one integrates the holomorphic two-form $du \wedge dx/y_2$ over the two-cycle $\Sigma_2(t)$ to obtain

$$\oiint_{\Sigma_2(t)} du \wedge \frac{dx}{y_2} = 2\pi \int_0^t \frac{du}{\sqrt{u\,(u-t)}}\; {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right| u \right) = 2i\pi^2\, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right| t \right), \tag{4.9}$$

where we used Equation (3.6) and Lemma 3.2.

To obtain from the family of K3 surfaces (4.7) a family of Calabi-Yau threefolds, one promotes the parameter t to an additional complex variable v and carries out yet another quadratic twist $y_2^2 \mapsto y_3^2/[v(v-t)]$ in Equation (4.7). One obtains the family

$$y_3^2 = x\,(x-1)\,(x-u)\,u\,(u-v)\,v\,(v-t). \tag{4.10}$$

This family constitutes a pencil of elliptically fibered Calabi-Yau threefolds, denoted by $\pi : X_t \to \mathbb{P}^1$. Each member X_t of the family is fibered by K3 surfaces of Picard rank 19 over $\mathbb{P}^1$ with affine coordinate u. As a consequence, the local system $R^3\pi_*\mathbb{C}_X$ of middle cohomology is irreducible and the transcendental cohomology group $H_\mathbb{Z}$ carries a pure Hodge structure of weight three and type $(1,1,1,1)$.

The natural Hodge structure on the parabolic cohomology group of X_t can be described in terms of periods of the holomorphic three-form $dv \wedge du \wedge dx/y_3$. A transcendental three-cycle $\Sigma_3(t)$ on each threefold X_t is obtained as Lefschetz thimble by tracing out the two-cycle $\Sigma_2(v)$ in the K3 fiber over the line segment between the cusps $v=0$ and $v=t$. If one integrates the holomorphic three-form $dv \wedge du \wedge dx/y_2$ over the cycle $\Sigma_3(t)$, one obtains for the holomorphic period

$$\oiiint_{\Sigma_3(t)} dv \wedge du \wedge \frac{dx}{y_3} = -2\pi^3\, {}_4F_3\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1,1,1 \end{array} \right| t \right), \tag{4.11}$$

where we have used Equation (3.6) and Lemma 3.2. The period is annihilated by the rank-four and degree-one Picard-Fuchs operator

$$L_t^{(4)}\left(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right); (1,1,1) \right) = \theta^4 - t\left(\theta + \frac{1}{2}\right)^4. \tag{4.12}$$

The Picard-Fuchs operator (4.12) is one of the 14 original Calabi-Yau operators mentioned in the introduction and was labelled "(3)" in the AESZ database [2]. We make the following:

Remark 4.1. For the construction of the cycles Σ_n we employed two different strategies, namely the use of either Pochhammer cycles or Lefschetz thimbles. For $n=1$, we used a family of A-cycles equivalent to a Pochhammer contour. For $n=2$ and $n=3$, we used a Lefschetz thimbles to form transcendental cycles with a non-trivial $(n-1)$-cycle in the elliptic or K3 fiber over a line segment between cusps. The reason can be traced back to the equivalent ways of defining Euler's beta function. The beta function is $\mathrm{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(t-1)^{\beta-1}dt$ for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. The beta function is then analytically continued for all values of α and β. This is achieved by converting the Euler integral into an integral over a Pochhammer contour $C_{\{0,1\}}$ around $t=0$ and $t=1$ to obtain

$$(1 - e^{2\pi i\alpha})(1 - e^{2\pi i\beta})\,\mathrm{B}(\alpha,\beta) = \oint_{C_{\{0,1\}}} t^{\alpha-1}(t-1)^{\beta-1}dt. \tag{4.13}$$

In the context of our iterative construction (see Section 5) it turns out that contour integration is easier to describe in the general setting.

4.2. Closely related examples

To obtain from the family of elliptic curves in Equation (4.3) a family of K3 surfaces of Picard rank 18 instead of Picard rank 19, one again promotes the family parameter t to an additional complex variable u, but carries out the quadratic twist $y_1^2 \mapsto y_2^2/[u^2-t^2]$ in Equation (4.3). One obtains a twisted Legendre pencil given by

$$y_2^2 = x\,(x-1)\,(x-u)\,(u^2-t^2). \tag{4.14}$$

The equation defines the Weierstrass model for a family of elliptically fibered K3 surfaces X_t of Picard rank 18 with section over $\mathbb{P}^1$, denoted by $\pi : X_t \to \mathbb{P}^1$. We define a transcendental two-cycle $\hat{\Sigma}_2(t)$ by by tracing out the A-cycle $\Sigma_1(u)$ in the fiber over the line segment between the cusps at $u=-t$ and $u=t$ (avoiding $u=0$ by using a small arc). Using Lemma 3.2 we obtain for the period integral

$$\begin{aligned} \oiint_{\hat{\Sigma}_2(t)} du \wedge \frac{dx}{y_2} &= 2\pi \int_{-t}^{t} \frac{du}{\sqrt{u^2-t^2}}\; {}_2F_1\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\right| u\right) \\ &= 2i\pi^2\, {}_1F_0\left(\left.\frac{1}{2}\right| t^2\right) \star {}_2F_1\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\right| t\right). \end{aligned} \tag{4.15}$$

The quadratic twist has turned Equation (4.3) into (4.14) and, similarly, the holomorphic period for the family of K3 surfaces is the Hadamard product of the function ${}_1F_0(t^2)$ — which accounts for the modified quadratic twist — and the holomorphic period (4.6). The Hadamard product can be evaluated explicitly and yields

$$ {}_1F_0\left(\frac{1}{2}\middle| t^2\right) \star {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle| t\right) = {}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,\frac{1}{2}\end{matrix}\middle| t^2\right). $$

The period (4.15) annihilated by the rank-four and degree-two Picard-Fuchs operator

$$ \theta^3\,(\theta-1) - \frac{t^2}{16}\,(2\,\theta+3)^2\,(1+2\,\theta)^2\,. $$

To obtain from the family of K3 surfaces in Equation (4.14) a family of Calabi-Yau threefolds, one promotes the family parameter t to an additional complex variable v and carries out a quadratic twist $y_2^2 \mapsto y_3^2/[v^2-t^2]$. One obtains the family

$$ y_3^2 = x\,(x-1)\,(x-u)\,(u^2-v^2)\,(v^2-t^2), \tag{4.16} $$

that constitutes a non-trivial pencil of elliptically fibered Calabi-Yau threefolds, denoted by $\pi : X_t \to \mathbb{P}^1$. As before, the natural Hodge structure on the parabolic cohomology group of X_t can be described in terms of periods and period relations of the holomorphic three-form $dv \wedge du \wedge dx/y_3$. We construct a transcendental three-cycle $\hat{\Sigma}_3(t)$ on X_t as Lefschetz thimble by tracing out the cycle $\hat{\Sigma}_2(v)$ in the K3 fiber over the line segment between the cusps $v=-t$ and $v=t$ (avoiding $v=0$ by using a portion of a small circle). If one integrates the holomorphic three-form $dv \wedge du \wedge dx/y_2$ over the cycle $\hat{\Sigma}_3(t)$, one obtains for the holomorphic period

$$ \oiiint_{\hat{\Sigma}_3(t)} dv \wedge du \wedge \frac{dx}{y_3} = -2\pi^3\,{}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{matrix}\middle| t^2\right), \tag{4.17} $$

where we have applied the cancellation formula (3.6) to conclude

$$ {}_1F_0\left(\frac{1}{2}\middle| t^2\right) \star {}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,\frac{1}{2}\end{matrix}\middle| t^2\right) = {}_4F_3\left(\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{matrix}\middle| t^2\right). $$

The period (4.17) is annihilated by the fourth-order and degree-two Picard-Fuchs operator (rescaled by 2^4)

$$ L^{(4)}_{t^2}\left(\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right);(1,1,1)\right) = \theta^4 - \frac{t^2}{16}\,(2\,\theta+1)^2\,(2\,\theta+3)^2\,. \tag{4.18} $$

The Picard-Fuchs operator (4.18) is another of the 14 original Calabi-Yau operators mentioned in the introduction and was labelled "(10)" in the AESZ database [2].

5. The twist construction

In this section we describe the construction of twisted families with generalized functional invariant in full generality, including several modified variants needed later, and the computation of period integrals.

5.1. Elliptic fibrations

In this section we will recall facts about the geometry of elliptically fibered varieties; we refer to the papers [16, 27, 62] for details. We also adopt the definition of terminal, canonical, and log-terminal singularity of a projective variety from aforementioned articles.

We define an elliptic fibration $\pi : X \to S$ to be a proper surjective morphism with connected fibers between normal complex varieties X and S whose general fibers are nonsingular elliptic curves. We assume that π is smooth over an open subset S^0 whose complement is a divisor with only normal crossings. Then, the local system $H^i_0 := R^i\pi_*\mathbb{Z}_X \mid_{S^0}$ forms a variation of Hodge structure over S^0. There is a canonical bundle formula for the elliptic fibration: with the fundamental line bundle denoted by $\mathcal{L} := \big(R^1\pi_*\mathcal{O}_X\big)^{-1}$, the canonical bundles $\boldsymbol{\omega}_X := \wedge^{\mathrm{top}}T^*X$ and $\boldsymbol{\omega}_S := \wedge^{\mathrm{top}}T^*S$ are related by

$$\boldsymbol{\omega}_X \cong \pi^*\Big(\boldsymbol{\omega}_S \otimes \mathcal{L}\Big) \otimes \mathcal{O}_X(D), \tag{5.1}$$

where D is a certain effective divisor on X that only depends on divisors of S over which π has multiple fibers and divisors of X that give (-1)-curves in the fibers of π. The existence of a section for the elliptic fibration $\pi : X \to S$ prevents the presence of multiple fibers. And the presence of (-1)-curves in the fibers is avoided by imposing a minimality criterion. In the case of an elliptic surface, we assume that the fibration is relatively minimal, i.e., that there are no (-1)-curves in the fibers of π. In the case of an elliptic threefold, we assume that no contraction of a surface in X is compatible with the fibration. X is a Calabi-Yau manifold if $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ and $\boldsymbol{\omega}_X \cong \mathcal{O}_X$. It is known [39] that for any elliptic fibration on a Calabi-Yau threefold, the base surface can have at worst log-terminal orbifold singularities. In this article we will take the base surface S always to be a blow-up of $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_k$.

It is a well-known that for any elliptic fibration $\pi : X \to S$ with section $\sigma : S \to X$, there always exists a Weierstrass model W over S, i.e., there is a complex variety W and a proper flat surjective morphism $p : W \to S$ with canonical section whose fibers are irreducible cubic curves in $\mathbb{P}^2$ together with a birational map from X to W that maps σ to the canonical section of the Weierstrass model. The map from X to W blows down all components of fibers which do not intersect $\sigma(S)$. It is also known that for a relatively minimal elliptic fibration with section, the morphism on the Weierstrass model is in fact a resolution of the singularities of W.

5.2. Weierstrass models

Let $\mathcal{L}$ be a line bundle on S, and g_2 and g_3 sections of $\mathcal{L}^4$ and $\mathcal{L}^6$, respectively, such that the discriminant $\Delta = g_2^3 - 27\, g_3^2$ is a section of $\mathcal{L}^{12}$ not identically zero. Define $\mathbf{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ and let $p : \mathbf{P} \to S$ be the natural projection and $\mathcal{O}_{\mathbf{P}}(1)$ be the tautological line bundle. We denote by x, y, and z the sections of $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^2$, $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^3$, and $\mathcal{O}_{\mathbf{P}}(1)$, respectively, which correspond to the natural injections of $\mathcal{L}^2$, $\mathcal{L}^3$, and $\mathcal{O}_S$ into $\pi_* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$. We denote by W the projective variety in $\mathbf{P}$ defined by the equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3. \tag{5.2}$$

Its canonical section $\sigma : S \to W$ is given by the point $[x : y : z] = [0 : 1 : 0]$ such that $\Sigma := \sigma(S) \subset W$ is a Cartier divisor on W, and its normal bundle is isomorphic to the fundamental line bundle by $p_* \mathcal{O}_{\mathbf{P}}\big(-\Sigma\big) \cong \mathcal{L}$. It then follows that W is normal if S is normal; and W is Gorenstein if S is, and the formula (5.1) for the dualizing sheaf reduces to

$$\boldsymbol{\omega}_W = \pi^* \Big(\boldsymbol{\omega}_S \otimes \mathcal{L}\Big). \tag{5.3}$$

Thus, the total space is a Calabi-Yau variety if and only if the line bundle $\mathcal{L}$ is the anti-canonical bundle of the base, i.e., $\mathcal{L} = \boldsymbol{\omega}_S^{-1} = \mathcal{O}_S(-K_S)$. For the Weierstrass model $p : W \to S$ of an elliptic fibration $\pi : X \to S$ with section we can compare the discriminant locus $\Delta(\pi)$, i.e., the points over which X_p is singular, with the vanishing locus of $\Delta(p) = g_2^3 - 27\, g_3^2$. We then have $\Delta(p) \subset \Delta(\pi)$. In fact, the morphism from X to W is a resolution of singularities if and only if $\Delta(p) = \Delta(\pi)$ in which case it is also a small resolution.

We call a Weierstrass model minimal if there is no prime divisor D on S such that $\mathrm{div}(g_2) \geq 4\,D$ and $\mathrm{div}(g_3) \geq 6\,D$. We call two minimal Weierstrass models that are smooth over an open subset $S_0 \subset S$ equivalent if there is an isomorphism of the Weierstrass models over S_0 which preserves the canonical sections. Every Weierstrass fibration is birationally isomorphic to a minimal Weierstrass fibration. A criterion for W having only rational singularities can then be stated as follows: If the reduced discriminant divisor $(\Delta)_{\mathrm{red}}$ has only normal crossings, then W has only rational Gorenstein singularities if and only if the Weierstrass model is minimal.

5.3. Families of Weierstrass models

We now turn to projective families of Calabi-Yau n-folds over $B = \mathbb{P}^1\backslash\{0, 1, \infty\}$ which we denote by $\pi : X \to B$. Each element $X_t = \pi^{-1}(t)$ of the family is a compact complex n-fold with trivial canonical bundle $\boldsymbol{\omega}_{X_t} \cong \mathcal{O}_{X_t}$ and is assumed to be elliptically fibered with section over a fixed normal complex variety S. Such a family is described as a family of minimal normal Weierstrass models. That is, each complex n-fold $\pi_t : X_t \to S$ is given as a minimal Weierstrass model $p_t : W_t \to S$. For each $t \in \mathbb{P}^1\backslash\{0, 1, \infty\}$ and the affine coordinate chart $u = (u_1, \ldots, u_{n-1}) \in \mathbb{C}^{n-1} \subset S$, the Weierstrass model W_t has the form

$$y^2 z = 4x^3 - g_2\,(t, u)\; xz^2 - g_3\,(t, u)\; z^3. \tag{5.4}$$

There is a holomorphic sub-bundle $\mathcal{H} \to B$ of the vector bundle $V = R^n \pi_* \mathbb{C}_X \to B$ whose fibers are $H^0(\boldsymbol{\omega}_{X_t}) \subset H^n(X_t, \mathbb{C})$. The vector bundle $\mathcal{V} = V \otimes \mathcal{O}_B$ carries a canonical flat connection ∇, called the the *Gauss-Manin connection* [42–45]. Representing the family X_t as the family of Weierstrass models W_t determines an explicit meromorphic section $\eta \in \Gamma(B, \mathcal{V})$ such that such that $\eta_t \in H^n(X_t, \mathbb{C})$ can be represented by the closed differential form

$$\eta_t = du_1 \wedge \cdots \wedge du_{n-1} \wedge \frac{dx}{y}, \tag{5.5}$$

in the affine chart $z = 1$ on W_t. A local parallel section Σ of the dual bundle $\mathcal{H}^*$ for some open $U \subset B$ is represented by a closed transcendental n-cycle $\Sigma(t)$ on each fiber X_t with $t \in U$. The *period sheaf* $\Pi(\mathcal{H}, \eta) \to B$ is the sheaf whose stalks are generated by local analytic functions obtained by paring the global section η with local parallel sections of $\mathcal{H}^*$, i.e., of the form

$$t \in U \mapsto \omega(t) = \langle \Sigma, \eta \rangle = \oint_{\Sigma(t)} \eta_t, \tag{5.6}$$

for open $U \subset B$ and the fiberwise Poincaré pairing $\langle ., . \rangle$. We call the local non-vanishing analytic function $\omega(t)$ a *period integral* or *period over* $\Sigma(t)$.

The Picard-Fuchs differential equation is then obtained as follows: we fix a meromorphic vector field $\frac{d}{dt}$ on the curve B for the choice of affine coordinate t on $\mathbb{P}^1$. The vector field $\frac{d}{dt}$ induces a covariant derivative operator $\nabla_{d/dt}$ on $\mathcal{V}$. Since $\mathcal{V}$ has rank n, the meromorphic section η and its derivatives $\nabla^i_{d/dt}\eta$ for $1 \leq 1 \leq n$ must be linearly dependent over the field of meromorphic functions on B, and there is a relation

$$\sum_{i=0}^{m} a_i(t)\, \nabla^i_{d/dt}\eta = 0,$$

where $m \leq n$ and we always normalize to have $a_m = 1$. Since ∇ vanishes on the parallel section Σ of $\mathcal{H}^*$, it follows that the period $\omega(t)$ satisfies the differential equation

$$\frac{d^m}{dt^m}\omega(t) + \sum_{i=0}^{m-1} a_i(t)\, \frac{d^i}{dt^i}\omega(t) = 0.$$

5.4. Twisted family of Weierstrass models

We define a *generalized functional invariant* to be a triple (i, j, α) with $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq 6$ and $\alpha \in \{\frac{1}{2}, 1\}$. The general notion of generalized functional invariant was first introduced in [30]. A generalized functional invariant defines a family of ramified covering maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $i + j$ given by

$$[v_0 : v_1] \mapsto \left[c_{ij} v_1^{i+j} \tilde{t} \ : v_0^i (v_0 + v_1)^j\right], \tag{5.7}$$

which is totally ramified over 0, ramified to degrees i and j over ∞, and has a simple ramification point over $\tilde{t}$.

For a family $\pi : X \to B$ with Weierstrass model (5.4) such that

$$0 \leq \deg_t(g_2) \leq \min\left(\frac{4}{i}, \frac{4\alpha}{j}\right), \quad 0 < \deg_t(g_3) \leq \min\left(\frac{6}{i}, \frac{6\alpha}{j}\right), \tag{5.8}$$

we construct a new family $\tilde{\pi} : \tilde{X} \to B$ such that each element $\tilde{X}_{\tilde{t}} = \tilde{\pi}^{-1}(\tilde{t})$ is a compact complex $(n+1)$-fold and also elliptically fibered with section over $\mathbb{P}^1 \times S$. We call this new family the *twisted family with generalized functional invariant* (i, j, α) *of* $X \to B$. For $\tilde{t} \in B$, there is a local coordinate chart $\{[v_0 : v_1], (u_1, \ldots, u_{n-1}) \in \mathbb{P}^1 \times S\}$ such that the Weierstrass model $\tilde{W}_{\tilde{t}}$ for $\tilde{X}_{\tilde{t}}$ is given by

$$
\begin{aligned}
\tilde{y}^2 \tilde{z} = 4\,\tilde{x}^3 &- g_2\left(\frac{c_{ij}\tilde{t}v_1^{i+j}}{v_0^i(v_0+v_1)^j},\, u\right) v_0^4 v_1^{4-4\alpha}(v_0+v_1)^{4\alpha}\tilde{x}\tilde{z}^2 \\
&- g_3\left(\frac{c_{ij}\tilde{t}v_1^{i+j}}{v_0^i(v_0+v_1)^j},\, u\right) v_0^6 v_1^{6-6\alpha}(v_0+v_1)^{6\alpha}\,\tilde{z}^3
\end{aligned}
\tag{5.9}
$$

with $c_{ij} = (-1)^i\, i^i\, j^j/(i+j)^{i+j}$. The Weierstrass model (5.9) is smooth over the open subset $D^0 \times S^0$ where D^0 is the complement in $\mathbb{P}^1$ of the curves $v_0 = 0$, $v_1 = 0$, $v_0 + v_1 = 0$, and $c_{ij}\tilde{t}v_1^{i+j} - v_0^i(v_0+v_1)^j = 0$. Conditions (5.8) ensure that the Weierstrass model is minimal and normal if the original Weierstrass model (5.4) is. Over the new base $\tilde{S} = \mathbb{P}^1 \times S$, the fundamental line bundle and canonical bundle are given by

$$
\tilde{\mathcal{L}} = \mathcal{M} \boxtimes \mathcal{L}, \quad \boldsymbol{\omega}_{\tilde{S}} = \boldsymbol{\omega}_{\mathbb{P}^1} \boxtimes \boldsymbol{\omega}_S. \tag{5.10}
$$

Here the external tensor product $\mathcal{M} \boxtimes \mathcal{L} = \mathrm{pr}_1^*\,\mathcal{M} \otimes \mathrm{pr}_2^*\,\mathcal{L}$ denotes the tensor product on $\tilde{S} = \mathbb{P}^1 \times S$ of the two pullback bundles to $\tilde{S}$, along the canonical projection maps $\mathrm{pr}_1 : \tilde{S} \to \mathbb{P}^1$ and $\mathrm{pr}_2 : \tilde{S} \to S$. Two $\mathbb{C}^*$-group actions, acting on Equation (5.9), are given by assigning weights to the defining variables as listed in Table 1 where *deg* denotes the total weight of Equation (5.9) and *sum* denotes the sum of weights of the defining variables. Since the total weight equals the sum of weights of the variables, we have $\mathcal{M} \otimes \boldsymbol{\omega}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}$. Therefore, the Calabi-Yau condition $\tilde{\mathcal{L}} = \boldsymbol{\omega}_{\tilde{S}}^{-1}$ is satisfied for the new family $\tilde{\pi} : \tilde{X} \to B$ such that $\boldsymbol{\omega}_{\tilde{X}_{\tilde{t}}} \cong \mathcal{O}_{\tilde{X}_{\tilde{t}}}$, if it is satisfied for the original family (5.4).

