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On the matter of $\mathcal{N} = 2$ matter

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Abstract

We introduce a variety of four-dimensional $\mathcal{N} = 2$ matter multiplets which have not previously appeared explicitly in the literature. Using these, we develop a class of supersymmetric actions supplying a context for a systematic exploration of $\mathcal{N} = 2$ matter theories, some of which include Hypermultiplet sectors in novel ways. We construct an $\mathcal{N} = 2$ supersymmetric field theory in which the propagating fields are realized off-shell exclusively as Lorentz scalars and Weyl spinors and which involves a sector with precisely the R-charge assignments characteristic of Hypermultiplets.
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There are various ways to realize four-dimensional $\mathcal{N} = 2$ Hypermultiplets [1] off-shell. The known examples have a nontrivial off-shell central charge [2], or have infinite independent degrees of freedom [3], or involve constrained vector fields [4]. It is a relevant and interesting open question whether a Hypermultiplet with no off-shell central charge, finite off-shell degrees of freedom, and no constrained vector fields, exists. In this Letter we introduce several $\mathcal{N} = 2$ matter multiplets, and a variety of corresponding supersymmetric actions, which facilitate a systematic exploration of this prospect. By way of partial solution, we construct an $\mathcal{N} = 2$ supersymmetric field theory in which the propagating fields are realized off-shell exclusively as Lorentz scalars and Weyl spinors and which involves a sector with precisely the R-charge assignments characteristic of Hypermultiplets.

In our discussion, a central role is played by constrained $\mathcal{N} = 2$ superfields which transform as symmetric rank-p tensor products of fundamental $SU(2)_R$ representations. The cases $p = 2, 3, 4$ and 5 describe a Triplet, a Quadruplet, a Quintet, and a Sextet, respectively, and we refer to the corresponding multiplets using these names. The well-known $\mathcal{N} = 2$ Tensor multiplet corresponds to a constrained Triplet superfield, and includes as off-shell components a dimension-one boson triplet, a dimension-three-half Weyl spinor doublet, and five dimension-two bosons transforming as a complex Lorentz scalar and a divergence-free Lorentz vector, the latter of which describes the Hodge dual of the three-form field strength associated with a two-form gauge potential.

In an early attempt to realize a Hypermultiplet off-shell as a "relaxed" version of a Tensor multiplet, the authors of [5] utilized the highest component of the Quintet superfield—a dimension-three real scalar boson—to supply the divergence of the erstwhile constrained vector component within the Tensor multiplet. A Dual Quintet superfield, which is a constrained scalar superfield admitting a supersymmetric coupling (i.e., an inner product) with the Quintet superfield, provides Lagrange

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multiplier terms which render the five dimension-one bosons within the Quintet multiplet non-propagating. What results is a supersymmetric field theory reproducing the on-shell state counting characteristic of Hypermultiplets, but with R-charge assignments which differ.

The four propagating bosons in a Hypermultiplet transform as a doublet under $SU(2)_R$ and also as a doublet under an additional symmetry $SU(2)_H$. The four propagating fermions assemble as a pair of Weyl spinors which transform as $SU(2)_R$ singlets and $SU(2)_H$ doublets. This structure admits a well-known generalization whereby $SU(2)_H$ is replaced with $Sp(2n)$ for $n \in \mathbb{N}$. A natural way to realize component fields with these representation assignments is obtained by starting with an $\mathcal{N} = 2$ Tensor multiplet and augmenting this by "tensoring" two additional indices, one corresponding to the fundamental $SU(2)_R$ representation and another to the fundamental $SU(2)_H$ representation. When the component fields are expressed in terms of irreducible representations of $SU(2)_R$, one obtains a standard presentment of what we call the Extended Tensor Multiplet.