$\mathbb{C}^*$	deg	x	y	z	v_0	v_1	Σ
λ_1	3	1	1	1	0	0	3
λ_2	12	4	6	0	1	1	12

Table 1: Weights of variables in Weierstrass equation.

5.4.1. Period integrals of twisted family. The differential form η_t in Equation (5.5) defines a flat section $\eta \in \Gamma(B, \mathcal{H})$ for the family in Equation (5.4). A flat section $\tilde{\eta} \in \Gamma(B, \tilde{\mathcal{H}})$ for the twisted family (5.9) is then given by the differential form

$$\tilde{\eta}_{\tilde{t}} = dv \wedge du_1 \wedge \cdots \wedge du_{n-1} \wedge \frac{d\tilde{x}}{\tilde{y}}, \tag{5.11}$$

in the affine chart $[v_0 : v_1] = [v : 1]$ and $\tilde{z} = 1$, such that

$$\tilde{\eta}_{\tilde{t}} = \frac{dv}{v(v+1)^\alpha} \wedge \eta_t, \tag{5.12}$$

where we have used $\tilde{x} = v_0^2 v_1^{2-2\alpha}(v_0 + v_1)^{2\alpha} x$ and $\tilde{y} = v_0^3 v_1^{3-3\alpha}(v_0 + v_1)^{3\alpha} y$. Thus, if $\omega(t)$ is a local section of the period sheaf $\Pi(\mathcal{H}, \eta) \to B$, a section of the new period sheaf $\Pi(\tilde{\mathcal{H}}, \tilde{\eta}) \to \tilde{B}$ is given by

$$\tilde{\omega}(\tilde{t}) = \oint_C \frac{dv}{v(v+1)^\alpha}\, \omega\left(\frac{c_{ij}\tilde{t}}{v^i(v+1)^j}\right), \tag{5.13}$$

where C is a non-contractible loop in the punctured v-plane. We have the following:

Proposition 5.1. *Let $\omega(t)$ be a period integral for the family (5.4) with absolutely convergent series $\omega(t) = \sum_{k\geq 0} f_k t^k$ for $|t| < 1$, and the twisted family (5.9) with generalized functional invariant (i, j, α) satisfy conditions (5.8). Let $C_{1/2}(0)$ be the circle $|v| = \frac{1}{2}$ in the v-plane with counterclockwise orientation. For every $\tilde{t} \in \mathbb{C}$ with $|\tilde{t}| < 1/(2^{i+j+1}|c_{ij}|)$ and $c_{ij} = (-1)^i\, i^i\, j^j/(i+j)^{i+j}$, the period integral (5.13) for $C = C_{1/2}(0)$ has an absolutely convergent series given by the Hadamard products:*

(5.14)

$$\text{if } \alpha = 1: \quad \tilde{\omega}(\tilde{t}) = (2\pi i)\ {}_{i+j-1}F_{i+j-2}\left(\begin{array}{c} \frac{1}{i+j} \quad \dots \quad \frac{i+j-1}{i+j} \\ \frac{1}{i} \ \dots \ \frac{i-1}{i} \ \frac{1}{j} \ \dots \ \frac{j-1}{j} \end{array} \middle|\, \tilde{t}\right) \star \omega(\tilde{t}),$$

$$\text{if } \alpha = \frac{1}{2}: \quad \tilde{\omega}(\tilde{t}) = (2\pi i)\ {}_{i+j}F_{i+j-1}\left(\begin{array}{c} \frac{\alpha}{i+j} \quad \dots \quad \frac{\alpha+i+j-1}{i+j} \\ \frac{1}{i} \ \dots \ \frac{i-1}{i} \ \frac{\alpha}{j} \ \dots \ \frac{\alpha+j-1}{j} \end{array} \middle|\, \tilde{t}\right) \star \omega(\tilde{t}).$$

Proof. For $|v| = \frac{1}{2}$ and $|\tilde{t}| < 1/(2^{i+j+1}|c_{ij}|)$ we have $|t| = |\frac{c_{ij}\,\tilde{t}}{v^i(v+1)^j}| < 1$. We use the absolutely and uniformly convergent series $\omega(t) = \sum_{k\geq 0} f_k t^k$ and carry out a term-by-term integration to obtain

$$\oint_{|v|=\frac{1}{2}} \frac{dv}{v(v+1)^{\alpha}}\, \omega\left(\frac{c_{ij}\,\tilde{t}}{v^i(v+1)^j}\right) = \sum_{k\geq 0} f_k\, \left(c_{ij}\tilde{t}\right)^k \oint_{|v|=\frac{1}{2}} \frac{dv}{v^{ik+1}(v+1)^{jk+\alpha}}.$$

Using the formula $(1+z)^{-k} = \sum_{l\geq 0} \frac{\Gamma(l+k)}{\Gamma(k)\Gamma(l+1)}(-z)^l$, we obtain from a residue computation the identity

$$\begin{aligned} \sum_{k\geq 0} f_k\, \left(c_{ij}\tilde{t}\right)^k \sum_{l\geq 0} \frac{\Gamma(jk+l+\alpha)}{\Gamma(jk+\alpha)\,\Gamma(l+1)}(-1)^l \oint_{|v|=\frac{1}{2}} \frac{dv}{v^{ik-l+1}} \\ = (2\pi i) \sum_{k\geq 0} f_k\, \frac{\prod_{m=0}^{i+j-1}\left(\frac{\alpha+m}{i+j}\right)_k}{k!\prod_{m=1}^{i-1}\left(\frac{m}{i}\right)_k \prod_{m=0}^{j-1}\left(\frac{m+\alpha}{j}\right)_k}\, \tilde{t}^k, \end{aligned} \tag{5.15}$$

where we used Gauss' multiplication formula for the Gamma function

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-\frac{1}{2}} \prod_{r=0}^{n-1} \Gamma\left(z+\frac{r}{n}\right). \tag{5.16}$$

Since we also have $|\tilde{t}|<1$ Equation (5.4.1) can be further simplified using Gamma-function identities and the Hadamard product. We obtain

$$\begin{aligned} (2\pi i) \sum_{k\geq 0} f_k\, \frac{\prod_{m=0}^{i+j-1}\left(\frac{\alpha+m}{i+j}\right)_k}{k!\prod_{m=1}^{i-1}\left(\frac{m}{i}\right)_k \prod_{m=0}^{j-1}\left(\frac{m+\alpha}{j}\right)_k}\, \tilde{t}^k \\ = (2\pi i)\ {}_{i+j}F_{i+j-1}\left(\left.\begin{array}{c} \frac{\alpha}{i+j},\quad \ldots \quad \frac{\alpha+i+j-1}{i+j} \\ \frac{1}{i},\ldots,\frac{i-1}{i},\frac{\alpha}{j},\ldots,\frac{\alpha+j-1}{j} \end{array}\right| \tilde{t}\right) \star \omega(\tilde{t}). \end{aligned} \tag{5.17}$$

The remaining equation is obtained by setting $\alpha=1$ and observing an obvious cancellation in the coefficients of the hypergeometric series. □

5.5. Variants of the twist construction

5.5.1. Twists with two parameters and subfamilies. The twist construction with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ can be generalized in a way that introduces two parameters. It is based on the two-parameter family of ramified covering maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree two given by

$$[v_0 : v_1] \mapsto \Big[4av_0(v_0+v_1)+(a-b)v_1^2 \ : \ 4v_0(v_0+v_1)\Big], \tag{5.18}$$

which is totally ramified over a and b. For $a=0$ and $b=\tilde{t}$, this family coincides with the covering maps used in the twist construction with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ in Equation (5.7).

We restrict the two-parameter family to the locus $a=-b=\tilde{t}$ instead, and obtain a new family $\tilde{\pi} : \tilde{X} \to B$ such that each element $\tilde{X}_{\tilde{t}} = \tilde{\pi}^{-1}(\tilde{t})$ is a compact complex $(n+1)$-fold with trivial canonical bundle and also elliptically fibered with section over $\mathbb{P}^1 \times S$. The Weierstrass model $\tilde{W}_{\tilde{t}}$ for $\tilde{X}_{\tilde{t}}$ is given by

$$\begin{aligned} \tilde{y}^2\tilde{z} = 4\,\tilde{x}^3 &- g_2\left(a+\frac{(a-b)v_1^2}{4v_0(v_0+v_1)},\,u\right)v_0^4(v_0+v_1)^4\tilde{x}\tilde{z}^2 \\ &- g_3\left(a+\frac{(a-b)v_1^2}{4v_0(v_0+v_1)},\,u\right)v_0^6(v_0+v_1)^6\,\tilde{z}^3\,, \end{aligned} \tag{5.19}$$

where we have set $a=-b=\tilde{t}$. Thus, if $\omega(t)$ is a local section of the period sheaf $\Pi(\mathcal{H},\eta)\to B$, a section of the new period sheaf $\Pi(\tilde{\mathcal{H}},\tilde{\eta})\to\tilde{B}$ is given by

$$\tilde{\omega}(\tilde{t}) = \oint_C \frac{dv}{v(v+1)}\,\omega\left(\tilde{t}(1+\frac{1}{2v(v+1)})\right), \tag{5.20}$$

where C is a non-contractible loop in the punctured v-plane. We have the following:

Proposition 5.2. *Let $\omega(t)$ be a period integral for the family (5.4) with absolutely convergent series $\omega(t)=\sum_{k\geq 0} f_k t^k$ for $|t|<1$, and the twisted family (5.19) with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ and $a=-b=\tilde{t}$ satisfy conditions (5.8). Let $C_{1/2}(0)$ be the circle in the v-plane $|v|=\frac{1}{2}$ oriented counterclockwise. For every $\tilde{t}\in\mathbb{C}$ with $|\tilde{t}|<1/2$, the period integral (5.20) for $C=C_{1/2}(0)$ has an absolutely convergent series given by the Hadamard product*

$$\tilde{\omega}(\tilde{t}) = (2\pi i)\sum_{n\geq 0}\frac{f_{2n}\,\left(\frac{1}{2}\right)_n}{n!}\,t^{2n} = (2\pi i)\,{}_1F_0\left(\frac{1}{2}\middle|\,t^2\right)\star\omega(t). \tag{5.21}$$

Proof. The proof is analogous to the proof of Proposition 5.1, where we use the additional identity

$$\sum_{l=0}^{n} \binom{n}{l}\binom{2l}{l}\left(-\frac{1}{2}\right)^{l} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{\left(\frac{1}{2}\right)_k}{k!} & \text{if } n = 2k \text{ is even.} \end{cases} \tag{5.22}$$

□

5.5.2. Relation to quadratic twists. Let $\pi : X \to B$ be a family with Weierstrass model (5.4) such that

$$0 \leq \deg_t(g_2) \leq 4, \quad 0 < \deg_t(g_3) \leq 6\,. \tag{5.23}$$

The *quadratic-twist family* $\hat{\pi} : \hat{X} \to B$ is constructed by promoting the family parameter t to an additional complex variable and carrying out a quadratic twist. Each element $\hat{X}_{\tilde{t}} = \hat{\pi}^{-1}(\tilde{t})$ is a compact complex $(n+1)$-fold and also elliptically fibered with section over $\mathbb{P}^1 \times S$. For $\tilde{t} \in B$ and local coordinate chart $\{[w_0 : w_1], (u_1, \dots, u_{n-1}) \in \mathbb{P}^1 \times S\}$, the Weierstrass model $\hat{W}_{\tilde{t}}$ for $\hat{X}_{\tilde{t}}$ is given by

$$\begin{aligned} \hat{y}^2 \hat{z} = 4\,\hat{x}^3 - g_2\left(\tilde{t}\,\frac{w_0}{w_1},\, u\right) w_1^4 w_0^2 (w_0 - w_1)^2\, \hat{x}\hat{z}^2 \\ - g_3\left(\tilde{t}\,\frac{w_0}{w_1},\, u\right) w_1^6 w_0^3 (w_0 - w_1)^3\, \hat{z}^3\,. \end{aligned} \tag{5.24}$$

The Weierstrass model (5.9) is smooth over an open subset of the form $D^0 \times S^0$ where D^0 is the complement in $\mathbb{P}^1$ of $w_0 = 0$, $w_1 = 0$, and $w_0 - \tilde{t}w_1 = 0$. Conditions (5.23) ensure that the Weierstrass model is minimal and normal if the Weierstrass model (5.4) is. Moreover, a similar argument as in Section 5.4 shows that the Calabi-Yau condition is satisfied such that $\boldsymbol{\omega}_{\hat{X}_{\tilde{t}}} \cong \mathcal{O}_{\hat{X}_{\tilde{t}}}$. We have the following:

Lemma 5.3. *The quadratic-twist family $\hat{\pi} : \hat{X} \to B$ with Weierstrass model (5.24) is birationally equivalent to the twisted family $\tilde{\pi} : \tilde{X} \to B$ with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ and Weierstrass model (5.9). In particular, the periods of the two families coincide.*

Proof. Setting

$$w = -\frac{1}{4v(v+1)}, \quad \hat{x} = \frac{(1+2v)^2\tilde{x}}{2^4 v^4 (v+1)^4}, \quad \hat{y} = \frac{(1+2v)^3\tilde{x}}{2^6 v^6 (v+1)^6},$$

in the affine charts $w_1 = \hat{z} = v_1 = \tilde{z} = 1$, $w_0 = w$ and $v_0 = v$, transforms Equation (5.24) into Equation (5.9) such that $\hat{\eta}_{\tilde{t}} = \tilde{\eta}_{\tilde{t}}$. □

Remark 5.4. The sequence of examples from Section 4.1 is precisely based on iteratively applying Equation (5.24) in the affine chart $\hat{z} = 1$ and $[w_0 : w_1] = [u : t]$ (or $[w_0 : w_1] = [v : t]$, etc.)

There is also a two-parameter family with Weierstrass models $\hat{W}_{a,b}$ given by

$$\begin{aligned}\hat{y}^2\hat{z} = 4\hat{x}^3 &- g_2\left(\frac{w_0}{w_1},\, u\right) w_1^4(w_0 - aw_1)^2(w_0 - bw_1)^2\,\hat{x}\hat{z}^2 \\ &- g_3\left(\frac{w_0}{w_1},\, u\right) w_1^6(w_0 - aw_1)^3(w_0 - bw_1)^3\,\hat{z}^3\,.\end{aligned} \tag{5.25}$$

The quadratic-twist family with Weierstrass model (5.25) is isomorphic to the family with Weierstrass model (5.19). This is seen by setting

$$w = a + \frac{a-b}{4\,v\,(v+1)}, \quad \hat{x} = \frac{(b-a)^2(1+2v)^2x}{2^4v^4(v+1)^4}, \quad \hat{y} = \frac{(b-a)^3(1+2v)^3y}{2^6v^6(v+1)^6}. \tag{5.26}$$

in the affine charts $w_1 = \hat{z} = v_1 = \tilde{z} = 1$, $w_0 = w$ and $v_0 = v$.

Let us explain the geometric relationship between the quadratic-twist family and twisted family with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ for a K3 surface. Let $X \to \mathbb{P}^1$ be the Jacobian rational elliptic surface given by

$$y^2z = 4x^3 - g_2(t)\,xz^2 - g_3(t)\,z^3.$$

If the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is given by $f(v) = a + (a-b)/(4v(v+1))$, then the pull-back $t = f(v)$ of the rational elliptic surface is a two-parameter family of Jacobian elliptic K3 surface $\tilde{X} \to \mathbb{P}^1$. On $\tilde{X}$, we have the deck transformation $\imath$ given by $v \mapsto -v-1$ and the elliptic involution $-\mathrm{id}$. The composition $\jmath = -\mathrm{id} \circ \imath$ is a Nikulin involution leaving the holomorphic two-form invariant, and the minimal resolution of the quotient $\tilde{X}/\jmath$ is the Jacobian elliptic K3 surface $\hat{X} \to \mathbb{P}^1$ given by

$$\hat{y}^2\hat{z} = 4\hat{x}^3 - g_2(t)\,(t-a)^2(t-b)^2\hat{x}\hat{z}^2 - g_3(t)\,(t-a)^3(t-b)^3\hat{z}^3.$$

That is, the K3 surface $\hat{X}$ is the quadratic-twist family of X having fibers of Kodaira-type I_0^* over the two ramification points of f. The situation is summarized in Figure 1. For each pair of K3 surfaces obtained as quadratic-twist family and twisted family with generalized functional invariant, we can easily check that the determinants of the discriminant groups of their transcendental lattices always differ by a square of the form $(1/2)^{2\alpha}$ with $0 \leq \alpha \leq 2$. As the transcendental lattices are related by the isometry given

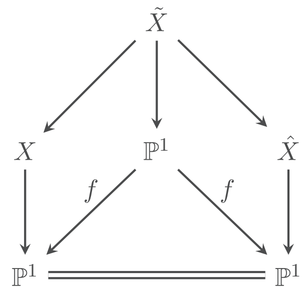

Figure 1: Relation between quadratic-twist family and twisted family.

in Equation (5.26) and is generally not a Hodge isometry, this is in perfect agreement with a general result obtained by Mehran [59].

Remark 5.5. The Picard-Fuchs systems for the full two-parameter family in Equations (5.19) and (5.25) are coupled linear partial differential equations in two complex variables; see [66, 67]. For families of elliptic curves, the Picard-Fuchs systems are obtained from the generalized hypergeometric system satisfied by the Appell function F_1 by rational pullback. For K3 surfaces, the Picard-Fuchs systems are obtained from the generalized hypergeometric system satisfied by the Appell function F_2 by rational pullback. Some explicit examples were determined in [22, 58].

5.5.3. Twists by rational surfaces. A rational elliptic surface $X' \to B = \mathbb{P}^1\backslash\{0, 1, \infty\}$ has a singular fiber of Kodaira-type IV^*, III^*, or II^* over $t = \infty$, if and only if its Weierstrass model is given by

$$y^2 z = 4x^3 - g_2'(t)\, xz^2 - g_3'(t)\, z^3, \tag{5.27}$$

and the degrees of the polynomials $g_2'(t)$ and $g_3'(t)$ are at most 1 and 2, respectively. In fact, we will focus on three cases X_k' for $k = 1, 2, 3$ in Table 2 where the rational elliptic surfaces are extremal and given in Table 5; the surfaces themselves will be discussed in Section 6.1, and the number μ associated with each X_k' surface will be explained in Corollary 6.8.

We fix any such rational surface X_k'. Given a second rational elliptic surface $X \to B$ with Weierstrass model

$$y^2 z = 4x^3 - g_2(t)\, xz^2 - g_3(t)\, z^3, \tag{5.28}$$

k	X'_k	μ	fiber at $t=\infty$	$\deg_t(g_2)$	$\deg_t(g_3)$
4	$X'_4 = X_{431}$	$\frac{1}{3}$	IV^*	1	2
3	$X'_3 = X_{321}$	$\frac{1}{4}$	III^*	1	1
2	$X'_2 = X_{211}$	$\frac{1}{6}$	II^*	0	1

Table 2: Rational surfaces used for twisting.

with degrees of $g_2(t)$ and $g_3(t)$ being at most 4 and 6, respectively, we define a family of Gorenstein threefolds $\tilde{\pi} : \tilde{X} \to B$ such that each element $\tilde{X}_{\tilde{t}} = \tilde{\pi}^{-1}(\tilde{t})$ is a compact complex threefold and also elliptically fibered with section over the Hirzebruch surface $\mathbb{F}_n$. We call this family the *twist family of* $X \to B$ *with* $X'_k \to B$. Here, the Hirzebruch surface is given by $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$ with $n = 0, \ldots, k$.

We define polynomials

$$g_2(t,u,v) := h(u,v)^4 v^4 g_2\left(\frac{t}{v}\right), \quad g_3(t,u,v) := h(u,v)^6 v^6 g_3\left(\frac{t}{v}\right), \tag{5.29}$$

where we have used $h(u,v)^2 = 4u^3 - g'_2(v)\,u - g'_3(v)$. We consider $\mathbb{F}_n$ the quotient space of $\mathbb{C}^4 \backslash \{u_0 = u_1 = v_0 = v_1 = 0\}$ by the action of $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$(\lambda_2, \lambda_3) \cdot [u_0 : u_1 : v_0 : v_1] = [\lambda_3 u_0 : \lambda_3 u_1 : \lambda_3^n \lambda_2 v_0 : \lambda_2 v_1],$$

It follows that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_1$ is the non-minimal surface obtained as the blow-up of $\mathbb{P}^2$ in one point. For $\tilde{t} \in \mathbb{P}^1$ and coordinates $[u_0 : u_1 : s_0 : s_1] \in \mathbb{F}_n$ and $[\tilde{x} : \tilde{y} : \tilde{z}] \in \mathbb{P}(2,3,1)$, let the Weierstrass model $\tilde{W}_{\tilde{t}}$ for $\tilde{X}_{\tilde{t}}$ be given by

$$\begin{aligned} \tilde{y}^2 \tilde{z} = 4\tilde{x}^3 &- g_2\left(\tilde{t}, \frac{u_0}{u_1}, \frac{v_0}{v_1\, u_1^n}\right) u_1^{4(n+2)} v_1^8 \tilde{x}\, \tilde{z}^2 \\ &- g_3\left(\tilde{t}, \frac{u_0}{u_1}, \frac{v_0}{v_1\, u_1^n}\right) u_1^{6(n+2)} v_1^{12} \tilde{z}^3, \end{aligned} \tag{5.30}$$

with $g_2(t,u,v)$ and $g_3(t,u,v)$ given in Equation (5.29). We have the following:

Lemma 5.6. *In the situation above, Equation (5.30) is a minimal and normal Weierstrass model over* $\mathbb{F}_n$ *for* $n = 0, \ldots, k$ *such that* $\boldsymbol{\omega}_{\tilde{X}_{\tilde{t}}} \cong \mathcal{O}_{\tilde{X}_{\tilde{t}}}$ *for* $\tilde{t} \in B$.

Proof. The first part follows by checking that the Weierstrass model obtained for each rational elliptic surface X'_k in Table 2 is minimal. Then, three $\mathbb{C}^*$-group actions on the defining variables in Equation (5.30) are given by the weights listed in Table 3 where *deg* denotes the total weight of Equation (5.30) and *sum* denotes the sum of weights of the defining variables. If the condition is satisfied that the total weight of Equation (5.30) equals the sum of weights of the defining variables for each acting $\mathbb{C}^*$ in Equation (5.30), then a Calabi-Yau threefold is obtained by removing the loci $\{u_0 = u_1 = 0\}$, $\{v_0 = v_1 = 0\}$, $\{x = y = z = 0\}$ from the solution set of Equation (5.30), and taking the quotient $(\mathbb{C}^*)^3$. Details are explained in [61]. □

$\mathbb{C}^*$	deg	x	y	z	v_0	v_1	u_0	u_1	sum
λ_1	3	1	1	1	0	0	0	0	3
λ_2	12	4	6	0	1	1	0	0	12
λ_3	$6(n+2)$	$2(n+2)$	$3(n+2)$	0	n	0	1	1	$6(n+2)$

Table 3: Weights of variables in Weierstrass equation.

The differential one-form $\eta_t = dx/y$ (and $\eta'_t = dx/y$) defines a flat section of the conormal bundle for the family in Equation (5.28) (resp. Equation (5.27)) in the affine chart $z = 1$. A flat section $\tilde{\eta} \in \Gamma(\mathbb{P}^1, \tilde{\mathcal{H}})$ for the twisted family (5.30) is then given by the differential form

$$\tilde{\eta}_{\tilde{t}} = dv \wedge du \wedge \frac{d\tilde{x}}{\tilde{y}}, \tag{5.31}$$

in the affine chart $[u_0 : u_1 : v_0 : v_1] = [u : 1 : v : 1]$ and $\tilde{z} = 1$, such that

$$\tilde{\eta}_{\tilde{t}} = \frac{dv}{v} \wedge \frac{du}{h(u,v)} \wedge \eta_{\tilde{t}/v} = \frac{dv}{v} \wedge \eta'_v \wedge \eta_{\tilde{t}/v}. \tag{5.32}$$

Thus, if $\omega(t)$ (and $\omega'(t)$) is a local section of the period sheaf of the rational elliptic surface $X \to \mathbb{P}^1$ (resp. $X' \to \mathbb{P}^1$), a section of the period sheaf $\Pi(\tilde{\mathcal{H}}, \tilde{\eta}) \to \mathbb{P}^1$ is given by

$$\tilde{\omega}(\tilde{t}) = \oint_C \frac{dv}{v}\, \omega'(v)\, \omega\left(\frac{\tilde{t}}{v}\right), \tag{5.33}$$

where C is a non-contractible loop in the punctured v-plane. We have the following:

Proposition 5.7. *Let $\omega(t)$ (and $\omega'(t)$) be a period integral for the rational elliptic surface $X \to \mathbb{P}^1$ (resp. $X' \to \mathbb{P}^1$ with a singular fiber of type IV^*, III^*, or II^* at $t = \infty$) with absolutely convergent series $\omega(t) = \sum_{k\geq 0} f_k t^k$ (resp. $\omega'(t) = \sum_{k\geq 0} f'_k t^k$) for $|t| < 1$. Let $C_{1/2}(0)$ be the circle $|v| = \frac{1}{2}$ in the v-plane with counterclockwise orientation. For every $\tilde{t} \in \mathbb{C}$ with $|\tilde{t}| < 1/2$, the period integral (5.33) for $C = C_{1/2}(0)$ has an absolutely convergent series given by the Hadamard product*

$$\tilde{\omega}(\tilde{t}) = (2\pi i)\, \omega'(\tilde{t}) \star \omega(\tilde{t}). \tag{5.34}$$

Proof. The proof is analogous to the proof of Proposition 5.1. For $C = C_{1/2}(0)$ a simple residue computation in Equation (5.33) yields

$$\tilde{\omega}(\tilde{t}) = \sum_{m,n} f'_m f_n t^n \oint_C \frac{dv}{v^{1-m+n}} = (2\pi i) \sum_{m} f'_m\, f_m t^m,$$

□

6. Modular elliptic families and related families

In this section, we will show that all extremal, modular elliptic families of elliptic curves or K3 surfaces with three singular fibers are in fact twisted families for some generalized functional invariant (i, j, α). In our iterative construction, the universal starting point is always the same pencil $\pi : X \to B$ of 'dimension zero' Calabi-Yau manifolds, namely the ramified family of pairs of points encountered before $\{\pm y\} \in X_t = \pi^{-1}(t)$ given by

$$y^2 = 1 - t \tag{6.1}$$

with $t \in \mathbb{C}$. For the family (6.1), we define $\Sigma_0(t)$ to be the point t for $t \in B$, take the branch cut branch cut along $\{\, t \,|\, 1 \leq t \leq \infty\}$, and consider the holomorphic 0-form $1/y_0$ with the period

$$\int_{\Sigma_0(t)} \frac{1}{y_0} = \frac{2}{y_0} = 2\, {}_1F_0\left(\left.\frac{1}{2}\right| t\right) = 2\left(1-t\right)^{-\frac{1}{2}}, \tag{6.2}$$

which is a solution to the hypergeometric differential equation $L_t^{(1)}(\frac{1}{2};)\, \omega(t) = 0$.

6.1. Extremal families of elliptic curves

Applying our twist construction to the family (6.1), we obtain families of one-dimensional Calabi-Yau manifolds, namely modular families of elliptic curves. We have the following:

Lemma 6.1. *For (i, j, α) with $1 \leq i \leq 2$, $1 \leq j \leq 2\alpha$ and $\alpha \in \{\frac{1}{2}, 1\}$, the twisted families with generalized functional invariant (i, j, α) given by*

$$\tilde{y}^2 = \left(1 - \frac{c_{ij}\tilde{t}}{\tilde{x}^i(\tilde{x}+1)^j}\right) \tilde{x}^2 \, (\tilde{x}+1)^{2\alpha}, \tag{6.3}$$

are families of genus-one curves. For $(i, j, \alpha) = (1, 1, 1)$, $(2, 1, 1)$, $(1, 1, \frac{1}{2})$, and $(2, 1, \frac{1}{2})$, the families admit a $\mathbb{C}(\tilde{t})$-rational Weierstrass point. The corresponding families of elliptic curves are rational elliptic surfaces with singular fibers given in Table 4; the explicit Weierstrass models and Mordell-Weil groups are given in Table 5.

Proof. The result follows by explicit computation. □

Remark 6.2. The points of maximal unipotent monodromy for an elliptic surface comprise the support of singular fibers of type I_n for $n \geq 1$ [29, Cor. 1].

Remark 6.3. The names of the Jacobian elliptic surfaces used in Table 4 and Table 5 coincide with the classical notation used by Herfurtner [47].

(i, j, α)	μ	singular fibers	notation
$(1, 1, 1)$	$\frac{1}{2}$	$I_1^*(\tilde{t}=\infty),\ I_4(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{141}
$(2, 1, 1)$	$\frac{1}{3}$	$IV^*(\tilde{t}=\infty),\ I_3(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{431}
$(1, 1, \frac{1}{2})$	$\frac{1}{4}$	$III^*(\tilde{t}=\infty),\ I_2(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{321}
$(2, 1, \frac{1}{2})$	$\frac{1}{6}$	$II^*(\tilde{t}=\infty),\ I_1(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{211}

Table 4: Families of elliptic curves.

Remark 6.4. Conditions (5.8) allow for a fifth case with $(i, j, \alpha) = (2, 2, 1)$. In this remaining case, the twisted family with generalized functional invariant $(i, j, \alpha) = (2, 2, 1)$ is a family of genus-one curves whose relative Jacobians coincide with the elliptic curves obtained for the twisted family with $(i, j, \alpha) = (1, 1, \frac{1}{2})$.

name, μ, G	g_2, g_3, Δ, J	ramification of J and singular fibers			
$\mathrm{MW}(\pi, \sigma)$	sections	t	J	$m(J)$	fiber
X_{141}	$g_2 = \frac{1}{3}\left(4t^2 - 64\,t + 64\right)$	$8 \pm 4\sqrt{3}$	0	3	smooth
$\mu = \frac{1}{2}$	$g_3 = \frac{8}{27}\left(2 - t\right)\left(32 - 32t - t^2\right)$	$-16 \pm 12\sqrt{2}, 2$	1	2	smooth
$\Gamma_0(4)$	$\Delta = -256\,t^4\,(t-1)$	0	∞	4	I_4 (A_3)
	$J = -\frac{(t^2-16t+16)^3}{108t^4(t-1)}$	1	∞	1	I_1
$\mathbb{Z}/4\mathbb{Z}$	$(X,Y)_1 = (\frac{2}{3}t - \frac{4}{3}, 0)$	∞	∞	1	I_1^* (D_5)
	$(X,Y)_{2,3} = (-\frac{1}{3}t - \frac{4}{3}, \pm 4\,i\,t)$				
X_{431}	$g_2 = 27 - 24\,t$	∞	0	1	IV^* (E_6)
$\mu = \frac{1}{3}$	$g_3 = 27 - 36\,t + 8\,t^2$	$\frac{9}{8}$	0	3	smooth
$\Gamma_0(3)$	$\Delta = -1728\,t^3\,(t-1)$	$\frac{9}{4} \pm \frac{3\sqrt{3}}{4}$	1	2	smooth
	$J = \frac{(-9+8\,t)^3}{64\,t^3(t-1)}$	0	∞	3	I_3 (A_2)
$\mathbb{Z}/3\mathbb{Z}$	$(X,Y)_{1,2} = (-\frac{3}{2}, \pm 2\sqrt{2}\,i\,t)$	1	∞	1	I_1
X_{321}	$g_2 = \frac{16}{3} - 4\,t$	$\frac{4}{3}$	0	3	smooth
$\mu = \frac{1}{4}$	$g_3 = -\frac{64}{27} + \frac{8}{3}\,t$	$\frac{8}{9}$	1	2	smooth
$\Gamma_0(2)$	$\Delta = -64\,t^2\,(t-1)$	∞	1	1	III^* (E_7)
	$J = \frac{(-4+3t)^3}{27\,t^2(t-1)}$	0	∞	2	I_2 (A_1)
$\mathbb{Z}/2\mathbb{Z}$	$(X,Y) = (\frac{2}{3}, 0)$	1	∞	1	I_1
X_{211}	$g_2 = 3$	∞	0	2	II^* (E_8)
$\mu = \frac{1}{6}$	$g_3 = -1 + 2\,t$	$\frac{1}{2}$	1	2	smooth
$\Gamma_0(1)$	$\Delta = -108\,t\,(t-1)$	0	∞	1	I_1
$\{\mathbb{I}\}$	$J = -\frac{1}{4\,t\,(t-1)}$	1	∞	1	I_1

Table 5: Extremal rational fibrations.

A Jacobian elliptic surfaces is called *extremal* if the rank of its Mordell-Weil group of sections vanishes. We have the following:

Corollary 6.5. *The elliptic surfaces in Table 4 and Table 5 constitute all extremal families of elliptic curves with three singular fibers and rational total space (up to quadratic twist and two-isogeny). Moreover,* $\tilde{t} = 0$ *is a point of maximal unipotent monodromy for each family.*

Proof. The extremal families of elliptic curves with three singular fibers were classified in [60, Tab. 5.2]. The surface X_{411} in [60] is obtained from X_{141} by quadratic twist. Similarly, X_{222} is obtained from X_{411} by fiberwise two-isogeny. The point of maximal unipotent monodromy in the base curve of the elliptic surfaces is the support of the singular fiber of type I_n for $n \geq 1$ [29, Cor. 1]. □

Application of results in [70] yields the following:

Corollary 6.6. *The families of elliptic curves in Table 4 with fiber of type I_n for $n = 2, 3, 4$ and $n = 1$ are the universal families of elliptic curves over the genus-zero modular curves for the congruence subgroups $\Gamma_0(n)$, and for $n = 1$ for the unique normal subgroup of index 2 in $\mathrm{PSL}(2,\mathbb{Z})$ denoted by $\Gamma_0(1)$. The family parameter $\tilde{t}$ is the corresponding Hauptmodul of the modular curve.*

The twist construction also provides us, near $t = 0$, with a family of non-contractible one-cycles for each family in Lemma 6.1. We have:

Lemma 6.7. *For $\tilde{t} \in \mathbb{C}$ with $|\tilde{t}| < 1/(2^{i+j+1}|c_{ij}|)$, the circle $C = C_{1/2}(0)$, given by $|\tilde{x}| = \frac{1}{2}$ in the $\tilde{x}$-plane with counterclockwise orientation, determines a family of non-contractible A-cycles $\Sigma_1(\tilde{t})$ for each family of genus-one curves in Lemma 6.1.*

Proof. For $|\tilde{t}| < 1/(2^{i+j+1}|c_{ij}|)$, each family $\tilde{X} \to \mathbb{P}^1 \ni \tilde{t}$ of genus-one curves in Lemma 6.1 has two branch points with $|\tilde{x}| < \frac{1}{2}$ and two branch points with $|\tilde{x}| > \frac{1}{2}$. Therefore, there is a non-contractible one-cycle $\Sigma_1(\tilde{t}) \subset \tilde{X}_{\tilde{t}}$ that projects onto the circle $C = C_{1/2}(0)$, i.e., $|\tilde{x}| = \frac{1}{2}$ with counterclockwise orientation, and varies continuously for $|\tilde{t}| < 1/(2^{i+j+1}|c_{ij}|)$. At $\tilde{t} = 0$, the two branch points inside C coalesce, and $\Sigma_1(\tilde{t})$ constitutes a family of A-cycles. □

We then have the following:

Corollary 6.8. *For the twisted families in Lemma 6.1 with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$, $(2, 1, 1)$, $(1, 1, \frac{1}{2})$, or $(2, 1, \frac{1}{2})$, the period integral (5.13) is annihilated by the Picard-Fuchs operator $L^{(2)}_{\tilde{t}}\big((\mu, 1-\mu); (1)\big)$. In particular, the period over $\Sigma_1(\tilde{t})$ is holomorphic at $\tilde{t} = 0$ and* *given by*

$$\omega(\tilde{t}) = (2\pi i)\ {}_2F_1\left(\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\middle|\ \tilde{t}\right) \tag{6.4}$$

with $\mu = \frac{\alpha}{i+j}$.

Proof. The proof follows by application of Proposition 5.1 for the period integral $\omega(t) = {}_1F_0(\frac{1}{2}|t)$ and generalized functional invariant (i, j, α). □

6.2. Families of M_n-lattice polarized K3 surfaces

The procedure described in Section 5.5.2 allows us to construct the quadratic twists of the rational elliptic surfaces in Lemma 6.1. The four resulting families of K3 surfaces realize precisely the families of K3 surfaces considered by Hoyt in [49]. These K3 surfaces are not modular elliptic surfaces, but rather rational covers of them. To obtain modular elliptic K3 surfaces, we consider the twisted families with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ of the Jacobian elliptic surfaces in Lemma 6.1 instead. We denote by M_n the lattices $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n\rangle$ for $n \in \mathbb{N}$. We have the following:

Lemma 6.9. *For* $(n, \mu) \in \big\{(1, \frac{1}{6}), (2, \frac{1}{4}), (3, \frac{1}{3}), (4, \frac{1}{2})\big\}$, *the twisted families with generalized functional invariant* $(i, j, \alpha) = (1, 1, 1)$ *given by*

$$\begin{aligned} Y^2 = 4\,X^3 &- \underbrace{g_2\left(-\frac{t}{4u(u+1)}\right)\Big(u\,(u+1)\Big)^4}_{=:\,g_2(t,u)} X \\ &- \underbrace{g_3\left(-\frac{t}{4u(u+1)}\right)\Big(u\,(u+1)\Big)^6}_{=:\,g_3(t,u)}, \end{aligned} \tag{6.5}$$

with $g_2(t)$ *and* $g_3(t)$ *determined by* (n, μ) *in Table 5, define families of Jacobian elliptic K3 surfaces of Picard rank 19 with two singular fibers of Kodaira-type* II^*, III^*, IV^*, *or* I_1^* *at* $u = 0$ *and* $u = -1$, *a fiber of Kodaira-type* I_{2n} *at* $u = \infty$, *and two singular fibers of type* I_1 *at* $u = -1/2 \pm \sqrt{1-t}/2$. *The families are families of* M_n*-lattice polarized K3 surfaces for* $n = 1, \ldots, 4$.

Proof. The proof amounts to checking the Kodaira-types of singular fibers from G_2, G_3, Δ and comparing with the list of all Jacobian elliptic surfaces given in the Shimada classification [68] of Jacobian elliptic K3 surfaces.

The fact that the constructed families of K3 surfaces are M_n-polarized was explained by Dolgachev in [28]. Moreover, it is easy to see that all torsion sections of the rational elliptic surfaces survive the mixed-twist construction. The torsion sections are listed in Table 6. □

Λ	torsion	sections
M_4	[4]	$(X,Y)_1 = \left(-\frac{1}{6}\,(u+1)\,u\left(8\,u^2+t+8\,u\right),0\right)$ $(X,Y)_{2,3} = \left(\frac{1}{12}\,(u+1)\,u\left(-16\,u^2+t-16\,u\right),\pm itu^2\,(u+1)^2\right)$
M_3	[3]	$(X,Y)_{1,2} = \left(-\frac{3}{2}\,u^2\,(u+1)^2,\pm\frac{i}{2}\sqrt{2}tu^2\,(u+1)^2\right)$
M_2	[2]	$(X,Y) = \left(\frac{2}{3}\,u^2\,(u+1)^2,0\right)$
M_1	[1]	–

Table 6: Torsions sections of Equation (6.5).

The configurations of singular fibers, the Mordell-Weil groups, the determinants of the discriminant groups, and the lattice polarizations are summarized in Table 7.

derived from		ρ	Configuration of singular fibers			$\mathrm{MW}(\pi,\sigma)$	disc Q	Λ
Srfc	μ		$u=\infty$	$u^2+u+t/4$	$u=0,-1$			
X_{141}	$\frac{1}{2}$	19	$I_8\ (A_7)$	$2\,I_1$	$2\,I_1^*\ (2\,D_5)$	[4]	2^3	M_4
X_{431}	$\frac{1}{3}$	19	$I_6\ (A_5)$	$2\,I_1$	$2\,IV^*\ (2\,E_6)$	[3]	$2\cdot 3$	M_3
X_{321}	$\frac{1}{4}$	19	$I_4\ (A_3)$	$2\,I_1$	$2\,III^*\ (2\,E_7)$	[2]	2^2	M_2
X_{211}	$\frac{1}{6}$	19	$I_2\ (A_1)$	$2\,I_1$	$2\,II^*\ (2\,E_8)$	[1]	2	M_1

Table 7: K3 surfaces from extremal rational surfaces.

Let us denote by $\Gamma_0(n)^+$ the modular group $\Gamma_0(n)$ extended by the Fricke involution, i.e., the element corresponding to $\tau \mapsto -1/(n\tau)$. It is a classical result due to Dolgachev that the moduli spaces of pseudo-ample M_n-polarized K3 surfaces are isomorphic to the rational modular curves that are the compactification of the curves $\mathbb{H}/\Gamma_0(n)^+$ [28]. We therefore have the following:

Corollary 6.10. *The twisted families with generalized functional invariant* $(1,1,1)$ *of the families of elliptic curves in Lemma 6.1 are families of* M_n*-lattice polarized K3 surfaces over the rational modular curves* $\mathbb{H}/\Gamma_0(n)^+$ *for* $n = 1,2,3,4$.