To be more explicit, the Tensor multiplet is specified by the superfield $\mathbb{L}_{ij} = \mathbb{L}_{(ij)}$, subject to the constraint $D_{\alpha(i}\mathbb{L}_{jk)} = 0$, where $D_{\alpha i}$ is a superspace derivative, and to the reality constraint $\mathbb{L}_{ij} = \varepsilon_{im}\varepsilon_{jn}\mathbb{L}^{mn}$, where complex conjugation is indicated by the raising or lowering of $SU(2)$ indices. The lowest components of the Tensor multiplet comprise a boson triplet L_{ij}. The lowest components of the Extended Tensor multiplet comprise the "tensored" analog $L_{ij}{}^{k\hat{i}}$, where the hatted index corresponds to the fundamental representation of $SU(2)_H$, subject to the reality constraint $L_{ij}{}^{p\hat{i}} = \varepsilon_{im}\varepsilon_{jn}\varepsilon^{pq}\varepsilon^{\hat{i}\hat{j}}L^{mn}{}_{q\hat{j}}$. We decompose $L_{ij}{}^{p\hat{i}}$ into $SU(2)_R$ irreps as

$$L_{ij}{}^{p\hat{i}} = \frac{2}{3}\delta_{(i}{}^{p}\phi_{j)}{}^{\hat{i}} + \varepsilon^{pk}u_{ijk}{}^{\hat{i}}, \tag{1}$$

where $\phi_i{}^{\hat{i}}$ is a doublet–doublet, and $u_{ijk}{}^{\hat{i}}$ is a quadruplet–doublet under $SU(2)_R \times SU(2)_H$. These fields are subject to the constraints $\phi_i{}^{\hat{i}} = \varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\phi^j{}_{\hat{j}}$ and $u_{ijk}{}^{\hat{i}} = \varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn}\varepsilon^{\hat{i}\hat{j}}u^{lmn}{}_{\hat{j}}$. It is straightforward to determine the supersymmetry transformation rules for a complete supermultiplet which utilizes $\phi_i{}^{\hat{i}}$ and $u_{ijk}{}^{\hat{i}}$ as its lowest components. These are given by

$$\delta_Q \phi_i{}^{\hat{i}} = i\bar{\epsilon}_i\psi^{\hat{i}} + i\varepsilon^{mn}\bar{\epsilon}_m\lambda_{ni}{}^{\hat{i}} + i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^j\psi_{\hat{j}}$$
$$+ i\varepsilon_{ij}\varepsilon_{mn}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^m\lambda^{nj}{}_{\hat{j}},$$

$$\delta_Q u_{ijk}{}^{\hat{i}} = i\bar{\epsilon}_{(i}\lambda_{jk)}{}^{\hat{i}} + i\varepsilon_{im}\varepsilon_{jn}\varepsilon_{kp}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^{(m}\lambda^{np)}{}_{\hat{j}},$$

$$\delta_Q \psi^{\hat{i}} = \frac{3}{2}\slashed{\partial}\phi_i{}^{\hat{i}}\epsilon^i + A_{ai}{}^{\hat{i}}\gamma^a\epsilon^i + \varepsilon^{mn}N_m{}^{\hat{i}}\epsilon_n,$$

$$\delta_Q \lambda_{ij}{}^{\hat{i}} = \frac{1}{3}\varepsilon_{k(i}\slashed{\partial}\phi_{j)}{}^{\hat{i}}\epsilon^k + 2\slashed{\partial}u_{ijk}{}^{\hat{i}}\epsilon^k$$
$$- \frac{2}{3}\varepsilon_{k(i}A_{aj)}{}^{\hat{i}}\gamma^a\epsilon^k - \frac{2}{3}N_{(i}{}^{\hat{i}}\epsilon_{j)},$$

$$\delta_Q N_i{}^{\hat{i}} = i\varepsilon_{ij}\bar{\epsilon}^j\slashed{\partial}\psi^{\hat{i}} - 3i\bar{\epsilon}^j\slashed{\partial}\lambda_{ij}{}^{\hat{i}},$$