Proof. The proof follows directly by checking that the singular fibers and Mordell-Weil groups for the families constructed in Lemma 6.9 agree with the ones given by Dolgachev in [28]. □

For each twisted family in Equation (6.5), we define a family of closed two-cycles $\Sigma_2(t)$ as follows: for $t \in \mathbb{C}$ with $|t| < 1/2$ let $C = C_{1/2}(0)$ be the circle $|u| = \frac{1}{2}$ in the u-plane with counterclockwise orientation. For every $u \in C$, a cycle $\Sigma_1'(t,u)$ in the elliptic fiber is obtained from $\Sigma_1(-\frac{t}{4u(u+1)})$ — where $\Sigma_1(t)$ was defined in Lemma 6.7 — by rescaling $(X,Y) \to (u^2(u+1)^2 X, u^3(u+1)^3 Y)$. For $t \in \mathbb{C}$ with $|t| < 1/2$, we obtain a continuously varying family of closed two-cycles as a warped product $\Sigma_2(t) = C \times_u \Sigma_1'(t,u)$. By warped product we mean that the cycle $\Sigma_1(-\frac{t}{4u(u+1)})$ is warped, i.e. it is rescaled by a function of the affine coordinate u. We then have the following:

Corollary 6.11. *For the twisted families with generalized functional invariant* $(1,1,1)$ *in Lemma 6.9, the period integral (5.13) is annihilated by the Picard-Fuchs operator* $L_t^{(3)}((\mu,1/2,1-\mu);(1,1))$*. In particular, the period over* $\Sigma_2(t)$ *is holomorphic at* $t = 0$ *and given by*

$$\omega = (2\pi i)^2 \; {}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1,1 \end{array} \middle| \, t \right). \tag{6.6}$$

Proof. Application of Proposition 5.1 for the period integral $\omega(t) = {}_2F_1(\mu, 1-\mu; 1 | \, t)$ and the twisted families of Weierstrass models in Equation (6.5) yields the following formula

$${}_1F_0\left(\frac{1}{2} \middle| \, t \right) \star {}_2F_1\left(\begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \middle| \, t \right) = {}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1,1 \end{array} \middle| \, t \right). \tag{6.7}$$

□

By Clausen's identity each period in Corollary 6.11 is a perfect square, namely

$${}_3F_2\left(\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1,1 \end{array} \middle| \, t \right) = \left[{}_2F_1\left(\begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2} \\ 1 \end{array} \middle| \, t \right) \right]^2. \tag{6.8}$$

Ratios of solutions of the hypergeometric differential equation with holomorphic solution ${}_2F_1\big(\frac{\mu}{2}, \frac{1-\mu}{2}; 1\big| t\big)$ are so-called Schwarzian s-maps, and the corresponding triangle groups are the modular groups $\Gamma_0(n)^+$ in Corollary 6.10 and listed in Table 8. The so-called *Kummer identity* relates the hypergeo-

n	μ	triangle group
n	μ	$(2, \frac{2}{1-2\mu}, \infty)$
1	$\frac{1}{6}$	$(2, 3, \infty)$
2	$\frac{1}{4}$	$(2, 4, \infty)$
3	$\frac{1}{3}$	$(2, 6, \infty)$
4	$\frac{1}{2}$	$(2, \infty, \infty)$

Table 8: Triangle groups.

metric function on the right hand side of (6.8) back to the original period, i.e., for $t = 4T(1-T)$ it follows

$$(6.9) \qquad {}_2F_1\left(\begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2} \\ 1 \end{array} \middle| \, t \right) = {}_2F_1\left(\begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \middle| \, T \right).$$

The geometric origin of Equation (6.8) and (6.9) is the fact that an M_n-polarized K3 surface admits a Shioda-Inose structure relating it to a Kummer surface associated with the product of two isogenous elliptic curves; see [29].

6.3. Families of K3 surfaces of Picard-rank 18

The twisted families with generalized functional invariant $(1,1,1)$ of the elliptic surfaces in Lemma 6.1 are restrictions of two-parameter families of K3 surfaces with affine parameters $a, b \in \mathbb{C}$, as explained in Section 5.5.1. The families of Section 6.2 are obtained for $a = 0$. The restriction $a = -b$ yields different one-parameter families of K3 surfaces of Picard rank 18. We denote the relevant rank-18 lattices by $M = H \oplus E_8 \oplus E_8$, $\tilde{M} = H \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}/\mathbb{Z}_2$, $M' = H \oplus E_6^{\oplus 2} \oplus A_2^{\oplus 2}/\mathbb{Z}_3$, and $\tilde{M}' = H \oplus D_5^{\oplus 2} \oplus A_3^{\oplus 2}/\mathbb{Z}_4$. We have the following:

Lemma 6.12. *For* $(n, \mu) \in \big\{(1, \frac{1}{6}), (2, \frac{1}{4}), (3, \frac{1}{3}), (4, \frac{1}{2})\big\}$*, the twisted families with generalized functional invariant* $(i, j, \alpha) = (1, 1, 1)$ *and* $a = -b = t$

given by

$$Y^2 = 4\,X^3 - \underbrace{g_2\Big(t\Big(1+\frac{1}{2\,u(u+1)}\Big)\Big)\,\Big(u\,(u+1)\Big)^4}_{=:\,g_2(t,u)} X \tag{6.10}$$
$$- \underbrace{g_3\Big(t\Big(1+\frac{1}{2\,u(u+1)}\Big)\Big)\,\Big(u\,(u+1)\Big)^6}_{=:\,g_3(t,u)},$$

with $g_2(t)$ and $g_3(t)$ determined by (n, μ) in Table 5, define families of Jacobian elliptic K3 surfaces of Picard rank 18 with two singular fibers of Kodaira-type II^*, III^*, IV^*, or I_1^* at $u = 0$ and $u = -1$, two fibers of Kodaira-type I_n at $2u^2 + 2u + 1 = 0$, and two singular fibers of type I_1 at the roots of $p(t, u) = 2(t-1)u(u+1) + t$. The families are polarized K3 surfaces with lattice polarization M, $\tilde{M}$, M', and $\tilde{M}'$ for $n = 1, \ldots, 4$.

Proof. The proof amounts to checking the Kodaira-types of singular fibers from G_2, G_3, Δ and comparing with the list of all Jacobian elliptic surfaces given in the Shimada classification [68] of Jacobian elliptic K3 surfaces. The torsion sections for the families in Equation (6.10) are listed in Table 9. $\square$

Λ	torsion	sections
M_4	[4]	$(X,Y)_1 = \left(\frac{1}{3}\,(u+1)\,u\left(2\,tu^2+2\,tu-4\,u^2+t-4\,u\right),0\right)$ $(X,Y)_{2,3} = \Big(-\frac{1}{6}\,(u+1)\,u\left(2\,tu^2+2\,tu+8\,u^2+t+8\,u\right),$ $\pm it\,(2\,u+1+i)\,(-2\,u-1+i)\,u^2\,(u+1)^2\Big)$
M_3	[3]	$(X,Y)_{1,2} = \Big(-\frac{3}{2}\,u^2\,(u+1)^2,$ $\pm\frac{i}{2}\sqrt{2}t\,(2\,u+1+i)\,(-2\,u-1+i)\,u^2\,(u+1)^2\Big)$
M_2	[2]	$(X,Y) = \left(\frac{2}{3}\,u^2\,(u+1)^2,0\right)$
M_1	[1]	–

Table 9: Torsions sections of Equation (6.10).

The configurations of singular fibers, the Mordell-Weil groups, the determinants of the discriminant groups, and lattice polarizations are summarized in Table 10. We make the following:

Remark 6.13. Families of lattice polarized K3 surfaces in Lemma 6.12 are restrictions of the general two-parameter families introduced in Section 5.5.1.

derived from		ρ	Configuration of singular fibers			$\mathrm{MW}(\pi,\sigma)$	disc Q	Λ
Srfc	μ		$2u^2+2u+1$	$p(t,u)=0$	$u=0,-1$			
X_{141}	$\frac{1}{2}$	18	$2\,I_4\ (2\,A_3)$	$2\,I_1$	$2\,I_1^*\ (2\,D_5)$	[4]	2^4	$\tilde{M}'$
X_{431}	$\frac{1}{3}$	18	$2\,I_3\ (2\,A_2)$	$2\,I_1$	$2\,IV^*\ (2\,E_6)$	[3]	3^2	M'
X_{321}	$\frac{1}{4}$	18	$2\,I_2\ (2\,A_1)$	$2\,I_1$	$2\,III^*\ (2\,E_7)$	[2]	2^2	$\tilde{M}$
X_{211}	$\frac{1}{6}$	18	$2\,I_1$	$2\,I_1$	$2\,II^*\ (2\,E_8)$	[1]	1	M

Table 10: K3 surfaces from extremal rational surfaces.

The two-parameter family of M-lattice polarized K3 surfaces $\tilde{X}$ admits a Shioda-Inose structure, relating it to Kummer surfaces $\hat{X}$ associated with two non-isogenous elliptic curves [20]. In fact, the latter admit a Jacobian elliptic fibration with singular fibers $2I_0^*, II^*, 2I_1$. It turns out that the double cover induced by the degree-two map (5.18) on the base $\mathbb{P}^1$ induces a Hodge isometry between transcendental lattices $T_{\tilde{X}}(2) \cong T_{\hat{X}}$; see Figure 1. Thus, in the case of M-lattice polarized K3 surfaces $\tilde{X}$ the period domain is

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z})\Big) \rtimes \mathbb{Z}/2\mathbb{Z}\backslash\Big(\mathbb{H}\times\mathbb{H}\Big),$$

the moduli space of two elliptic curves, with a generator of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z})$ by exchanging the two sides.

For each twisted family with generalized functional invariant $(i,j,\alpha) = (1,1,1)$ and $a=-b=t$ in Equation (6.10), we define a family of closed two-cycles $\Sigma_2(t)$ as follows: for $t\in\mathbb{C}$ with $|t|<1/2$ let $C=C_{1/2}(0)$ be the circle given by $|u|=\frac{1}{2}$ in the u-plane with counterclockwise orientation. For every $u\in C$, a cycle $\Sigma_1'(t,u)$ in the elliptic fiber is obtained from $\Sigma_1(t(1+\frac{1}{2\,u(u+1)}))$ — where $\Sigma_1(t)$ was defined in Lemma 6.7 — by rescaling $(X,Y)\to(u^2(u+1)^2X, u^3(u+1)^3Y)$. For $t\in\mathbb{C}$ with $|t|<1/2$, we obtain a continuously varying family of closed two-cycles as a warped product $\Sigma_2(t) = C\times_u \Sigma_1'(t,u)$. We have the following:

Corollary 6.14. *For the twisted families in Lemma 6.12 with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ and $a=-b=t$, the period integral (5.13) is annihilated by the Picard-Fuchs operator*

$$L^{(4)}_{t^2}\left(\left(\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2}\right);\left(1,1,\frac{1}{2}\right)\right).$$

In particular, the period over $\Sigma_2(t)$ is holomorphic at $t = 0$ and given by

$$\omega = (2\pi i)^2 \, {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1 - \frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right). \tag{6.11}$$

Proof. We apply Proposition 5.2 to the period integral $\omega(t) = {}_2F_1\big(\mu, 1 - \mu; 1 \big| \, t\big)$ and the twisted families in Equation (6.5), and use the identity

$$ {}_1F_0\left(\left. \frac{1}{2} \right| t^2 \right) \star {}_2F_1\left(\left. \begin{array}{c} \mu, 1-\mu \\ 1 \end{array} \right| t \right) = {}_4F_3\left(\left. \begin{array}{c} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1 - \frac{\mu}{2} \\ 1, 1, \frac{1}{2} \end{array} \right| t^2 \right). \tag{6.12}$$

□

7. Elliptic fibrations on the mirror families

Non-trivial generalized functional invariants can be used to analyze elliptic fibrations on the mirror families obtained from the Dwork pencil, i.e., the one-parameter family of deformed Fermat hypersurfaces in $\mathbb{P}^n = \mathbb{P}(X_0, \dots, X_n)$ given by

$$X_0^{n+1} + X_1^{n+1} + \dots + X_n^{n+1} + (n+1)\,\lambda\, X_0 X_1 \cdots X_n = 0. \tag{7.1}$$

For each integer $n \in \mathbb{N}$, Equation (7.1) constitutes a family of $(n-1)$-dimensional Calabi-Yau hypersurfaces $X_\lambda^{(n-1)}$. For $n = 4$ Equation (7.1) is the quintic family of Candelas et al. [14]. For the family (7.1) the discrete group of symmetries for the Greene-Plesser orbifolding procedure is easily identified: it is generated by the action $(X_0, X_j) \mapsto (\zeta_{n+1}^n X_0, \zeta_{n+1} X_j)$ for $1 \le j \le n$ with $\zeta_{n+1} = \exp\left(\frac{2\pi i}{n+1}\right)$. In virtue of the fact that the product of all generators multiplies the homogeneous coordinates by a common phase, the symmetry group is $G_{n-1} = (\mathbb{Z}/(n+1)\,\mathbb{Z})^{n-1}$. The new affine variables

$$\begin{aligned} t &= \frac{(-1)^{n+1}}{\lambda^{n+1}}, \quad x_1 = \frac{X_1^n}{(n+1)\, X_0 \cdot X_2 \cdots X_n\, \lambda}, \\ x_2 &= \frac{X_2^n}{(n+1)\, X_0 \cdot X_1 \cdot X_3 \cdots X_n\, \lambda}, \; \dots, \end{aligned}$$

are invariant under the action of G_{n-1}. Hence, they descend to coordinates on the quotient $X_\lambda^{(n-1)}/G_{n-1}$. A second family of hypersurfaces $Y_t^{(n-1)}$ is then defined in terms of the new variables $x_1, \ldots, x_n$ by the remaining relation between those, namely the equation

$$f_n(x_1, \ldots, x_n, t) = x_1 \cdots x_n \Big(x_1 + \cdots + x_n + 1 \Big) + \frac{(-1)^{n+1}\, t}{(n+1)^{n+1}} = 0. \tag{7.2}$$

It was proved in [5] that the family of special Calabi-Yau hypersurfaces $Y_t^{(n-1)}$ of degree $(n+1)$ in $\mathbb{P}^n$ obtained from Equation (7.2) is in fact the mirror family of a general hypersurface $\mathbb{P}^n$ of degree $(n+1)$ and co-dimension one in $\mathbb{P}^n$. The subspace of the cohomology $H^{n-1}(X_\lambda^{(n-1)}, \mathbb{Q})$ invariant under the obvious action of G_{n-1} or, equivalently, the cohomology $H^{n-1}(Y_t^{(n-1)}, \mathbb{Q})$ has dimension n and the Hodge numbers $(1, \ldots, 1)$. We have the following:

Lemma 7.1. *For every $n \geq 2$ the family of hypersurfaces $Y_t^{(n-1)}$ given by Equation (7.2) is a fibration over $\mathbb{P}^1$ by hypersurfaces $Y_{\tilde{t}}^{(n-2)}$ where x_n is the affine base coordinate, and*

$$t = -\frac{n^n}{(n+1)^{n+1} x_n \, (x_n+1)^n} \, \tilde{t}. \tag{7.3}$$

Proof. For each $x_n \neq 0, -1$ substituting $\tilde{x}_i = x_i/(x_n+1)$ for $1 \leq i \leq n-1$ and $\tilde{t} = -n^n \, t/((n+1)^{n+1} x_n \, (x_n+1)^n)$ defines a fibration of the hypersurface (7.2) by $f_{n-1}(\tilde{x}_1, \ldots, \tilde{x}_{n-1}\tilde{t}) = 0$ since

$$f_n(x_1, \ldots, x_n, t) = x_n \, (x_n+1)^n \, f_{n-1}(\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{t}\,) = 0. \tag{7.4}$$

□

The rational function on the right hand side of Equation (7.3) relating t to $\tilde{t}$ has precisely the characteristic form of a generalized functional invariant (5.7) with $(i, j) = (n, 1)$. The unique holomorphic $(n-1)$-form on $Y_t^{(n-1)}$ is given by

$$\eta_t^{(n-1)} = \frac{dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n}{\partial_{x_1} f_n(x_1, \ldots, x_n, t)}. \tag{7.5}$$

One defines an $(n-1)$-cycle $\Sigma_{(n-1)}$ on $Y_t^{(n-1)}$ by requiring that the period integral of η_t over $\Sigma_{(n-1)}$ emerges as residue in x_1 in the integral over the torus $T^n = S^1 \times \cdots \times S^1$. The corresponding section of the period sheaf is given by

$$\omega_{n-1}(t) = \oint_{\Sigma_{n-1}} \frac{dx_2 \wedge \cdots \wedge dx_n}{\partial_{x_1} f_n(x_1, \ldots, x_n, t)}. \tag{7.6}$$

We have the following:

Proposition 7.2. *For $n \geq 1$ and $|t| \leq 1$, there is a transcendental $(n-1)$-cycle Σ_{n-1} on $Y_t^{(n-1)}$ such that*

$$\omega_{n-1}(t) = \oint_{\Sigma_{n-1}} \eta_t^{(n-1)} = (2\pi i)^{n-1} \, {}_nF_{n-1}\left(\begin{matrix} \frac{1}{n+1} & \cdots & \frac{n}{n+1} \\ 1 & \ldots & 1 \end{matrix} \middle| \, t \right). \tag{7.7}$$

The iterative relation (7.3) induces an iterative relation between periods, namely

$$\omega_{n-1}(t) = (2\pi i) \, {}_nF_{n-1}\left(\begin{matrix} \frac{1}{n+1} & \cdots & \frac{n}{n+1} \\ \frac{1}{n} & \ldots & \frac{n-1}{n} \end{matrix} \middle| \, t \right) \star \omega_{n-2}(t) \quad \textit{for } n \geq 2. \tag{7.8}$$

where the cycle Σ_{n-1} is determined by $T^n(r_n) := \frac{n}{n+1} \cdot \left(T^{n-1}(r_{n-1}) \times S^1_{r_{n-1}}\right)$ where $r_n = 1 - \frac{n}{n+1}$ and $\frac{n}{n+1} \cdot \left(T^{n-1}(r_{n-1}) \times S^1_{r_{n-1}}\right)$ means rescaling by $\frac{n}{n+1}$.

Proof. Rescaling and writing Equation (7.2) in the form $1 - \tau\phi(w_1, \ldots, w_n) = 0$ with $\tau = \frac{1}{n+1} t^{\frac{1}{n+1}}$ and $\phi = w_1 + \cdots + w_n + \frac{1}{w_1 \cdots w_n}$ leads to the residue period

$$\frac{\omega_{n-1}}{(2\pi i)^{n-1}} = \frac{1}{(2\pi i)^n} \oint_{T^n} \frac{\frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_n}{w_n}}{1 - \tau \, \phi(w_1, \ldots, w_n)} = \sum_{l \geq 0} [\phi^l]_0 \, \tau^l,$$

where $[\phi^l]_0$ for the constant term in ϕ^l. Using Equation (5.16) we obtain the series expansion of the hypergeometric function in Equation (7.7).

For $x_n \neq 0, -1$ the coordinate transformation $\tilde{x}_i = x_i/(x_n + 1)$ for $1 \leq i \leq n-1$ and $\tilde{t} = -n^n \, t/((n+1)^{n+1} x_n \, (x_n + 1)^n)$ yields

$$\eta_t^{(n-1)} = \tilde{\eta}_{\tilde{t}}^{(n-2)} \wedge \frac{dx_n}{x_n(x_n + 1)}. \tag{7.9}$$

For any x_n on $S^1_{r_n}$ with $r_n = 1 - \frac{n}{n+1}$ write $1/(x_n + 1) = R \, e^{i\varphi}$ with $\frac{n+1}{n+2} \leq R \leq \frac{n+1}{n}$, the transformation $\tilde{x}_i = x_i/(x_n + 1)$ maps the circle $\tilde{x}_i = r_{n-1} e^{it}$

to the circle $x_i = R\, r_{n-1} e^{i(t+\varphi)}$ with $0 < R\, r_{n-1} \leq \frac{n+1}{n^2} < 1$ as $n \geq 2$. We obtain

$$\underbrace{\int \cdots \int}_{T^n(r_n)} \frac{dx_1 \wedge \cdots \wedge dx_n}{f_n(x_1, \ldots, x_n, t)} = \oint_{|x_n| = \frac{1}{2}} \frac{dx_n}{x_n(x_n+1)} \times \underbrace{\int \cdots \int}_{T^{n-1}(r_{n-1})} \frac{d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_{n-1}}{f_{n-1}(\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{t})}. \tag{7.10}$$

Using Proposition 5.1 Equation (7.8) follows. □

Remark 7.3. The iterative relation between periods in Equation (7.8) in the special case $n = 4$, i.e., the series expansion of the equation

$$ {}_4F_3\left(\begin{array}{c} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ 1, 1, 1 \end{array} \middle| \, t \right) = {}_4F_3\left(\begin{array}{c} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \end{array} \middle| \, t \right) \star {}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{array} \middle| \, t \right), \tag{7.11}$$

was the "surprise found [...] when the coefficients are calculated and substituted" in [14, Eq. (3.7)]. Our Lemma 7.1 explains that this is in fact a general feature of the iterative fibration structure on the mirror family of the Dwork pencil.

7.1. Mirror family of pairs of points

For $n = 1$ the family $Y_t^{(0)}$ is a family of pairs of points in $\mathbb{P}^1$ given in an affine chart by the equation

$$x_1 \Big(x_1 + 1 \Big) + \frac{t}{4} = 0 \tag{7.12}$$

with $t \in \mathbb{P} \backslash \{0, 1, \infty\}$. For $n = 1$ the deformed quadratic Fermat pencil X_λ and Y_t are equivalent. That is, the family in Equation (7.12) satisfies, $X_\lambda^{(0)} \cong Y_t^{(0)}$ with $t = \lambda^2$. Moreover, if we set $x_1 = (y_0 - 1)/2$ in Equation (7.12), we obtain Equation (4.1), that is, precisely the universal starting point of our twist construction.

7.2. Mirror family of elliptic curves

For $n = 2$ the family $Y_t^{(1)}$ is equivalent to the elliptic modular surface over the rational modular curve for $\Gamma_0(3)$. In fact, using the birational transformation

$$x_1 = \frac{4\,t}{3\,(2\,X+3)}, \qquad x_2 = \frac{i\sqrt{2}\,Y - 4\,t}{6\,(2\,X+3)} - \frac{1}{2} \tag{7.13}$$

in Equation (7.2), we recover the Weierstrass normal form given by

$$Y^2 = 4X^3 - \underbrace{(27-24\,t)}_{g_2(t)}\,X - \underbrace{(27-36\,t+8\,t^2)}_{g_3(t)}. \tag{7.14}$$

Equation (7.14) is the Weierstrass model for X_{431} obtained in Lemma 6.1. Corollary 6.8 proves that the period integral of dX/Y over a suitable family of one-cycles $\Sigma_1(t)$ equals the hypergeometric function

$$\frac{\omega(t)}{2\pi i} = {}_2F_1\left(\begin{matrix}\mu, 1-\mu\\ 1\end{matrix}\middle|\, t\right) \tag{7.15}$$

with $\mu = \frac{1}{3}$. In other words, the extremal family of elliptic curves X_{431} is obtained from the family of pairs of points in Equation (4.1) using our twist construction with generalized functional invariant $(i, j, \alpha) = (2, 1, 1)$ as proved in Lemma 6.1. On the level of periods, this fact manifests as application of the cancellation formula (3.6) in the Hadamard product

$${}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\middle|\, t\right) = {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ \frac{1}{2}\end{matrix}\middle|\, t\right) \star {}_1F_0\left(\begin{matrix}\frac{1}{2}\end{matrix}\middle|\, t\right). \tag{7.16}$$

Since the transformation (7.13) maps the holomorphic one-form $3\sqrt{2}idX/Y$ to $\eta_t = dx_2/f_{2,x_1}$ in Equation (7.5), the period (7.15) is the period for the mirror cubic family.

7.3. Mirror families of K3 surfaces

For $n=3$ the family $Y_t^{(2)}$ is equivalent to a family of minimal Weierstrass models given by the equation

$$Y^2 = 4X^3 - \underbrace{g_2\left(-\frac{3^3\,t}{4^4\,u^3(u+1)}\right)\Big(u(u+1)\Big)^4}_{=g_2(t,u)} X \\ - \underbrace{g_3\left(-\frac{3^3 t}{4^4\,u^3(u+1)}\right)\Big(u(u+1)\Big)^6}_{=g_3(t,u)}, \tag{7.17}$$

where we choose for $g_2(t)$ and $g_3(t)$ the Weierstrass coefficients in Equation (7.14). This is seen by applying the birational transformation

(7.18)

$$\begin{aligned} x_1 &= -\frac{9\,(u+1)\,t}{64\,\left(3\,u^4+6\,u^3+3\,u^2+2\,X\right)},\\ x_2 &= \frac{-64\,i\,\sqrt{2}\,Y+9\,\left(64\,u^5+128\,u^4+64\,u^3+\left(3t+\frac{128}{3}\right)\,u+t\right)(u+1)}{1152\left(u^4+2\,u^3+u^4+\frac{2}{3}\,u\right)(u+1)},\\ x_3 &= -\,(u+1), \end{aligned}$$

in Equation (7.2). We have the following:

Lemma 7.4. *Equation (7.17) defines a family of M_2-polarized K3 surfaces.*

Proof. Equation (7.17) defines a family of Jacobian elliptic K3 surfaces of Picard rank 19 with a singular fiber of Kodaira-type IV^* over $u=0$, a singular fiber of Kodaira-type I_{12} over $u=\infty$, and four fibers of Kodaira-type I_1. The Mordell-Weil group is pure three-torsion generated by the sections $(X,Y)=\left(-3/2\,u^2\,(u+1)^2\,,\pm\frac{27\,i}{128}\sqrt{2}tu^2\right)$. It follows that the determinant of the discriminant group equals $3\cdot 12/3^2=2^2$. □

In other words, the family of Jacobian elliptic K3 surfaces in Equation (7.17) is the twisted family with generalized functional invariant $(i,j,\alpha)=(3,1,1)$ of the elliptic curves in Equation (7.14). Application of Proposition 5.1 together with the cancellation formula (3.6) proves that the period integral of the holomorphic two-form $du \wedge dX/Y$ over a suitable two-cycle $\Sigma_2(t)$ equals the hypergeometric function

$$\frac{\omega(t)}{(2\pi i)^2} = {}_3F_2\left(\left.\begin{array}{c}\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\\ 1, 1\end{array}\right| t\right) = {}_3F_2\left(\left.\begin{array}{c}\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\\ \frac{1}{3}, \frac{2}{3}\end{array}\right| t\right) \star {}_2F_1\left(\left.\begin{array}{c}\frac{1}{3}, \frac{2}{3}\\ 1\end{array}\right| t\right). \tag{7.19}$$

Since the transformation (7.18) maps the holomorphic two-form $3\sqrt{2}i du \wedge dX/Y$ to $\eta_t = dx_2 \wedge dx_3/f_{3,x_1}$ in Equation (7.5), the period in Equation (7.19) is the period for the mirror quartic family. We make the following:

Remark 7.5. Equation (7.17) defines a family of M_2-lattice polarized K3 surfaces Y with transcendental lattice $T(Y) = H \oplus \langle 4 \rangle$. Following Dolgachev [28] its mirror partner $Y^\vee$ is the family of generic quartic surfaces in $\mathbb{P}^3$ with $\mathrm{NS}(Y^\vee) = \langle 4 \rangle$ since $T(Y) = H \oplus \mathrm{NS}(Y^\vee)$. Equivalently, it was proved in [63] that the mirror quartic is the family of the Calabi-Yau varieties arising from the polytope P_0^* in dimension 3. The family X is the family of the Calabi-Yau varieties arising from the reflexive polytope P_0 and is the family of generic quartic surfaces in $\mathbb{P}^3$.

7.4. Mirror families of Calabi-Yau threefolds

For $n = 4$ the family $Y_t^{(3)}$ is equivalent to the family of minimal Weierstrass models given by the equation

$$\begin{aligned} Y^2 = 4\,X^3 &- \underbrace{g_2\left(\frac{3^3\,t}{5^5\,u(u+1)\,v^3\,(v+1)^2}\right)\Big(u\,(u+1)\,v\,(v+1)\Big)^4}_{=:\,g_2(t,u,v)} X \\ &- \underbrace{g_3\left(\frac{3^3\,t}{5^5\,u\,(u+1)\,v^3\,(v+1)^2}\right)\Big(u\,(u+1)\,v\,(v+1)\Big)^6}_{=:\,g_3(t,u,v)}, \end{aligned} \tag{7.20}$$

where we choose for $g_2(t)$ and $g_3(t)$ the Weierstrass coefficients in Equation (7.14). This is seen by applying the birational transformation

(7.21)

$$x_1 = \frac{36\,t\,u(u+1)}{9375\,v^2(v+1)^2u^4+18750\,v^2(v+1)^2u^3+9375\,v^2(v+1)^2u^2+6250\,X},$$

$$x_2 = \frac{-3125\,i\sqrt{2}Y+28125\left(v^3(v+1)^2u^4+2\,v^3(v+1)^2u^3+\left(v^5+2\,v^4+v^3-\frac{12\,t}{3125}\right)u^2-\frac{12\,ut}{3125}+\frac{2}{3}\,v\,X\right)(u+1)u(v+1)}{\left(56250\,v^2(v+1)^2u^4+112500\,v^2(v+1)^2u^3+56250\,v^2(v+1)^2u^2+37500\,X\right)(u+1)u(v+1)},$$

$$x_3 = u\,(v+1)\,, \qquad x_4 = -(u+1)\,(v+1),$$

in Equation (7.2). Equation (7.20) defines a family of minimal Weierstrass models over the two-dimensional base $\mathbb{P}^1 \times \mathbb{P}^1$ with affine coordinates u and v. Restricted to any generic v-slice we obtain a family of Jacobian elliptic K3 surfaces with M_3-polarization.

By inspection, the family in Equation (7.20) is obtained from the family of elliptic curves in Equation (7.14) by applying a sequence of twists, with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ first, and $(i, j, \alpha) = (3, 2, 1)$ second. Application of Proposition 5.1 and the cancellation formula (3.6) proves that the period integral of the holomorphic three-form $dv \wedge du \wedge dX/Y$ over a suitable three-cycle $\Sigma_3(t)$ equals the hypergeometric function

$$\begin{aligned}
(7.22) \qquad \frac{\omega(t)}{(2\pi i)^3} &= {}_4F_3\left(\left. \begin{array}{c} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ 1, 1, 1 \end{array} \right| t \right) \\
&= {}_4F_3\left(\left. \begin{array}{c} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \end{array} \right| t \right) \star {}_1F_0\left(\left. \frac{1}{2} \right| t \right) \star {}_2F_1\left(\left. \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \end{array} \right| t \right).
\end{aligned}$$

Since the transformation (7.21) maps the holomorphic two-form $3\sqrt{2} i dv \wedge du \wedge dX/Y$ to $\eta_t = dx_2 \wedge dx_3 \wedge dx_4 / f_{4,x_1}$ in Equation (7.5), the period in Equation (7.22) is the period for the mirror quintic family. The period in Equation (7.22) is annihilated by the fourth-order Picard-Fuchs operator $L_t^{(4)}\big((\frac{1}{5}, \dots, \frac{4}{5}); (1, \dots, 1)\big)$. The Picard-Fuchs operator is one of the 14 original Calabi-Yau operators mentioned in the introduction and labelled "(1)" in the AESZ database [2].

Remark 7.6. It was shown in [14] that the family of Calabi-Yau threefolds $Y_t^{(3)}$ has a general fiber with Hodge numbers $h^{2,1}(Y_t^{(3)}) = 1$ and $h^{1,1}(Y_t^{(3)}) = 101$. Following [14] its mirror $Y^\vee$ is the general family of quintic surfaces in $\mathbb{P}^4$ with Hodge numbers $h^{1,1}(Y^\vee) = 1$ and $h^{2,1}(Y^\vee) = 101$.

8. Combining twists and base transformations

In this section, we apply linear and quadratic transformations to the rational parameter space of the twisted families of elliptic curves, K3 surfaces, and Calabi-Yau threefolds already constructed.

8.1. Transformations of extremal families of elliptic curves

We apply the linear transformation $t \mapsto \frac{t}{t-1}$ to the rational deformation space of any extremal family of elliptic curves in Lemma 6.1 to obtain the

Weierstrass models

$$Y^2 = 4X^3 - \underbrace{g_2\left(\frac{t}{t-1}\right)(1-t)^4}_{=:\,\tilde{g}_2(t)} X - \underbrace{g_3\left(\frac{t}{t-1}\right)(1-t)^6}_{=:\,\tilde{g}_3(t)}, \tag{8.1}$$

where $g_2(t)$ and $g_3(t)$ are given in Table 5. The transformation produces isomorphic families of elliptic curves that we denote by $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, and $\tilde{X}_{211}$. They have the same number/type of singular fibers and Mordell-Weil groups as the families X_{141}, X_{431}, X_{321}, and X_{211}, but with the singular fibers over $t = 1$ and $t = \infty$ interchanged. The families X_{141}, X_{431}, X_{321}, or X_{211} (and hence $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, or $\tilde{X}_{211}$) are labelled by $\mu = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or $\mu = \frac{1}{6}$ (and by $\tilde{\mu} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or $\tilde{\mu} = \frac{1}{6}$). Let $\tilde{\Sigma}_1(t)$ be the family of one-cycles obtained from $\Sigma_1(\tilde{t})$ in Lemma 6.7 with $\tilde{t} = \frac{t}{t-1}$ by rescaling $(X, Y) \mapsto ((1-t)^2 X, (1-t)^3 Y)$. For $t \in \mathbb{C}$ with $|\tilde{t}| < 1/2$, $\tilde{\Sigma}_1(t)$ defines a family of A-cycles on $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, and $\tilde{X}_{211}$ in the neighborhood of $t = 0$. We have the following:

Corollary 8.1. *For the families of elliptic curves $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, and $\tilde{X}_{211}$ in Equation (8.1), period integrals of dX/Y are annihilated by the Picard-Fuchs operator*

$$\tilde{L}_t^{(2)} = \theta^2 - t\left(2\,\theta^2 + 2\,\theta + \tilde{\mu}^2 - \tilde{\mu} + 1\right) + t^2\left(\theta + 1\right)^2. \tag{8.2}$$

In particular, the period over $\tilde{\Sigma}_1(t)$ is holomorphic at $t = 0$ and given by

$$\tilde{\omega} = \frac{2\pi i}{(1-t)^{1-\tilde{\mu}}}\ {}_2F_1\left(\left.\begin{array}{c}\tilde{\mu}, \tilde{\mu}\\ 1\end{array}\right| \frac{t}{t-1}\right) \tag{8.3}$$

with $\tilde{\mu} \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$.

Proof. The proof follows from the following well-known identity for the Gauss hypergeometric function, namely

$$\frac{2\pi i}{1-t}\ {}_2F_1\left(\left.\begin{array}{c}\tilde{\mu}, 1-\tilde{\mu}\\ 1\end{array}\right| \frac{t}{t-1}\right) = \frac{2\pi i}{(1-t)^{1-\tilde{\mu}}}\ {}_2F_1\left(\left.\begin{array}{c}\tilde{\mu}, \tilde{\mu}\\ 1\end{array}\right| \frac{t}{t-1}\right).$$

□

Remark 8.2. For the above families of elliptic curves, twisted families can be constructed as in Section 6.2. The twisted families with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ of $\tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, and $\tilde{X}_{211}$ are families of M_n-lattice polarized K3 surfaces. A continuously varying family of closed two-cycles $\tilde{\Sigma}_2(t)$ can be constructed in each case such that the period over $\tilde{\Sigma}_2(t)$ is given by

$$\tilde{\omega} = (2\pi i)^2 \, {}_1F_0\left(\begin{array}{c} \frac{1}{2} \end{array}\middle| t\right) \star \left(\frac{1}{1-t} \, {}_2F_1\left(\begin{array}{c} \tilde{\mu}, 1-\tilde{\mu} \\ 1 \end{array}\middle| \frac{t}{t-1}\right)\right) \tag{8.4}$$

with $\tilde{\mu} \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$.

8.2. Transformations of lattice polarized K3 surfaces

We apply a linear or quadratic transformations, denoted by $t \mapsto f_k(t)$ with $k = 1, \ldots, 5$, to the rational parameter space of the families of M_n-lattice polarized K3 surfaces in Lemma 6.1 to obtain new Weierstrass models given by

$$Y^2 = 4X^3 - \underbrace{g_2\big(f_k(t), u\big)\, h_k(t)^2}_{=: \tilde{g}_2^{(k)}(t,u)} X - \underbrace{g_3\big(f_k(t), u\big)\, h_k(t)^3}_{=: \tilde{g}_3^{(k)}(t,u)}, \tag{8.5}$$

where $g_2(t,u)$ and $g_3(t,u)$ are given in Lemma 6.5, and the polynomials $f_k(t)$ and $h_k(t)$ are given in Table 11. It is readily checked that $\tilde{g}_2^{(k)}(t,u)$ and $\tilde{g}_3^{(k)}(t,u)$ define families of minimal Weierstrass model. By construction, these new families remain families of M_n-lattice polarized K3 surfaces.

Let $\tilde{\Sigma}_2(t)$ be the family of two-cycles obtained from the family $\Sigma_2(f_k(t))$ — where $\Sigma_2(t)$ was given in Section 6.2 — by rescaling

$$(u, X, Y) \mapsto (u, h_k(t)X, h_k(t)^{3/2}Y).$$

For $t \in \mathbb{C}$ with $|f_k(t)| < 1/2$, $\tilde{\Sigma}_2(t)$ defines a continuously varying family of two-cycles. We have the following:

Corollary 8.3. *For the families of elliptic K3 surfaces in Equation (8.5), the period integrals of $du \wedge dX/Y$ are annihilated by the Picard-Fuchs operator*

$$\begin{aligned} \tilde{L}_t^{(3)} = \theta^3 &- t\,(2\,\theta+1)\,\big(\theta^2+\theta+2pq-p-q+1\big) \\ &+ t^2\,(\theta+1)\,(\theta+1+q-p)\,(\theta+1-q+p)\,. \end{aligned} \tag{8.6}$$

In particular, the period over $\tilde{\Sigma}_2(t)$ is holomorphic at $t = 0$ and given by

$$\tilde{\omega} = (2\pi i)^2 \left(\frac{1}{(1-t)^{\frac{1-p-q}{2}}} \, {}_2F_1\left(\begin{array}{c} p, q \\ 1 \end{array}\middle| t\right)\right)^2, \tag{8.7}$$

k / (p,q)	$f_k(t)$ / $h_k(t)$	$\tilde{\omega}/(2\pi i)^2$
1	t	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, t\right)$
$\left(\frac{\mu}{2}, \frac{1-\mu}{2}\right)$	$1-t$	$= \left(\frac{1}{(1-t)^{\frac{1-\mu/2-(1-\mu)/2}{2}}}\, {}_2F_1\left(\frac{\mu}{2}, \frac{1-\mu}{2}; 1 \middle\vert\, t\right)\right)^2$
2	$\frac{t}{t-1}$	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, \frac{t}{t-1}\right)$
$\left(\frac{\mu}{2}, \frac{1+\mu}{2}\right)$	$1-t$	$= \left(\frac{1}{(1-t)^{\frac{1-\mu/2-(1+\mu)/2}{2}}}\, {}_2F_1\left(\frac{\mu}{2}, \frac{1+\mu}{2}; 1 \middle\vert\, t\right)\right)^2$
3	$4t(1-t)$	${}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, 4\,t\,(1-t)\right)$
$(\mu, 1-\mu)$	1	$= \left(\frac{1}{(1-t)^{\frac{1-\mu-(1-\mu)}{2}}}\, {}_2F_1\left(\mu, 1-\mu; 1 \vert\, t\right)\right)^2$
4	$\frac{t^2}{4(t-1)}$	$\frac{1}{\sqrt{1-t}}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, \frac{t^2}{4(t-1)}\right)$
$\left(\mu, \frac{1}{2}\right)$	$1-t$	$= \left(\frac{1}{(1-t)^{\frac{1-\mu-1/2}{2}}}\, {}_2F_1\left(\mu, \frac{1}{2}; 1 \vert\, t\right)\right)^2$
5	$-\frac{4\,t}{(1-t)^2}$	$\frac{1}{1-t}\, {}_3F_2\left(\mu, 1-\mu, \frac{1}{2}; 1, 1 \middle\vert\, -\frac{4\,t}{(1-t)^2}\right)$
(μ, μ)	$(1-t)^2$	$= \left(\frac{1}{(1-t)^{\frac{1-\mu-\mu}{2}}}\, {}_2F_1\left(\mu, \mu; 1 \vert\, t\right)\right)^2$

Table 11: Rational transformations and periods of new family.

where μ and (p, q) for $k = 1, \ldots, 5$ are given in Table 11.

Proof. The construction is an application of the general construction in Section 5.4. The proof amounts to checking some classical and well-known hypergeometric function identities listed in Table 11. The identities allow us to write each period as a symmetric square. □

8.3. Threefolds by combining twists and base transformations

To obtain families of elliptic Calabi-Yau threefolds, we start with a family of Jacobian elliptic K3 surfaces $X \to \mathbb{P}^1$, given as Weierstrass model

$$Y^2 = 4X^3 - g_2(t,u)\,X - g_3(t,u). \tag{8.8}$$

We will restrict ourselves to the cases where this K3 surface is chosen from Section 6.2, Section 8.2, or Remark 8.2. Applying our twist construction, we obtain new Weierstrass models for twisted families with generalized functional invariant (k,l,β) with $1 \le k,l \le 6$, $\beta \in \{\frac{1}{2},1\}$, and $c_{kl} = (-1)^k\,k^k\,l^l/(k+l)^{k+l}$ that are families of Gorenstein threefolds. We have the following:

Lemma 8.4. *For every family of elliptic K3 surfaces from Section 6.2, Section 8.2, or Remark 8.2, the twisted family with generalized functional invariant (k,l,β), given by the Weierstrass equation*

$$\begin{aligned} Y^2 = 4X^3 - \underbrace{g_2\left(\frac{c_{kl}t}{v^k(v+1)^l}, u\right)\, v^4(v+1)^{4\beta}}_{=:\, g_2(t,u,v)} X \\ - \underbrace{g_3\left(\frac{c_{kl}t}{v^k(v+1)^l}, u\right)\, v^6(v+1)^{6\beta}}_{=:\, g_3(t,u,v)}, \end{aligned} \tag{8.9}$$

defines a family (over B) of Jacobian elliptic Calabi-Yau threefolds over $\mathbb{P}^1 \times \mathbb{P}^1$. For K3 surfaces from Lemma 6.9, we assumed $1 \le k \le 1/\mu$ and $1 \le l \le \beta/\mu$ with $\beta \in \{\frac{1}{2},1\}$, and for K3 surfaces from Section 8.2 or Remark 8.2 we set $(k,l,\beta) = (1,1,1)$.

Proof. The construction is an application of the general construction in Section 5.4. □

For each twisted family in Equation (8.9), we define a family of closed three-cycles $\Sigma_3(t)$ as follows: For $t \in \mathbb{C}$ with $|t| < 1/(2^{k+l+1}|c_{kl}|)$, we start with the circle $C = C_{1/2}(0)$, given by $|v| = \frac{1}{2}$ in the v-plane with counterclockwise orientation. For every $v \in C$, a two-cycle $\Sigma_2'(t,v)$ in the K3-fiber is obtained from $\Sigma_2(\frac{c_{kl}t}{v^k(v+1)^l})$, where $\Sigma_2(t)$ was defined in Section 6.2, by rescaling $(u,X,Y) \to (u, v^2(v+1)^{2\beta}X, v^3(v+1)^{3\beta}Y)$. For $t \in \mathbb{C}$ with $|t| < 1/(2^{i+j+1}|c_{ij}|)$, we obtain a continuously varying family of closed three-cycles as a warped product $\Sigma_3(t) = C \times_v \Sigma_2'(t,v)$.