$$\delta_Q A_{ai}{}^{\hat{i}} = \frac{1}{2}i\bar{\epsilon}_i\gamma_{ab}\partial^b\psi^{\hat{i}} - \frac{3}{2}i\varepsilon^{mn}\bar{\epsilon}_m\gamma_{ab}\partial^b\lambda_{ni}{}^{\hat{i}}$$
$$+ \frac{1}{2}i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^j\gamma_{ab}\partial^b\psi_{\hat{j}} - \frac{3}{2}i\varepsilon_{ij}\varepsilon_{mn}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^m\gamma_{ab}\partial^b\lambda^{nj}{}_{\hat{j}}, \tag{2}$$

where $\psi^{\hat{i}}$ is an $SU(2)_H$ doublet of Weyl spinors, $\lambda_{ij}{}^{\hat{i}} = \lambda_{(ij)}{}^{\hat{i}}$ is an assembly of six Weyl spinors transforming as a triplet-doublet under $SU(2)_R \times SU(2)_H$, $N_i{}^{\hat{i}}$ describes four complex bosons transforming as a doublet–doublet, and $A_{ai}{}^{\hat{i}}$ is a Lorentz vector transforming as a doublet–doublet, subject to the reality constraint $A_{ai}{}^{\hat{i}} = \varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}A_a{}^j{}_{\hat{j}}$, and also constrained to be divergence-free, $\partial^a A_{ai}{}^{\hat{i}} = 0$. The Extended Tensor transformation rules (2) describe $32 + 32$ off-shell degrees of freedom, and represent the $\mathcal{N} = 2$ susy algebra with no central charge.

An action functional invariant under (2) is given by

$$S_{\text{ET}} = \int d^4x \left(-\frac{3}{4}\phi_i{}^{\hat{i}}\Box\phi^i{}_{\hat{i}} - \frac{9}{2}u_{ijk}{}^{\hat{i}}\Box u^{ijk}{}_{\hat{i}} + N_i{}^{\hat{i}}N^i{}_{\hat{i}} \right.$$
$$\left. - A_{ai}{}^{\hat{i}}A^{ai}{}_{\hat{i}} - i\bar{\psi}^{\hat{i}}\slashed{\partial}\psi_{\hat{i}} - \frac{9}{2}i\bar{\lambda}_{ij}{}^{\hat{i}}\slashed{\partial}\lambda^{ij}{}_{\hat{i}} \right). \tag{3}$$

This model describes $16 + 16$ propagating degrees of freedom, including the "Hypermultiplet" fields $\phi_i{}^{\hat{i}}$ and $\psi^{\hat{i}}$, but also includes extra matter fields $u_{ijk}{}^{\hat{i}}$, $A_{ai}{}^{\hat{i}}$, and $\lambda_{ij}{}^{\hat{i}}$. In order to realize an off-shell Hypermultiplet conforming to our specified criteria, we require a mechanism to "relax" the divergence-free constraint imposed on $A_{ai}{}^{\hat{i}}$ and also to render this field auxiliary. Thus, we seek another supermultiplet which can properly supply the divergence $\partial^a A_{ai}{}^{\hat{i}}$.

A suitable multiplet for relaxing the Extended Tensor multiplet should have as its highest components four dimension-two Lorentz scalar bosons transforming under $SU(2)_R \times SU(2)_H$ as a doublet–doublet and subject to a reality constraint $X_i{}^{\hat{i}} = \varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}X^j{}_{\hat{j}}$. This requirement is fulfilled by the Sextet multiplet, which describes $64 + 64$ off-shell degrees of freedom, and has the following component transformation rules,

$$\delta_Q Z_{ijklm}{}^{\hat{i}} = i\varepsilon_{n(i}\bar{\epsilon}^n\Sigma_{jklm)}{}^{\hat{i}}$$
$$+ i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon_{ls}\varepsilon_{mt}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{n(p}\bar{\epsilon}_n\Sigma^{qrst)}{}_{\hat{j}},$$