8.3.1. Calabi-Yau operators of the hypergeometric case. Applying our twist construction to the elliptic K3 surfaces from Section 6.2, we obtain the following:

Corollary 8.5. *For the twisted families in Lemma 8.4 with generalized functional invariant (k, l, β) of the M_n-lattice polarized K3 surfaces from Section 6.2, the period integral (5.13) is annihilated by the Picard-Fuchs operator*

$$ {}_1L_t^{(4)}(p,q) = \theta^4 - t\,\big(\theta+p\big)\big(\theta+q\big)\big(\theta+1-q\big)\big(\theta+1-p\big). \tag{8.10} $$

In particular, the period over $\Sigma_3(t)$ is holomorphic at $t=0$ and given by

$$ \omega = (2\pi i)^3 \; {}_4F_3\left(\left. \begin{array}{c} p,q,1-q,1-p \\ 1,1,1 \end{array} \right| t \right). \tag{8.11} $$

The values (p,q) resulting from a twist with generalized functional invariant (k, l, β) of a family M_n-lattice polarized K3 surface with $1 \le n \le 4$ are given in Table 12.

Proof. The proof follows by applying Equation (5.14) to the periods $\omega(t)$ computed in Corollary 6.11. One then checks for which generalized functional invariants (k, l, β) within the range provided by Lemma 8.4, the Hadamard product in Proposition 5.1 produces a hypergeometric function of type ${}_4F_3$ using Equation (3.6). □

Twisted families of the Jacobian elliptic K3 surfaces from Section 6.3 can also be obtained from generalized functional invariants $(k, l, \beta) = (\frac{m}{2}, \frac{m}{2}, 1)$ in Equation (8.9) where m is an odd integer. In fact, if we set

$$ v = -\frac{1}{1+\tilde{v}^2}, \quad X = \frac{\tilde{v}^2 \tilde{X}}{(1+\tilde{v}^2)^4}, \quad Y = \frac{\tilde{v}^3 \tilde{Y}}{(1+\tilde{v}^2)^6}, \tag{8.12} $$

we obtain $dv \wedge du \wedge dX/Y = 2d\tilde{v} \wedge du \wedge d\tilde{X}/\tilde{Y}$, and Equation (8.9) becomes the minimal and normal Weierstrass model given by

$$ \tilde{Y}^2 = 4\tilde{X}^3 - \underbrace{g_2\left(\frac{t(1+\tilde{v}^2)^m}{(2\tilde{v})^m}, u\right)\tilde{v}^4}_{=:\tilde{g}_2(t,u,\tilde{v})} \tilde{X} - \underbrace{g_3\left(\frac{t(1+\tilde{v}^2)^m}{(2\tilde{v})^m}, u\right)\tilde{v}^6}_{=:\tilde{g}_3(t,u,\tilde{v})}. \tag{8.13} $$

We have the following:

Lemma 8.6. *For every family of elliptic K3 surfaces from Section 6.3, the twisted family with generalized functional invariant* $(k,l,\beta)=(\frac{m}{2},\frac{m}{2},1)$, *given by the Weierstrass equation (8.13), defines a family of elliptic Calabi-Yau threefolds over* $\mathbb{P}^1\times\mathbb{P}^1$. *Here, we set* $m=1$, *except for the family of* M'*-lattice polarized K3 surfaces in Lemma 6.10 where we have* $m\in\{1,3\}$.

The construction of a continuously varying family of closed three-cycles $\tilde{\Sigma}_3(t)$ is analogous to the construction of $\Sigma_3(t)$ above. We also have the following:

Corollary 8.7. *For the twisted families in Lemma 8.13 with generalized functional invariant* $(k,l,\beta)=(\frac{m}{2},\frac{m}{2},1)$ *of the elliptic K3 surfaces from Section 6.3, the period integral of* $d\tilde{v}\wedge du\wedge d\tilde{X}/\tilde{Y}$ *is annihilated by the Picard-Fuchs operator*

$$ {}_1L^{(4)}_{t^2}(p,q)=\theta^4-t^2\,\big(\theta+2p\big)\,\big(\theta+2q\big)\,\big(\theta+2-q\big)\,\big(\theta+2-p\big). \tag{8.14}$$

In particular, the period over $\tilde{\Sigma}_3(t)$ *is holomorphic at* $t=0$ *and given by*

$$\tilde{\omega}=(2\pi i)^3\,{}_4F_3\left(\left.\begin{array}{c}p,q,1-q,1-p\\1,1,1\end{array}\right|t^2\right), \tag{8.15}$$

for the values (p,q) *given in Table 12.*

Proof. We evaluate the period of the holomorphic three-from $d\tilde{v}\wedge du\wedge d\tilde{X}/\tilde{Y}$ over $\tilde{\Sigma}_3(t)$ by a residue computation. By construction of $\tilde{\Sigma}_3(t)$, it follows that for $|t|<1$ and $(\tilde{v},u,\tilde{X},\tilde{Y})\in\tilde{\Sigma}_3(t)$ we have

$$\left|\frac{t\,(1+\tilde{v}^2)^m}{(2\tilde{v})^m}\,\left(1+\frac{1}{2\,u\,(u+1)}\right)\right|,\;\left|\frac{t\,(1+\tilde{v}^2)^m}{(2\tilde{v})^m}\right|<1.$$

Using Corollary 6.14, we obtain the period integral from the following computation

(8.16)

$$\begin{aligned}\frac{\tilde{\omega}}{(2\pi i)^3}&==\frac{1}{2\pi i}\oint_{C_{1/2}(0)}\frac{d\tilde{v}}{\tilde{v}}\,{}_4F_3\left(\left.\begin{array}{c}\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2},1-\frac{\mu}{2}\\1,1,\frac{1}{2}\end{array}\right|\frac{t^2\,(1+\tilde{v}^2)^{2m}}{(2\tilde{v})^{2m}}\right)\\&={}_{4+m}F_{3+m}\left(\left.\begin{array}{c}\frac{\mu}{2},\frac{1-\mu}{2},\frac{1+\mu}{2},1-\frac{\mu}{2},\frac{1}{2m},\frac{3}{2m},\dots,\frac{2m-1}{2m}\\\frac{1}{2},\frac{1}{m},\frac{2}{m},\dots,\frac{m}{m},1,1\end{array}\right|t^2\right).\end{aligned}$$

We observe that for the given parameters m,μ there is a cancellation in the coefficients of the hypergeometric series, and we obtain Equation (8.15). For

$m = 1$, or $m = 3, \mu = \frac{1}{3}$, the hypergeometric series reduce to

$$ {}_4F_3\left(\begin{matrix} \frac{\mu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, 1-\frac{\mu}{2} \\ 1,1,1 \end{matrix} \middle| t^2 \right), \quad \text{or} \quad {}_4F_3\left(\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1,1,1 \end{matrix} \middle| t^2 \right). $$

□

Remark 8.8. The Calabi-Yau operators (8.10) obtained in Corollary 8.5 and 8.7 for parameters $(p, q, q' = 1-q, p' = 1-p)$, with their classification number in the AESZ database [2], are summarized in Table 12. The Calabi-Yau operators have degree one and are called Calabi-Yau operators of the hypergeometric case. In particular, Table 12 includes the generalized functional invariants that were found in [31] to construct threefolds fibered by M_n-polarized K3 surfaces using toric geometry.

Remark 8.9. It was proved in [34] that the regular singular points $t = 0, 1, \infty$ of the Picard-Fuchs operator in Equation (8.10) correspond to the conifold limit, large complex structure limit, and the orbifold point, respectively. In particular, the monodromy around $t = \infty$ is maximally unipotent.

8.3.2. Calabi-Yau operators in the extra case. Applying our twist construction to the elliptic K3 surfaces in Corollary 8.3, we obtain the following:

Corollary 8.10. *For the twisted families in Equation (8.9) with generalized functional invariant $(k, l, \beta) = (1, 1, 1)$ of the elliptic K3 surfaces in Corollary 8.3, the period integral (5.13) is annihilated by the Picard-Fuchs operator*

$$ \begin{aligned} (8.17)\quad {}_2L_t^{(4)}(p,q) = \theta^4 &- 2t\left(\theta+\frac{1}{2}\right)^2\left(\theta^2+\theta+2pq-p-q+1\right) \\ &+ t^2\left(\theta+\frac{1}{2}\right)\left(\theta+\frac{3}{2}\right)\left(\theta+1+p-q\right)\left(\theta+1-p+q\right). \end{aligned} $$

In particular, the period over $\Sigma_3(t)$ is holomorphic at $t = 0$ and given by

$$ (8.18)\qquad \omega = (2\pi i)^3\, {}_1F_0\left(\frac{1}{2} \middle| t \right) \star \left(\frac{1}{(1-t)^{\frac{1-p-q}{2}}}\, {}_2F_1\left(\begin{matrix} p, q \\ 1 \end{matrix} \middle| t \right) \right)^2, $$

for the values (p, q) given in Table 13.

AESZ	(p,q)	twist with $(k,l,\beta=1)$				twist with $(k,l,\beta=\frac{1}{2})$				twist with $(\frac{m}{2},\frac{m}{2},1)$			
		M_4	M_3	M_2	M_1	M_4	M_3	M_2	M_1	$\tilde{M}'$	M'	$\tilde{M}$	M
(3)	$(\frac{1}{2},\frac{1}{2})$	$(1,1)$											
(5)	$(\frac{1}{3},\frac{1}{2})$	$(2,1)$	$(1,1)$										
(4)	$(\frac{1}{3},\frac{1}{3})$		$(2,1)$										
(6)	$(\frac{1}{4},\frac{1}{2})$	$(2,2)$	$(3,1)$	$(1,1)$		$(1,1)$							
(11)	$(\frac{1}{4},\frac{1}{3})$		$(2,2)$	$(2,1)$			$(1,1)$						
(10)	$(\frac{1}{4},\frac{1}{4})$			$(2,2)$				$(1,1)$		(1)			
(14)	$(\frac{1}{6},\frac{1}{2})$		$(3,3)$		$(1,1)$	$(2,1)$		$(1,2)$					
(8)	$(\frac{1}{6},\frac{1}{3})$			$(4,2)$	$(2,1)$		$(2,1)$				(1)		
(12)	$(\frac{1}{6},\frac{1}{4})$				$(2,2)$			$(2,1)$	$(1,1)$				
(13)	$(\frac{1}{6},\frac{1}{6})$								$(2,1)$		(3)		
(1)	$(\frac{1}{5},\frac{2}{5})$		$(3,2)$	$(4,1)$									
(7)	$(\frac{1}{8},\frac{3}{8})$			$(4,4)$			$(3,1)$	$(2,2)$	$(1,3)$			(1)	
(2)	$(\frac{1}{10},\frac{3}{10})$							$(4,1)$	$(2,3)$				
(9)	$(\frac{1}{12},\frac{5}{12})$							$(4,2)$					(1)

Table 12: Twist parameters for operators ${}_1L_t^{(4)}(p,q)$ in the 'hypergeometric case'.

Proof. The proof follows by applying Equation (5.14) to the periods $\omega(t)$ computed in Corollary 8.3. □

Remark 8.11. The Calabi-Yau operators (8.17) obtained in Corollary 8.10 for parameters (p,q), with classification number (and any alternative name used) in the AESZ database [2], are summarized in Table 13. The Calabi-Yau operators are called Calabi-Yau operators of the extra case.

8.3.3. Calabi-Yau operators in the even case. Applying our twist construction to the elliptic K3 surfaces in Remark 8.2, we obtain the following:

#	AESZ	Name	(p,q)	twist of K3 with (M_n,k)
1	(17)	$35, 3^*$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(M_4,3),(M_4,4),(M_4,5)$
2	–	–	$\left(\frac{1}{3},\frac{1}{2}\right)$	$(M_3,4)$
3	(66)	6^*	$\left(\frac{1}{4},\frac{1}{2}\right)$	$(M_2,4)$
4	–	14^*	$\left(\frac{1}{6},\frac{1}{2}\right)$	$(M_1,4)$
5	(39)	4^*	$\left(\frac{1}{3},\frac{1}{3}\right)$	$(M_3,5)$
6	(20)	$46, 4^{**}$	$\left(\frac{1}{3},\frac{2}{3}\right)$	$(M_3,3)$
7	(45)	8^*	$\left(\frac{1}{6},\frac{1}{3}\right)$	$(M_3,1)$
8	(34)	8^{**}	$\left(\frac{1}{6},\frac{2}{3}\right)$	$(M_3,2)$
9	(38)	10^*	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(M_2,5),(M_4,1)$
10	(32)	$111, 10^{**}$	$\left(\frac{1}{4},\frac{3}{4}\right)$	$(M_2,3),(M_4,2)$
11	(40)	13^*	$\left(\frac{1}{6},\frac{1}{6}\right)$	$(M_1,5)$
12	(21)	$47, 13^{**}$	$\left(\frac{1}{6},\frac{5}{6}\right)$	$(M_1,3)$
13	(44)	7^*	$\left(\frac{1}{8},\frac{3}{8}\right)$	$(M_2,1)$
14	(41)	7^{**}	$\left(\frac{1}{8},\frac{5}{8}\right)$	$(M_2,2)$
15	(43)	9^*	$\left(\frac{1}{12},\frac{5}{12}\right)$	$(M_1,1)$
16	(42)	9^{**}	$\left(\frac{1}{12},\frac{7}{12}\right)$	$(M_1,2)$

Table 13: Twist parameters for operators ${}_2L_t^{(4)}(p,q)$ in the 'extra case'.

Corollary 8.12. *For the twisted families in Lemma 8.9 with generalized functional invariant $(k,l,\beta)=(1,1,1)$ of the M_n-lattice polarized K3 surfaces in Remark 8.2, the period integral (5.13) is annihilated by the Picard-Fuchs operator*

$$(8.19)\qquad {}_3L_t^{(4)}\big(\mu,\tilde\mu\big) = \theta^4 - t\,\left(2\,\theta^2+2\theta+\tilde\mu^2-\tilde\mu+1\right)(\theta+\mu)\,(\theta-\mu+1) + t^2\,\left(\theta+2-\mu\right)(\theta+1+\mu)\,(\theta+\mu)\,(\theta+1-\mu)$$

with $\mu=\frac{1}{2}$ and $\tilde\mu\in\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$. In particular, the period over $\Sigma_3(t)$ is holomorphic at $t=0$ and given by

$$(8.20)\qquad \omega=(2\pi i)^3\;{}_2F_1\left(\left.\begin{array}{c}\frac{1}{2},\frac{1}{2}\\ 1\end{array}\right| t\right)\star\left(\frac{1}{1-t}\;{}_2F_1\left(\left.\begin{array}{c}\tilde\mu,1-\tilde\mu\\ 1\end{array}\right|\frac{t}{t-1}\right)\right).$$

Proof. The proof follows by applying Equation (5.14) to the periods $\omega(t)$ computed in Remark 8.2, and then using Equation (3.6). $\square$

To obtain the Calabi-Yau operators in Equation (8.19) with $\mu = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ as Picard-Fuchs operators, we use the variant of our twist construction in Section 5.5.3. For the families of elliptic curves $X = \tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, $\tilde{X}_{211}$, and X'_k in Table 2, we already constructed families of A-cycles $\tilde{\Sigma}_1(t)$ and $\Sigma_1(t)$, respectively, for $|t| < 1$.

For the twisted family in Equation (5.30), we define a family of closed three-cycles $\hat{\Sigma}_3(t)$ as follows: Applying Lemma 6.7 to the elliptic curve $h^2 = 4u^3 - g'_2(v)u - g'_3(v)$, we obtain a family of A-cycles $\Sigma_1(v) \ni (u, h)$, such that $\Sigma_1(v)$ projects onto the circle $C_u = C_{1/2}(0)$, i.e., the circle $|u| = \frac{1}{2}$ in the u-plane with counterclockwise orientation, for every $|v| < 1$. For $t \in \mathbb{C}$ and every $(v, u) \in C_v \times C_u$, a cycle $\hat{\Sigma}_1(t, v, u)$ in the elliptic fiber of Equation (5.30) is obtained from $\hat{\Sigma}_1(\frac{t}{v})$, by rescaling $(X, Y) \to (h^2v^2X, h^3v^3Y)$ such that $(u, h) \in \Sigma_1(v)$. For $t \in \mathbb{C}$ with $|t| < 1/2$, we obtain a continuously varying family of closed three-cycles as a warped product $\hat{\Sigma}_3(t) = C_v \times C_u \times_{(v,u)} \hat{\Sigma}_1(t, v, u)$. We have the following:

Corollary 8.13. *For $X = \tilde{X}_{141}$, $\tilde{X}_{431}$, $\tilde{X}_{321}$, or $\tilde{X}_{211}$, the twist family of X with X'_k in Table 2 given by Equation (5.30) for $k = 2, 3, 4$, is a family over B of Jacobian elliptic Calabi-Yau threefolds over $\mathbb{F}_n$ with $n = 0, \ldots, k$. The period integral (5.33) is annihilated by the Picard-Fuchs operator ${}_3L_t^{(4)}(\mu, \tilde{\mu})$ in Equation (8.19). In particular, the period over $\hat{\Sigma}_3(t)$ is holomorphic at $t = 0$ and given by*

$$\hat{\omega} = (2\pi i)^3 \, {}_2F_1\left(\begin{matrix} \mu, 1-\mu \\ 1 \end{matrix} \middle| \, t\right) \star \left(\frac{1}{1-t} \, {}_2F_1\left(\begin{matrix} \tilde{\mu}, 1-\tilde{\mu} \\ 1 \end{matrix} \middle| \, \frac{t}{t-1}\right)\right) \tag{8.21}$$

with $\mu \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ and $\tilde{\mu} \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$.

Proof. The proof follows by applying Proposition 5.7 and Equation (5.34) to the periods $\omega(t)$ computed in Corollary 8.1. $\square$

Remark 8.14. The Calabi-Yau operators (8.19) obtained in Corollary 8.12 and Corollary 8.13 for parameters $(\mu, \tilde{\mu})$, with their classification number (and any alternative name used) in the AESZ database [2], are summarized in Table 14. The Calabi-Yau operators are called Calabi-Yau operators of the even case.

#	AESZ	Name	$(\mu, \tilde{\mu})$
1	(32)	111	$\left(\frac{1}{2}, \frac{1}{2}\right)$
2	(31)	110	$\left(\frac{1}{3}, \frac{1}{2}\right)$
3	(15)	30	$\left(\frac{1}{4}, \frac{1}{2}\right)$
4	(33)	112	$\left(\frac{1}{6}, \frac{1}{2}\right)$
5	(34)	141, 8**	$\left(\frac{1}{2}, \frac{1}{3}\right)$
6	(35)	142	$\left(\frac{1}{3}, \frac{1}{3}\right)$
7	-	196	$\left(\frac{1}{4}, \frac{1}{3}\right)$
8	(36)	143	$\left(\frac{1}{6}, \frac{1}{3}\right)$

#	AESZ	Name	$(\mu, \tilde{\mu})$
9	(41)	189, 7**	$\left(\frac{1}{2}, \frac{1}{4}\right)$
10	(46)	194	$\left(\frac{1}{3}, \frac{1}{4}\right)$
11	(48)	197	$\left(\frac{1}{4}, \frac{1}{4}\right)$
12	(50)	199	$\left(\frac{1}{6}, \frac{1}{4}\right)$
13	(42)	190, 9**	$\left(\frac{1}{2}, \frac{1}{6}\right)$
14	(47)	195	$\left(\frac{1}{3}, \frac{1}{6}\right)$
15	(49)	198	$\left(\frac{1}{4}, \frac{1}{6}\right)$
16	(23)	61	$\left(\frac{1}{6}, \frac{1}{6}\right)$

Table 14: Twist parameters for the operators ${}_3L_t^{(4)}(p,q)$ in the 'even case'.

8.4. Calabi-Yau operators in the odd case

In this section we describe a fourth step in our iterative construction that produces families of (singular) elliptic Calabi-Yau fourfolds with section over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that realize all 14 one-parameter variations of Hodge structure of weight four and type $(1,1,1,1,1)$ over a one-dimensional rational deformation space of the so-called odd case. The families arise as twisted families of the elliptic Calabi-Yau threefolds of the hypergeometric case, previously obtained in Section 8.3.1. Applying our twist construction, we obtain their Weierstrass model as twisted families with generalized functional invariant $(m,n,\gamma) = (1,1,1)$. We have the following:

Lemma 8.15. *For every family of threefolds from Section 8.3.1, the twisted family with generalized functional invariant* $(1,1,1)$*, given by the Weierstrass equation*

$$Y^2 = 4X^3 - \underbrace{g_2\left(-\frac{t}{w(w+1)}, u, v\right)\, v^4(v+1)^4}_{=:\, g_2(t,u,v,w)} X - \underbrace{g_3\left(-\frac{t}{w(w+1)}, u\right)\, w^6(w+1)^6}_{=:\, g_3(t,u,v,w)}, \tag{8.22}$$

defines a family (over B) of Jacobian elliptic Calabi-Yau fourfolds over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The construction is an application of the general construction in Section 5.4. One first checks that Equation (8.22) defines a minimal Weierstrass model for every family of threefolds from Section 8.3.1 with affine coordinates $u, v, w \in \mathbb{C}$ and $t \in B = \mathbb{P}^1 \backslash \{0, 1, \infty\}$. The Weierstrass equation (8.22) extends to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, since we obtain a minimal and normal Weierstrass equation when introducing projective variables $[u_0 : u_1] \in \mathbb{P}^1$, $[v_0 : v_1] \in \mathbb{P}^1$, $[w_0 : w_1] \in \mathbb{P}^1$, and $[x : y : z] \in \mathbb{P}(2, 3, 1)$ and writing each fiber as the hypersurface

$$\begin{aligned} y^2 z = 4x^3 &- g_2\left(t, \frac{u_0}{u_1}, \frac{v_0}{v_1}, \frac{w_0}{w_1}\right) u_1^8 v_1^8 w_1^8 x z^2 \\ &- g_3\left(t, \frac{u_0}{u_1}, \frac{v_0}{v_1}, \frac{w_0}{w_1}\right) u_1^{12} v_1^{12} w_1^{12} z^3. \end{aligned} \tag{8.23}$$

Four $\mathbb{C}^*$-groups act on the defining variables in Equation (8.23) and are given by the weights listed in Table 15 where *deg* denotes the total weight of Equation (8.23) and *sum* denotes the sum of weights of the defining variables. Since the conditions are satisfied that for each $\mathbb{C}^*$-group the total weight equals the sum of weights, a Calabi-Yau fourfold is obtained by removing the loci $\{s_0 = s_1 = 0\}$, $\{u_0 = u_1 = 0\}$, $\{v_0 = v_1 = 0\}$, $\{x = y = z = 0\}$ from the solution set of Equation (8.23) and taking the quotient $(\mathbb{C}^*)^4$. $\square$

$\mathbb{C}^*$	deg	x	y	z	u_0	u_1	v_0	v_1	w_0	w_1	Σ
λ_1	3	1	1	1	0	0	0	0	0	0	3
λ_2	12	4	6	0	1	1	0	0	0	0	12
λ_3	12	4	6	0	0	0	1	1	0	0	12
λ_4	12	4	6	0	0	0	0	0	1	1	12

Table 15: Weights of variables in Weierstrass equation.

For each twisted family in Equation (8.22), we define a family of closed four-cycles $\Sigma_4(t)$ as follows: For $t \in \mathbb{C}$ with $|t| < 1/2$, we start with the circle $C = C_{1/2}(0)$, given by $|w| = \frac{1}{2}$ in the w-plane with counterclockwise orientation. For every $w \in C$, a three-cycle $\Sigma_3'(t, v)$ in the fiber is obtained from $\Sigma_3(-\frac{t}{w(w+1)})$, $\Sigma_3(t)$ was defined in Section 8.3, by rescaling $(u, v, X, Y) \to (u, v, w^2(w+1)^2 X, w^3(w+1)^3 Y)$. For $t \in \mathbb{C}$ with $|t| < 1/2$, we obtain a continuously varying family of closed four-cycles as a warped product $\Sigma_4(t) = C \times_w \Sigma_3'(t, w)$. We have the following:

Corollary 8.16. *For the twisted families in Lemma 8.15 with generalized functional invariant* $(1, 1, 1)$*, the period integral (5.13) is annihilated by the self-adjoint, rank-five Picard-Fuchs operator*

$$L_t^{(5)}(p,q) = \theta^5 - t\Big(\theta + \frac{1}{2}\Big)\Big(\theta + p\Big)\Big(\theta + q\Big)\Big(\theta + 1 - p\Big)\Big(\theta + 1 - q\Big). \tag{8.24}$$

In particular, the period over $\Sigma_4(t)$ *is holomorphic at* $t = 0$ *and given by*

$$\omega = (2\pi i)^4 \; {}_5F_4\left(\begin{array}{c} p, q, \frac{1}{2}, 1-q, 1-p \\ 1, 1, 1, 1 \end{array} \middle| \, t \right) \tag{8.25}$$

for the values (p, q) *given in Table 12.*

Proof. The proof follows by applying Equation (5.14) to the periods $\omega(t)$ computed in Corollary 8.5. One then checks that the Hadamard product in Proposition 5.1 produces a hypergeometric function of type ${}_5F_4$ using Equation (3.6). □

We have the following:

Corollary 8.17. *The differential operators* ${}_4L_t^{(p,q)}$*, given by*

$$\begin{aligned} {}_4L_t^{(4)}(p,q) = \theta^4 - \frac{1}{4}\, t \Big(& 8\,\theta^4 + 16\,\theta^3 - 2\,(p^2 + q^2 - p - q - 9)\,\theta^2 \\ & - 2\,(p^2 + q^2 - p - q - 5)\,\theta + 2 + p + q \\ & - pq - p^2 - q^2 + p^2 q + p\,q^2 + p^2 q^2 \Big) \\ + \frac{1}{16}\, t^2 \; & (2\,\theta + 2 + p - q)\,(2\,\theta + 1 + p + q) \\ \times & (2\,\theta + 2 - p + q)\,(2\,\theta + 3 - p - q)\,, \end{aligned} \tag{8.26}$$

are the Yifan-Yang pullbacks of the operators $L_t^{(5)}(p,q)$ *in Corollary 8.16 of minimal degree (in* t*), for the values* (p, q) *given in Table 16.*

Proof. The proof follows directly from Proposition 3.16. □

Remark 8.18. The Calabi-Yau operators (8.26) obtained in Corollary 8.17 for parameters (p, q), with their classification number (and any alternative name used) in the AESZ database [2], are summarized in Table 16. The Calabi-Yau operators are called Calabi-Yau operators of the odd case.

#	AESZ	Name	(p,q)
1	(51)	$\tilde{3}, 204$	$\left(\frac{1}{2},\frac{1}{2}\right)$
2	(92)	$\tilde{5}$	$\left(\frac{1}{3},\frac{1}{2}\right)$
3	(91)	$\tilde{4}$	$\left(\frac{1}{3},\frac{1}{3}\right)$
4	(93)	$\tilde{6}$	$\left(\frac{1}{4},\frac{1}{2}\right)$
5	(98)	$\tilde{11}$	$\left(\frac{1}{4},\frac{1}{3}\right)$
6	(97)	$\tilde{10}$	$\left(\frac{1}{4},\frac{1}{4}\right)$
7	(101)	$\tilde{14}$	$\left(\frac{1}{6},\frac{1}{2}\right)$
8	(95)	$\tilde{8}$	$\left(\frac{1}{6},\frac{1}{3}\right)$
9	(99)	$\tilde{12}$	$\left(\frac{1}{6},\frac{1}{4}\right)$
10	(100)	$\tilde{13}$	$\left(\frac{1}{6},\frac{1}{6}\right)$
11	(89)	$\tilde{1}$	$\left(\frac{1}{5},\frac{2}{5}\right)$
12	(94)	$\tilde{7}$	$\left(\frac{1}{8},\frac{3}{8}\right)$
13	(90)	$\tilde{2}$	$\left(\frac{1}{10},\frac{3}{10}\right)$
14	(96)	$\tilde{9}$	$\left(\frac{1}{12},\frac{5}{12}\right)$

Table 16: Twist parameters for the operators ${}_4L_t^{(4)}(p,q)$ in the 'odd case'.

9. Proof of Theorem 2.1

In Section 5 we have defined an iterative construction that produces families of elliptically fibered Calabi-Yau n-folds with section from families of elliptic Calabi-Yau varieties of one dimension lower by a combination of a quadratic twist and a rational base transformation encoded in the generalized functional invariant. Moreover, all Weierstrass models are obtained through a sequence of constructions that start with the quadric pencil in Equation (2.1). Each step $n = 1, 2, 3, 4$ of our iterative construction has also provided a family of a closed transcendental n-cycle for each family of n-folds as the warped product of the corresponding transcendental cycle in dimension $n-1$. Upon integration of this cycle with the holomorphic n-form we obtain a period for the family of elliptically fibered Calabi-Yau n-folds with section. By construction, the period is holomorphic on the unit disk about the point $t = 0$ of maximally unipotent monodromy. Each holomorphic period is then annihilated by a Picard-Fuchs operator which is a Calabi-Yau operator in the sense of [3].

The proof of Theorem 2.1 proceeds as follows: Bogner and Reiter classified all $\mathrm{Sp}(4,\mathbb{C})$-rigid, quasi-unipotent local systems which have a maximal unipotent element and are induced by fourth-order Calabi-Yau operators. In particular, they obtained explicit expressions for all Calabi-Yau operators and closed formulas for special solutions of them. We prove that we have realized all of these operators and holomorphic periods. There are the four cases:

1) The *hypergeometric case* consist of 14 operators called $P_1(4, 10, 4)$ [12, Theorem 6.1]. These operators precisely coincide with the 14 operators of Equations (8.10) and (8.14) obtained by the twist construction for parameters given in Table 12.
2) The *extra case* consist of 16 operators called $P_2(4, 6, 6)$ [12, Theorem 6.3]. These operators precisely coincide with the 16 operators of Equation (8.17) obtained by the twist construction for parameters listed in Table 13.
3) The *even case* consist of 16 operators called $P_2(4, 6, 8)$ [12, Theorem 6.4]. These operators precisely coincide with the 16 operators of Equation (8.20) obtained by the twist construction for parameters listed in Table 14.
4) The *odd case* consist of 14 operators called $P_1(4, 8, 4)$ [12, Theorem 6.2]. These operators precisely coincide with the Yifan-Yang pullbacks in Equations (3.21) of the 14 operators of Equations (8.24) obtained by the twist construction for the parameters listed in Table 16. □

Remark 9.1. We constructed all symplectically rigid Calabi-Yau operators as Picard-Fuchs operators of families of Calabi-Yau varieties. These operators are rank-four, degree-two, irreducible Calabi-Yau operators with three regular singular points. In addition to these operators, there are four additional rank-four, degree-two, irreducible Calabi-Yau operators with three regular singular points in the AESZ database [2]. The additional cases, 84, 254, 255, 406, have as degree-one term an irreducible polynomial (over $\mathbb{Q}[t]$) of degree four; see Remark 10.5.

10. Beyond symplectically rigid Calabi-Yau operators

10.1. Calabi-Yau operators from Heun's equation

Heun's equation is the rank-two, linear ordinary differential equation of the form

$$\left(\frac{d^2}{dt^2} + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{u-a}\right)\frac{d}{dt} + \frac{\alpha\beta t - q}{t(t-1)(t-a)}\right)\omega(t) = 0, \tag{10.1}$$

such that $\epsilon = \alpha + \beta - \gamma - \delta + 1$ to ensure that the point at infinity is a regular singular point. The parameter $q \in \mathbb{C}$ is called the accessory parameter. For $a \in \mathbb{C}$ and $a \neq 0, 1$, Heun's equation has four regular singular points at

$0, 1, a, \infty$ and the Riemann symbol

$$\mathcal{P}\left(\begin{array}{cccc|c} 0 & 1 & a & \infty & \\ \hline 1-\gamma & 1-\delta & 1-\epsilon & \alpha & t \\ 0 & 0 & 0 & \beta & \end{array}\right). \tag{10.2}$$

Every rank-two linear ordinary differential equation with at most four regular singular points can be transformed into this equation by a change of variable. The function $H\ell(a, q; \alpha, \beta, \gamma, \delta \,|t)$ is the unique solution of Heun's differential equation that is holomorphic and 1 at the singular point $t = 0$. We have the following:

Lemma 10.1. *The function $\omega(t) = H\ell(a, q; 1, 1, 1, 1 \,|t)$ is the unique solution of $L_t^{(2)}\omega(t) = 0$, holomorphic and 1 at $t = 0$, with*

$$L_t^{(2)}(a, q) = \theta^2 - \frac{t}{a}\big((a+1)\theta^2 + (a+1)\theta + q\big) + \frac{t^2}{a}(\theta+1)^2. \tag{10.3}$$

For $\alpha \in (0, 1) \cap \mathbb{Q}$, the function $\omega(t) = H\ell(a, \frac{q}{4}; \alpha, 1-\alpha, 1, \frac{1}{2} \,|t)^2$ is the unique solution of $L_t^{(3)}\omega(t) = 0$, holomorphic and 1 at $t = 0$, with

$$\begin{aligned} L_t^{(3)}(\alpha; a, q) = \theta^3 &- \frac{t}{2a}(2\theta+1)\big((a+1)\theta^2 + (a+1)\theta + q\big) \\ &+ \frac{t^2}{a}(\theta+2\alpha)\big(\theta+2(1-\alpha)\big)(\theta+1). \end{aligned} \tag{10.4}$$

We also have the following identity involving a Hadamard product:

$$H\ell\left(a, \frac{q}{4}; \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} \,\middle|\, t\right)^2 = {}_1F_0\left(\frac{1}{2} \,\middle|\, t\right) \star H\ell(a, q; 1, 1, 1, 1 \,|t). \tag{10.5}$$

Proof. The proof follows by explicit computation. □

To obtain Calabi-Yau operators, the following lemma is essential:

Lemma 10.2. *For $\alpha \in (0, 1) \cap \mathbb{Q}$, the function*

$$\omega(t) = {}_1F_0\left(\frac{1}{2} \,\middle|\, t\right) \star H\ell\left(a, \frac{q}{4}; \alpha, 1-\alpha, 1, \frac{1}{2} \,\middle|\, t\right)^2$$

is the unique solution of ${}_1L_t^{(4)}\omega(t) = 0$*, holomorphic and 1 at* $t = 0$*, with*

$$(10.6)\qquad {}_1L_t^{(4)}\big(\alpha; a, q\big) = \theta^4 - \frac{t}{4a}\big(2\theta+1\big)^2\big((a+1)\theta^2 + (a+1)\theta + q\big) + \frac{t}{4a}\big(2\theta+1\big)\big(2\theta+3\big)\big(\theta+2\beta\big)\big(\theta+2(1-\beta)\big).$$

For $\alpha \in (0,1) \cap \mathbb{Q}$*, the function*

$$\omega(t) = {}_2F_1\left(\begin{matrix}\alpha, 1-\alpha\\ 1\end{matrix}\middle|\, t\right) \star H\ell(a, q; 1, 1, 1, 1\,|\, t)$$

is the unique solution of ${}_2L_t^{(4)}\omega(t) = 0$*, holomorphic and 1 at* $t = 0$*, with*

$$(10.7)\quad {}_2L_t^{(4)}\big(\alpha; a, q\big) = \theta^4 - \frac{t}{a}\big(\theta+\alpha\big)\big(\theta+1-\alpha\big)\big((a+1)\theta^2 + (a+1)\theta + q\big) + \frac{t^2}{a}\big(\theta+\alpha\big)\big(\theta+1-\alpha\big)\big(\theta+\alpha+1\big)\big(\theta+2-\alpha\big).$$

Proof. The proof follows by explicit computation. □

The rank-four operators in Equations (10.6) and (10.7) have four regular singular points at $0, 1, a, \infty$. In particular, they are not symplectically rigid operators. Notice that rescaling $t \mapsto \lambda a t$ leaves the operator θ invariant and allows us to clear denominators. We then have the following:

Proposition 10.3. *The rank-four operators* ${}_iL_{\lambda a t}^{(4)}\big(\alpha; a, q\big)$ *with* $i = 1, 2$ *in Equation (10.6) and Equation (10.7), with parameters* $(\alpha; a; q; \lambda)$ *given in Table 17 and 18, constitute all 33 rank-four, degree-two Calabi-Yau operators in the AESZ database [2] with four regular singular points whose degree-one term is not an irreducible polynomial (over* $\mathbb{Q}[t]$*) of degree four.*

Proof. The proof follows by explicit computation. □

Remark 10.4. The fact that there are two solutions for each entry in Table 17 and 18 is due to the following identity for Heun functions

$$H\ell(a, q; 1, 1, 1, 1|t) = H\ell\left(\frac{1}{a}, \frac{q}{a}; 1, 1, 1, 1\middle|\frac{t}{a}\right).$$

Remark 10.5. In the AESZ database [2], there are 36 rank-four, degree-two, irreducible Calabi-Yau operators with four regular singular points. Three additional cases, 18, 182, 205, that do not appear in Table 17 and 18 have as degree-one term an irreducible polynomial (over $\mathbb{Q}[t]$) of degree four.

AESZ	$(\alpha; a, q; \lambda)$ in ${}_1L^{(4)}_{\lambda at}(\alpha; a, q)$
16	$\left(\frac{1}{2}, 4, 2, 64\right), \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 16\right)$
29	$\left(\frac{1}{2}, 577 \pm 408\sqrt{2}, 170 \pm 120\sqrt{2}, 68 \pm 48\sqrt{2}\right)$
41	$\left(\frac{1}{2}, \frac{17}{81} \pm i\frac{56}{81}\sqrt{2}, \frac{14}{27} \pm i\frac{8}{27}\sqrt{2}, 28 \pm i\,16\sqrt{2}\right)$
42	$\left(\frac{1}{2}, 17 \pm 12\sqrt{2}, 6 \pm 4\sqrt{2}, 48 \pm 32\sqrt{2}\right)$
184	$\left(\frac{1}{2}, \frac{117}{125} \pm i\frac{44}{125}, \frac{22}{25} \pm i\frac{4}{25}, 44 \pm i\,8\right)$
185	$\left(\frac{1}{2}, -7 \pm 4\sqrt{3}, -2 \pm \frac{4}{3}\sqrt{3}, 36 \mp 24\sqrt{3}\right)$
25	$\left(\frac{1}{4}, \frac{-123\pm55\sqrt{5}}{2}, \frac{-33\pm15\sqrt{5}}{2}, 88 \mp 40\sqrt{5}\right)$
36	$\left(\frac{1}{4}, 2, 1, 128\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 64\right)$
45	$\left(\frac{1}{4}, -8, -2, 128\right), \left(\frac{1}{4}, -\frac{1}{8}, \frac{1}{4}, -16\right)$
58	$\left(\frac{1}{4}, 9, 3, 144\right), \left(\frac{1}{4}, \frac{1}{9}, \frac{1}{3}, 16\right)$
133	$\left(\frac{1}{4}, \frac{1\pm i\sqrt{3}}{2}, \frac{3\pm i\sqrt{3}}{6}, 72 \pm 34\,i\sqrt{3}\right)$
137	$\left(\frac{1}{4}, \frac{9}{8}, \frac{3}{4}, 144\right), \left(\frac{1}{2}, \frac{8}{9}, \frac{2}{3}, 128\right)$
18	$\left(\frac{3}{8}, -4, -1, 64\right), \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, -16\right)$
183	$\left(\frac{3}{8}, \frac{4}{3}, 1, 64\right), \left(\frac{3}{8}, \frac{3}{4}, \frac{3}{4}, 48\right)$
26	$\left(\frac{1}{3}, -27, -8, 108\right), \left(\frac{1}{3}, -\frac{1}{27}, \frac{8}{27}, -4\right)$

Table 17: Non-rigid Calabi-Yau operators from Lemma 10.2.

10.2. Realizing non-rigid Calabi-Yau operators

The twisted families of Section 6.1 are precisely the extremal families of elliptic curves with three singular fibers and rational total space (up to quadratic twist and two-isogeny) classified in [60, Tab. 5.2]. Miranda and Persson also classified the extremal families of elliptic curves with rational total space and four singular fibers, the highest number that can occur, in [60, Tab. 5.3]. There are six of them, in the notation of Herfurtner denoted as X_{5511}, X_{6321}, X_{4422}, X_{8211}, X_{3333}, and X_{9111}. Analogous to Corollary 6.6, they are the modular elliptic surfaces for the subgroups $\Gamma_1(5)$, $\Gamma_0(6)$, $\Gamma_0(4) \cap \Gamma(2)$, $\Gamma_0(8)$, $\Gamma(3)$, and $\Gamma_0(9)$, respectively.

The latter four, X_{4422}, X_{8211}, X_{3333}, and X_{9111}, are easily understood in terms of the construction in Section 8. For example, X_{8211} and X_{9111}

AESZ	$(\alpha; a, q; \lambda)$ in ${}_2L^{(4)}_{\lambda a t}(\alpha; a, q)$
48	$\left(\frac{1}{3}, 2, 1, 216\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, 108\right)$
38	$\left(\frac{1}{4}, 2, 1, 512\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 256\right)$
65	$\left(\frac{1}{6}, 2, 1, 3456\right), \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, 1728\right)$
134	$\left(\frac{1}{3}, \frac{1\pm i\sqrt{3}}{2}, \frac{3\pm i\sqrt{3}}{6}, \frac{243\pm 81\, i\sqrt{3}}{2}\right)$
135	$\left(\frac{1}{4}, \frac{1\pm i\sqrt{3}}{2}, \frac{3\pm i\sqrt{3}}{6}, 288 \pm 96\, i\sqrt{3}\right)$
136	$\left(\frac{1}{6}, \frac{1\pm i\sqrt{3}}{2}, \frac{3\pm i\sqrt{3}}{6}, 1944 \pm 648\, i\sqrt{3}\right)$
24	$\left(\frac{1}{3}, \frac{-123\pm 55\sqrt{5}}{2}, \frac{-33\pm 15\sqrt{5}}{2}, \frac{297\mp 135\sqrt{5}}{2}\right)$
51	$\left(\frac{1}{4}, \frac{-123\pm 55\sqrt{5}}{2}, \frac{-33\pm 15\sqrt{5}}{2}, 352 \mp 160\sqrt{5}\right)$
63	$\left(\frac{1}{6}, \frac{-123\pm 55\sqrt{5}}{2}, \frac{-33\pm 15\sqrt{5}}{2}, 2376 \mp 1080\sqrt{5}\right)$
15	$\left(\frac{1}{3}, -8, -2, 216\right), \left(\frac{1}{3}, -\frac{1}{8}, \frac{1}{4}, -27\right)$
68	$\left(\frac{1}{4}, -8, -2, 512\right), \left(\frac{1}{4}, -\frac{1}{8}, \frac{1}{4}, -64\right)$
62	$\left(\frac{1}{6}, -8, -2, 3456\right), \left(\frac{1}{6}, -\frac{1}{8}, \frac{1}{4}, -432\right)$
70	$\left(\frac{1}{3}, 9, 3, 243\right), \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{3}, 27\right)$
69	$\left(\frac{1}{4}, 9, 3, 576\right), \left(\frac{1}{4}, \frac{1}{9}, \frac{1}{3}, 64\right)$
64	$\left(\frac{1}{6}, 9, 3, 3888\right), \left(\frac{1}{6}, \frac{1}{9}, \frac{1}{3}, 432\right)$
138	$\left(\frac{1}{3}, \frac{9}{8}, \frac{3}{4}, 243\right), \left(\frac{1}{3}, \frac{8}{9}, \frac{2}{3}, 216\right)$
139	$\left(\frac{1}{4}, \frac{9}{8}, \frac{3}{4}, 576\right), \left(\frac{1}{4}, \frac{8}{9}, \frac{2}{3}, 512\right)$
140	$\left(\frac{1}{6}, \frac{9}{8}, \frac{3}{4}, 3888\right), \left(\frac{1}{6}, \frac{8}{9}, \frac{2}{3}, 3456\right)$

Table 18: Non-rigid Calabi-Yau operators from Lemma 10.2.

are pull-backs of modular elliptic surfaces X_{141} or X_{431} in Table 5, along the map $t \mapsto t^k$ with $k = 2$ or $k = 3$, respectively. Similar arguments apply to X_{4422} and X_{3333}. Weierstrass models for the elliptic surfaces X_{5511} and X_{6321} for subgroups $\Gamma_1(5)$, $\Gamma_0(6)$ are given in Table 19.