$$\delta_Q \Sigma_{ijkl}{}^{\hat{i}} = -2\varepsilon^{mn}\slashed{\partial}Z_{ijklm}{}^{\hat{i}}\epsilon_n + K_{a(ijk}{}^{\hat{i}}\gamma^a\epsilon_{l)} + \varepsilon_{m(i}P_{jkl)}{}^{\hat{i}}\epsilon^m,$$

$$\delta_Q P_{ijk}{}^{\hat{i}} = 2i\varepsilon^{mn}\bar{\epsilon}_m\Sigma_{nijk}{}^{\hat{i}} + i\bar{\epsilon}_{(i}\xi_{jk)}{}^{\hat{i}},$$

$$\delta_Q K_{aijk}{}^{\hat{i}} = -\frac{1}{5}i\bar{\epsilon}^m(5\slashed{\partial}\gamma_a + 2\partial_a)\Sigma_{mijk}{}^{\hat{i}} - \frac{1}{2}i\varepsilon_{m(i}\bar{\epsilon}^m\gamma_a\xi_{jk)}{}^{\hat{i}}$$
$$- \frac{1}{5}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_m(5\slashed{\partial}\gamma_a + 2\partial_a)\Sigma^{mpqr}{}_{\hat{j}}$$
$$- \frac{1}{2}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{m(p}\bar{\epsilon}_m\gamma_a\xi^{qr)}{}_{\hat{j}},$$

$$\delta_Q \xi_{ij}{}^{\hat{i}} = \frac{3}{2}\slashed{\partial}P_{ijm}{}^{\hat{i}}\epsilon^m + \frac{1}{2}\varepsilon^{mn}(3\slashed{\partial}\gamma^a + 8\partial^a)K_{aijm}{}^{\hat{i}}\epsilon_n + X_{(i}{}^{\hat{i}}\epsilon_{j)},$$

$$\delta_Q X_i{}^{\hat{i}} = \frac{4}{3}i\bar{\epsilon}^m\slashed{\partial}\xi_{mi}{}^{\hat{i}} + \frac{4}{3}i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_m\slashed{\partial}\xi^{mj}{}_{\hat{j}}. \tag{4}$$

As a rule, adjacent $SU(2)_R$ indices are always symmetrized. For example, the lowest component $Z_{ijklm}{}^{\hat{i}} = Z_{(ijklm)}{}^{\hat{i}}$ transforms as a sextet–doublet under $SU(2)_R \times SU(2)_H$. All bosons except $P_{ijk}{}^{\hat{i}}$ satisfy reality constraints, namely $X_i{}^{\hat{i}} = \varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}X^j{}_{\hat{j}}$, $K_{aijk}{}^{\hat{i}} = \varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn}{}^{\hat{i}\hat{j}}K_a^{lmn}{}_{\hat{j}}$, and $Z_{ijklm}{}^{\hat{i}} = \varepsilon_{in}\varepsilon_{jp}\varepsilon_{kq}\varepsilon_{lr}\varepsilon_{ms}{}^{\hat{i}\hat{j}} \times Z^{npqrs}{}_{\hat{j}}$.

We can "relax" the Extended Tensor multiplet using the Sextet multiplet, by identifying the component $X_i{}^{\hat{i}}$ with $\partial^a A_{ai}{}^{\hat{i}}$. Under this circumstance the supersymmetry algebra closes on

the component fields only if the rule $\delta_Q A_{ai}{}^{\hat{i}}$ is augmented by $\xi_{ij}{}^{\hat{i}}$ dependent terms. These modifications are easy to determine from the transformation rule $\delta_Q \xi_{ij}{}^{\hat{i}}$. Furthermore, the commutator $\delta_Q^2 N_i{}^{\hat{i}}$ is proportional to $\partial^a A_{ai}{}^{\hat{i}}$. Since this is no longer zero, this indicates the need for a $\xi_{ij}{}^{\hat{i}}$ dependent term to be added to $\delta_Q N_i{}^{\hat{i}}$. This process continues to other components, and can be resolved completely. The fully amended transformation rules for the Relaxed Extended Tensor multiplet, which describes $96 + 96$ component degrees of freedom, are