It is straight forward to work out corresponding versions of Corollary 6.8. In fact, the period integral of dX/Y over a family of suitable A-cycles $\Sigma_1(t)$

name, G	g_2, g_3, Δ, J, sections	Ramification of J and singular fibers			
$\mathrm{MW}(\pi,\sigma)$		t	J	$m(j)$	fiber
X_{5511}	$g_2 = \frac{3}{4}c^4t^4 - 9c^3t^3 + \frac{21}{2}c^2t^2 + 9ct + \frac{3}{4}$	$p_1(ct) = 0$	0	3	smooth
$\Gamma_0(5)$	$g_3 = -\frac{1}{6}c^6t^6 + \frac{9}{4}c^5t^5 - \frac{75}{8}c^4t^4 - \frac{75}{8}c^2t^2 - \frac{9}{4}ct - \frac{1}{8}$	$\pm\frac{i}{c}$	1	2	smooth
	$\Delta = 729\,c^5t^5(c^2t^2 - 11ct - 1)$	$p_2(ct) = 0$	1	2	smooth
	$J = \frac{(c^4t^4-12c^3t^3+14c^2t^2+12ct+1)^3}{1728\,c^5t^5(c^2t^2-11ct-1)}$	$0, \infty$	∞	5	$2\,I_5\,(A_4)$
$\mathbb{Z}/5\mathbb{Z}$	$(X,Y)_{1,2} = (\frac{1}{4}c^2t^2 + \frac{3}{2}ct + \frac{1}{4}, \pm 3\sqrt{3}c^2t^2)$	$(t^2 - 11t - 1 = 0)/c$	∞	1	$2\,I_1$
	$(X,Y)_{3,4} = (\frac{1}{4}c^2t^2 - \frac{3}{2}ct + \frac{1}{4}, \pm 3\sqrt{3}ct)$				
X_{6321}	$g_2 = \frac{3}{4}\big(t-4\big)\big(t^3 + 12t^2 + 48t - 64\big)$	$4, 4(1-\sqrt[3]{2})$	0	3	smooth
$\Gamma_0(6)$	$g_3 = -\frac{1}{8}\big(t^2 + 4t - 8\big)\big(t^4 + 8t^3 + 512t - 512\big)$	$-2\sqrt[3]{2}(1 \pm i\sqrt{3}) - 4$	0	3	smooth
	$\Delta = -729\,t^6\big(t-1\big)\big(t+8\big)^2$	$4\big(t^4 + 2t^3 + 8t - 2 = 0\big)$	1	2	smooth
	$J = -\frac{(t-4)^3(t^3+12t^2+48t-64)^3}{1728t^6(t-1)(t+8)^2}$	$2 \pm 2\sqrt{3}$	1	2	smooth
$\mathbb{Z}/6\mathbb{Z}$	$(X,Y)_1 = (-\frac{1}{2}t^2 - 2t + 4, 0)$	0	∞	6	$I_6\ (A_5)$
	$(X,Y)_{2,3} = (\frac{1}{4}t^2 - 2t + 4, \pm 3\sqrt{3}t^2)$	1	∞	1	I_1
	$(X,Y)_{4,5} = (\frac{1}{4}t^2 + 4t + 4, \pm 3\sqrt{3}t(t+8))$	-8	∞	2	$I_2\ (A_1)$
		∞	∞	3	$I_3\ (A_2)$

Table 19: Extremal rational fibrations (with polynomials $p_1(t) = t^4 - 12t^3 + 14t^2 + 12t + 1$, $p_2(t) = t^4 - 18t^3 + 74t^2 + 18t + 1$ and $c = \frac{11\mp 5\sqrt{5}}{2}$, $c' = \frac{11\pm 5\sqrt{5}}{2}$).

in the neighborhood of $t = 0$ are Heun functions, and we use rational transformations on the parameter curve to re-arrange the four singular points if necessary. We have the following:

Corollary 10.6. *For the families of elliptic curves over $\mathbb{P}^1\backslash\{0,1,a,\infty\}$, X_{5511}, X_{6321}, X_{8211}, and X_{9111}, the period integrals of dX/Y are annihilated by the Picard-Fuchs operator $L_t^{(2)}(a,q)$ in Equation (10.3). For each family, the period over $\Sigma_1(t)$ is $\omega(t) = H\ell(a,q;1,1,1,1\,|t)$ and holomorphic at $t = 0$, with parameters (a,q) and singular fibers at $t = 0, 1, a, \infty$ given in Table 20.*

Proof. For the families of Weierstrass models we use dX/Y as the holomorphic one-form on each regular fiber. It is well-known (cf. [73]) that the Picard-Fuchs equation is given by the Fuchsian system

$$\frac{d}{dt}\begin{pmatrix}\omega_1\\ \eta_1\end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\ln\Delta}{dt} & \frac{3\,\delta}{2\,\Delta} \\ -\frac{g_2\,\delta}{8\,\Delta} & \frac{1}{12}\frac{d\ln\Delta}{dt}\end{pmatrix}\cdot\begin{pmatrix}\omega_1\\ \eta_1\end{pmatrix}, \tag{10.8}$$

where $\omega_1 = \oint_{\Sigma_1}\frac{dX}{Y}$ and $\eta_1 = \oint_{\Sigma_1}\frac{X\,dX}{Y}$ for each one-cycle Σ_1 and with $\delta = 3\,g_3\,g_2' - 2\,g_2\,g_3'$. The rest follows by explicit computation. □

(a,q)	Configuration	Surface
$(2,1)$	$I_1(t=0)\oplus I_2(t=1)\oplus I_1(t=a)\oplus I_8(t=\infty)$	X_{8211}
$\left(\frac{1}{2},\frac{1}{2}\right)$	$I_8(t=0)\oplus I_2(t=1)\oplus I_1(t=a)\oplus I_1(t=\infty)$	X_{8211}
$\left(\frac{1\pm i\sqrt{3}}{2},\frac{3\pm i\sqrt{3}}{6}\right)$	$I_1(t=0)\oplus I_1(t=1)\oplus I_1(t=a)\oplus I_9(t=\infty)$	X_{9111}
$\left(-(c')^2,-\frac{3}{c}\right)$	$I_5(t=0)\oplus I_1(t=1)\oplus I_1(t=a)\oplus I_5(t=\infty)$	X_{5511}
	$c=\frac{11\mp5\sqrt{5}}{2},\ c'=\frac{11\pm5\sqrt{5}}{2}$	
$(-8,-2)$	$I_6(t=0)\oplus I_1(t=1)\oplus I_2(t=a)\oplus I_3(t=\infty)$	X_{6321}
$\left(-\frac{1}{8},\frac{1}{4}\right)$	$I_3(t=0)\oplus I_1(t=1)\oplus I_2(t=a)\oplus I_6(t=\infty)$	X_{6321}
$(9,3)$	$I_1(t=0)\oplus I_6(t=1)\oplus I_2(t=a)\oplus I_3(t=\infty)$	X_{6321}
$\left(\frac{1}{9},\frac{1}{3}\right)$	$I_3(t=0)\oplus I_6(t=1)\oplus I_2(t=a)\oplus I_1(t=\infty)$	X_{6321}
$\left(\frac{9}{8},\frac{3}{4}\right)$	$I_2(t=0)\oplus I_6(t=1)\oplus I_1(t=a)\oplus I_3(t=\infty)$	X_{6321}
$\left(\frac{8}{9},\frac{2}{3}\right)$	$I_2(t=0)\oplus I_1(t=1)\oplus I_6(t=a)\oplus I_3(t=\infty)$	X_{6321}

Table 20: Parameters of Heun functions for extremal elliptic surfaces.

Analogous to Lemma 6.5, it follows that twisted families with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ of X_{5511}, X_{6321}, X_{8211}, and X_{9111} are families of M_n-lattice polarized K3 surfaces with $n=5,6,8,9$, respectively. As before, we also obtain a continuously varying family of closed two-cycles $\Sigma_2(t)$. We therefore have the following:

Corollary 10.7. *The twisted families with generalized functional invariant $(i,j,\alpha)=(1,1,1)$ given by Equation (6.5), of the families in Corollary 10.6 are families over the rational modular curves $\mathbb{H}/\Gamma_0(n)^+$ of M_n-lattice polarized K3 surfaces with $n=5,6,8,9$. For each family, the period integral (5.13) is annihilated by the Picard-Fuchs operator $L_t^{(3)}(\frac{1}{4};a,q)$ in Equation (10.4). In particular, the period over $\Sigma_2(t)$ is holomorphic at $t=0$ and given by*

$$\omega=(2\pi i)^2\ {}_1F_0\left(\left.\frac{1}{2}\right|t\right)\star H\ell(a,q;1,1,1,1\,|t), \tag{10.9}$$

where parameters (a,q) and singular fibers over $t=0,1,a,\infty$ (before twisting) are given in Table 20.

Proof. The proof follows directly by checking that the singular fibers and Mordell-Weil groups for the families constructed in Lemma 6.9 agree with the ones given by Dolgachev in [28]. The rest of the proof is analogous to the proof of Corollary 6.11. □

Remark 10.8. Identity (10.5) reflects the well-known decomposition of the Picard-Fuchs operator into a symmetric square for families of M_n-lattice polarized K3 surfaces of Picard-rank 19. If one considers non-rigid, smooth Calabi-Yau threefolds (non-isotrivially) fibered by K3 surfaces admitting a M_n-lattice polarization, then it wash shown in [33] that $1 \leq n \leq 11$, $n \neq 10$ and that all such n can be realized. Our twist construction provides explicit examples for such Calabi-Yau threefolds for $n \in \{1, 2, 3, 4, 5, 6, 8, 9\}$.

Applying our twist construction again yields families of Calabi-Yau threefolds whose Picard-Fuchs operators realize non-rigid Calabi-Yau operators with four regular singular points. As in Section 8.3, we also obtain a continuously varying family of closed three-cycles $\Sigma_3(t)$. We have the following:

Corollary 10.9. *The twisted families with generalized functional invariant* $(k, l, \beta) = (1, 1, 1)$, *given by Equation (8.9), of the elliptic K3 surfaces in Corollary 10.7 are families over* $\mathbb{P}^1 \backslash \{0, 1, a, \infty\}$ *of Jacobian elliptic Calabi-Yau threefolds over* $\mathbb{P}^1 \times \mathbb{P}^1$. *For each family, the period integral (5.13) is annihilated by the Picard-Fuchs operator* ${}_1L_t^{(4)}(\frac{1}{4}; a, q)$ *in Equation (10.6). In particular, the period over* $\Sigma_3(t)$ *is holomorphic at* $t = 0$ *and given by*

$$\omega = (2\pi i)^3 \; {}_1F_0\left(\frac{1}{2}\middle| t\right) \star H\ell\left(a, \frac{q}{4}; \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} \middle| t\right)^2 \tag{10.10}$$

with parameters (a, q) *given in Table 20.*

Proof. The proof is analogous to the proof of Lemma 8.4 and Corollary 8.5. □

Applying base transformations between twists again greatly improves the scope of our twist construction. We make the following:

Remark 10.10. Special function identities for the Heun function can be used to realize Picard-Fuchs operators ${}_1L_t^{(4)}(\alpha; a, q)$ in Equation (10.6) for values other than $\alpha = \frac{1}{4}$ in Table 19. As an example, we consider the case $\alpha = \frac{1}{2}$ where we use a sequence of identities for the Heun function found in [54]. For $\beta \in (0,1) \cap \mathbb{Q}$ and $a \neq 1$, we use the linear identity

$$H\ell\left(a, q; 1, 1, 1, 1 \middle| x\right) = \frac{1}{1 - \frac{x}{a}}\, H\ell\left(1-a, 1-q; 1, 1, 1, 1 \middle| T_1(x)\right) \tag{10.11}$$

with $T_1(x) = \frac{(1-a)\,x}{x-a}$, the quadratic identity

$$H\ell\left(a, \frac{q}{4}; \beta, 1-\beta, 1, \frac{1}{2} \middle| T_2(x)\right) = (1-x)^\beta\, H\ell\left(a', q'; 2\beta, 1, 1, 2\beta \middle| x\right) \tag{10.12}$$

with $T_2(x) = \frac{R\,x\,(a-x)}{1-x}$, where a and a' are related by

$$(a')^2\,(1-a)^2 - 16\,(1-a')\,a = 0, \tag{10.13}$$

and $q = 4\,R\,(q' - \beta a')$ and $R = \frac{1+a}{2(2-a')}$, combined with the bi-quadratic quartic identity

$$H\ell\left(a, \frac{q}{4}; \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} \middle| T_4(x)\right) = \left(1 - \frac{x^2}{a}\right)^{1/2} H\ell\left(a, q; 1, 1, 1, 1 \middle| x\right), \tag{10.14}$$

with $T_4(x) = \frac{4\,a\,x\,(1-x)\,(a-x)}{(a-x^2)^2}$. This implies that a Heun function of the form

$$H\ell\left(a, \frac{q}{4}; \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2} \middle| T_2(x)\right)$$

is related to the Heun function

$$\frac{\sqrt{1-x}}{(1-\frac{x}{a})\sqrt{1-T_1(x)^2/a}}\, H\ell\left(1-a', \frac{1-q'}{4}; \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} \middle| T_4\big(T_1(x)\big)\right).$$

In turn, the latter is realized as holomorphic period of an extremal rational surface after pullback by a base transformation and twist. The Picard-Fuchs operator of the twisted family, constructed analogously to Corollary 10.9, then realizes the Calabi-Yau operators ${}_1L_t^{(4)}(\frac{1}{2}; a, q)$ for parameters (a, q) given in Table 20.

Applying the variant of the twist construction for Section 5.5.3 yields other families of Calabi-Yau threefolds whose Picard-Fuchs operators realize more non-rigid Calabi-Yau operators with four regular singular points. As in Section 8.3.3, we also obtain a continuously varying family of closed three-cycles $\hat{\Sigma}_3(t)$. We have the following:

Corollary 10.11. *For every family $X \to \mathbb{P}^1\backslash\{0,1,a,\infty\}$ in Corollary 10.6, the twist family of X with X'_k in Table 2 given by Equation (5.30) for $k=2,3,4$, is a family over $\mathbb{P}^1\backslash\{0,1,a,\infty\}$ of Jacobian elliptic Calabi-Yau threefolds over $\mathbb{F}_n$ with $n=0,\dots,k$. The period integral (5.33) is annihilated by the Picard-Fuchs operator ${}_2L_t^{(4)}(\mu;a,q)$ in Equation (10.7). In particular, the period over $\hat{\Sigma}_3(t)$ is holomorphic at $t=0$ and given by*

$$\hat{\omega} = (2\pi i)^3\, {}_2F_1\left(\left.\begin{array}{c}\mu, 1-\mu\\ 1\end{array}\right| t\right) \star H\ell(a,q;1,1,1,1\,|t), \tag{10.15}$$

where parameters (a,q) are given in Table 20 and $\mu \in \{\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$.

Proof. The proof is analogous to the proof of Corollary 8.13 in Section 8.3.3. □

Remark 10.12. Corollary 10.9, Corollary 10.11, and Remark 10.10 realize 30 non-rigid Calabi-Yau operators with four regular singular points as Picard-Fuchs operators of families of Calabi-Yau threefolds obtained by our twist construction, namely all operators ${}_1L_t^{(4)}(\mu;a,q)$ in Equation (10.6) with $\mu \in \{\frac{1}{2},\frac{1}{4}\}$ and operators ${}_2L_t^{(4)}(\mu;a,q)$ in Equation (10.7) with $\mu \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$, for parameters (a,q) given in Table 20 up to rescaling $t \mapsto \lambda a t$ of the affine base coordinate according to Table 17 and 18.

11. Discussion and outlook

We introduced a twist construction to iteratively obtain families of Calabi-Yau n-folds over $\mathbb{P}^1\backslash\{0,1\infty\}$, internally elliptically fibered by Calabi-Yau $(n-1)$-folds. Our construction is a geometric generalization of Katz's middle convolution combined with an additional rational pullback operation on the internal fibration. By computing the periods of a holomorphic top-form over explicit topological cycles and expressing the results in hypergeometric terms, we produced Weierstrass models whose Picard-Fuchs operators realize all 60 Calabi-Yau operators inducing $\mathrm{Sp}(4,\mathbb{C})$-rigid, quasi-unipotent local systems of weight three and rank four having a maximal unipotent element. This is important because the usual middle convolution only is guaranteed to produce the $\mathrm{GL}(4,\mathbb{C})$-rigid monodromy tuples. Our iterative construction provides a unifying construction for many examples of elliptic curves, K3 surfaces, and Calabi-Yau threefolds considered in the context of mirror symmetry, e.g., families considered in [32, 63, 80] and examples in [82]. Moreover, by restricting the family parameter to special values one readily obtains elliptic curves, K3 surfaces, and Calabi-Yau threefolds with properties such as CM, admitting a Shioda-Inose structure associated with abelian surfaces with quaternionic multiplication, or rigidity. Some of these arithmetic properties were investigated in [78], and their fiberwise Picard-Fuchs equations were computed in [79]. These results are all reproduced by our iterative construction.

We also used our iterative construction to obtain families with four singular fibers, such as all extremal Jacobian rational elliptic surfaces with four singular fibers from the Miranda-Persson list [60], and models for families of M_n-lattice polarized K3 surfaces for $n \leq 9$ with $n \neq 7$. On the level of periods, the role of the Gauss hypergeometric function was then replaced by the Heun function. Identities for the hypergeometric function were replaced by identities for the Heun equation, for example relations found in [54, 74]. In this way, our iterative construction again provided a unified geometric approach for many differential equations associated with K3 surfaces studied in isolation [10, 11, 64, 65, 72]. Our iterative construction also reproduced many of the classical examples of threefolds investigated in the context of mirror symmetry, for example in [52, 53]. We hope that the obtained new families and our iterative technique itself could be of interest for "global mirror symmetry" frameworks, e.g., see [18, 19], curve-counting on 3-folds, F-theory, for studying thin vs. arithmetic monodromy, and maybe in the future even homological mirror symmetry.

However, the geometric realization of the Calabi-Yau operators in the odd case in Theorem 2.1 is not yet completely satisfactory: instead of producing families of threefolds whose Picard-Fuchs operators realize the fourth-order Calabi-Yau operators of the odd case directly, we constructed families of fourfolds instead, such that the Yifan-Yang pullback of their rank-five Picard-Fuchs operators realized the Calabi-Yau operators. The observant reader might have noticed that a similar situation already occurred at lower dimension in our iterative construction. The twist construction applied to any family of elliptic curves from Table 5 produced families of K3 surfaces whose Picard-Fuchs operators were symmetric squares of rank-two and degree-one Calabi-Yau operators. In fact, Clausen's identity (6.8) expresses the holomorphic K3 periods as squares of Gauss hypergeometric functions. Using the hypergeometric function identity (6.9), we were able to relate the (symmetric) square root back to the holomorphic solution of the Picard-Fuchs equation before the twist. In this sense, carrying out a twist and taking a (symmetric) square root is equivalent to carrying out a quadratic transformation on the parameter space of the original family. Hodge-theoretically this is due to the fact that the K3 surfaces of Picard-rank 19 or 18 admit Shioda-Inose structures relating them to Kummer surfaces associated to products of elliptic curves. We have already proved that, at least in one case, a similar Hodge-theoretic interpretation exists for the exterior square root of the simplest Calabi-Yau operator in the odd case as well.

Moreover, we expect that our iterative construction of a transcendental cycle for the holomorphic period can be extended to obtain a full basis of transcendental cycles. Such bases would in turn allow us to construct the integral monodromy matrices for each family. This is important because the full period lattices are a powerful tool to distinguish examples of different geometric variations of Hodge structure over $\mathbb{Z}$ that are isomorphic over $\mathbb{R}$. For example, the quintic-mirror and the quintic-mirror twin family share the exact same Picard-Fuchs equation, but they have different ranks in the even dimensional cohomologies [34]. These results will be the subject of a forthcoming article.

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