$$\delta_Q \phi_i{}^{\hat{i}} = i\bar{\epsilon}_i \psi^{\hat{i}} + i\varepsilon^{mn}\bar{\epsilon}_m \lambda_{ni}{}^{\hat{i}} + i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^j \psi_{\hat{j}}$$
$$+ i\varepsilon_{ij}\varepsilon_{mn}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^m \lambda^{nj}{}_{\hat{j}},$$

$$\delta_Q u_{ijk}{}^{\hat{i}} = i\bar{\epsilon}_{(i}\lambda_{jk)}{}^{\hat{i}} + i\varepsilon_{im}\varepsilon_{jn}\varepsilon_{kp}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^{(m}\lambda^{np)}{}_{\hat{j}}$$
$$- \frac{16}{15}i\bar{\epsilon}^l \Sigma_{ijkl}{}^{\hat{i}} - \frac{16}{15}i\varepsilon_{im}\varepsilon_{jn}\varepsilon_{kl}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_r \Sigma^{rmnl}{}_{\hat{j}},$$

$$\delta_Q Z_{ijklm}{}^{\hat{i}} = i\varepsilon_{n(i}\bar{\epsilon}^n \Sigma_{jklm)}{}^{\hat{i}}$$
$$+ i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon_{ls}\varepsilon_{mt}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{n(p}\bar{\epsilon}_n \Sigma^{qrst)}{}_{\hat{j}},$$

$$\delta_Q \psi^{\hat{i}} = \frac{3}{2}\partial\!\!\!/\phi_i{}^{\hat{i}}\epsilon^i + A_{ai}{}^{\hat{i}}\gamma^a\epsilon^i + \varepsilon^{mn}N_m{}^{\hat{i}}\epsilon_n,$$

$$\delta_Q \lambda_{ij}{}^{\hat{i}} = \frac{1}{3}\varepsilon_{k(i}\partial\!\!\!/\phi_{j)}{}^{\hat{i}}\epsilon^k + 2\partial\!\!\!/ u_{ijk}{}^{\hat{i}}\epsilon^k - \frac{2}{3}\varepsilon_{k(i}A_{aj)}{}^{\hat{i}}\gamma^a\epsilon^k$$
$$- \frac{2}{3}N_{(i}{}^{\hat{i}}\epsilon_{j)} - \frac{4}{3}\varepsilon_{im}\varepsilon_{jn}\varepsilon^{\hat{i}\hat{j}}P^{mnl}{}_{\hat{j}}\epsilon_l + \frac{4}{3}K_{aijm}{}^{\hat{i}}\gamma^a\epsilon^m,$$

$$\delta_Q \Sigma_{ijkl}{}^{\hat{i}} = -2\varepsilon^{mn}\partial\!\!\!/ Z_{ijklm}{}^{\hat{i}}\epsilon_n + K_{a(ijk}{}^{\hat{i}}\gamma^a\epsilon_{l)} + \varepsilon_{m(i}P_{jkl)}{}^{\hat{i}}\epsilon^m,$$

$$\delta_Q N_i{}^{\hat{i}} = i\varepsilon_{ij}\bar{\epsilon}^j\partial\!\!\!/\psi^{\hat{i}} - 3i\bar{\epsilon}^j\partial\!\!\!/\lambda_{ij}{}^{\hat{i}} - \frac{8}{3}i\varepsilon_{ij}\varepsilon_{mn}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^m \xi^{nj}{}_{\hat{j}},$$

$$\delta_Q A_{ai}{}^{\hat{i}} = \frac{1}{2}i\bar{\epsilon}_i \gamma_{ab}\partial^b\psi^{\hat{i}} - \frac{3}{2}i\varepsilon^{mn}\bar{\epsilon}_m\gamma_{ab}\partial^b\lambda_{ni}{}^{\hat{i}}$$
$$+ \frac{4}{3}i\bar{\epsilon}^m\gamma_a\xi_{mi}{}^{\hat{i}} + \frac{1}{2}i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^j\gamma_{ab}\partial^b\psi_{\hat{j}}$$
$$- \frac{3}{2}i\varepsilon_{ij}\varepsilon_{mn}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}^m\gamma_{ab}\partial^b\lambda^{nj}{}_{\hat{j}}$$
$$+ \frac{4}{3}i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_m\gamma_a\xi^{mj}{}_{\hat{j}},$$

$$\delta_Q P_{ijk}{}^{\hat{i}} = 2i\varepsilon^{mn}\bar{\epsilon}_m\partial\!\!\!/\Sigma_{nijk}{}^{\hat{i}} + i\bar{\epsilon}_{(i}\xi_{jk)}{}^{\hat{i}},$$

$$\delta_Q K_{aijk}{}^{\hat{i}} = -\frac{1}{5}i\bar{\epsilon}^m(5\partial\!\!\!/\gamma_a + 2\partial_a)\Sigma_{mijk}{}^{\hat{i}} - \frac{1}{2}i\varepsilon_{m(i}\bar{\epsilon}^m\gamma_a\xi_{jk)}{}^{\hat{i}}$$
$$- \frac{1}{5}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_m(5\partial\!\!\!/\gamma_a + 2\partial_a)\Sigma^{mpqr}{}_{\hat{j}}$$
$$- \frac{1}{2}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{m(p}\bar{\epsilon}_m\gamma_a\xi^{qr)}{}_{\hat{j}},$$

$$\delta_Q \xi_{ij}{}^{\hat{i}} = \frac{3}{2}\partial\!\!\!/ P_{ijm}{}^{\hat{i}}\epsilon^m + \frac{1}{2}\varepsilon^{mn}(3\partial\!\!\!/\gamma^a + 8\partial^a)K_{aijm}{}^{\hat{i}}\epsilon_n$$
$$+ \partial^a A_{a(i}{}^{\hat{i}}\epsilon_{j)}. \tag{5}$$

These transformation rules represent the $\mathcal{N} = 2$ supersymmetry algebra on all fields, without a central charge. The Relaxed Extended Tensor multiplet amends the Extended Tensor multiplet by removing the divergence-free constraint on $A_{ai}{}^{\hat{i}}$, and does so by amalgamating this multiplet with the Sextet multiplet.

The action for the Extended Tensor multiplet given in (3) is not invariant if we allow $\partial^a A_{ai}{}^{\hat{i}} \neq 0$. One must add several terms which together restore supersymmetry in the clas-

sical action, rendering the resulting combined action invariant under (5). The construction obtained in this way resolves the issue of the divergence-free vector, but not without cost. On the one hand this introduces more ostensibly propagating fields, namely the bosons $Z_{ijklm}{}^{\hat{i}}$ and the fermions $\Sigma_{ijkl}{}^{\hat{i}}$. But the corresponding action has deeper problems. A thorough analysis, in which a proper basis is identified by field redefinitions, shows that the redefined fields $(\phi_i{}^{\hat{i}}|\psi^{\hat{i}})$ contribute to the Hamiltonian with opposite signs as compared to the contributions from $(u_{ijk}{}^{\hat{i}}, Z_{ijklm}{}^{\hat{i}}|\lambda_{ij}{}^{\hat{i}}, \Sigma_{ijkl}{}^{\hat{i}})$. This does not represent a mathematical inconsistency in the classical theory.[1] However, the putative self-action associated with (5) thereby presents a problematic obstruction to consistent quantization.

The theory is improved significantly by coupling a Dual Sextet multiplet as a Lagrange multiplier. This multiplet has the following transformation rules,

$$\delta_Q H_i{}^{\hat{i}} = i\bar{\epsilon}^j \Omega_{ij}{}^{\hat{i}} + i\varepsilon_{ij}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_k \Omega^{jk}{}_{\hat{j}},$$

$$\delta_Q \Omega_{ij}{}^{\hat{i}} = \frac{4}{3}\partial\!\!\!/ H_{(i}{}^{\hat{i}}\epsilon_{j)} + \varepsilon^{mn}T_{aijm}{}^{\hat{i}}\gamma^a\epsilon_n + R_{ijm}{}^{\hat{i}}\epsilon^m,$$

$$\delta_Q R_{ijk}{}^{\hat{i}} = \frac{3}{2}i\bar{\epsilon}_{(i}\partial\!\!\!/\Omega_{jk)}{}^{\hat{i}} + i\varepsilon^{mn}\bar{\epsilon}_m\eta_{nijk}{}^{\hat{i}},$$

$$\delta_Q T_{aijk}{}^{\hat{i}} = \frac{1}{4}i\varepsilon_{m(i}\bar{\epsilon}^m(3\partial\!\!\!/\gamma_a - 2\partial_a)\Omega_{jk)}{}^{\hat{i}} + \frac{1}{2}i\bar{\epsilon}^m\gamma_a\eta_{mijk}{}^{\hat{i}}$$
$$+ \frac{1}{4}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{m(p}\bar{\epsilon}_m(3\partial\!\!\!/\gamma_a - 2\partial_a)\Omega^{qr)}{}_{\hat{j}}$$
$$+ \frac{1}{2}i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon^{\hat{i}\hat{j}}\bar{\epsilon}_m\gamma_a\eta^{mpqr}{}_{\hat{j}},$$

$$\delta_Q \eta_{ijkl}{}^{\hat{i}} = 2\varepsilon_{m(i}\partial\!\!\!/ R_{jkl)}{}^{\hat{i}}\epsilon^m + \frac{2}{5}(5\partial\!\!\!/\gamma^a + 8\partial^a)T_{a(ijk}^{\hat{i}}\epsilon_{l)}$$
$$+ \varepsilon^{mn}Y_{ijklm}{}^{\hat{i}}\epsilon_n,$$

$$\delta_Q Y_{ijklm}{}^{\hat{i}} = -2i\varepsilon_{n(i}\bar{\epsilon}^n\partial\!\!\!/\eta_{jklm)}{}^{\hat{i}}$$
$$- 2i\varepsilon_{ip}\varepsilon_{jq}\varepsilon_{kr}\varepsilon_{ls}\varepsilon_{mt}\varepsilon^{\hat{i}\hat{j}}\varepsilon^{n(p}\bar{\epsilon}_n\partial\!\!\!/\eta^{qrst)}{}_{\hat{j}}. \tag{6}$$

Note that the highest component $Y_{ijklm}{}^{\hat{i}}$ has the appropriate $SU(2)_R \times SU(2)_H$ structure to couple invariantly to the field $Z_{ijklm}{}^{\hat{i}}$, so as to render $Z_{ijklm}{}^{\hat{i}}$ auxiliary. This can be done supersymmetrically; indeed it is this feature which gives this multiplet its name. As an extra bonus, the modified action has a healthy kinetic sector in the sense that the Hamiltonian is properly positive definite, so as not to preclude quantization.

The structure of this theory is rendered helpfully perspicuous using Adinkra diagrams [6–10], as shown in Fig. 1. These represent the transformation rules graphically; white nodes correspond to off-shell bosons and black nodes correspond to off-shell fermions. The state counting is indicated by the numerals in the nodes, and inter-node edge connections represent terms in the supersymmetry transformation rules. The vertical placement of the nodes correspond faithfully to the relative engineering dimensions of the corresponding fields; "low-

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[1] This circumstance is analogous to the similarly problematic action obtained by coupling two chiral $\mathcal{N} = 1$ superfields Φ_1 and Φ_2 using $\int d^4\theta \, (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2)$, resulting in a model which exhibits supersymmetry invariance at the classical level but is plagued with non-unitarity at the quantum level.
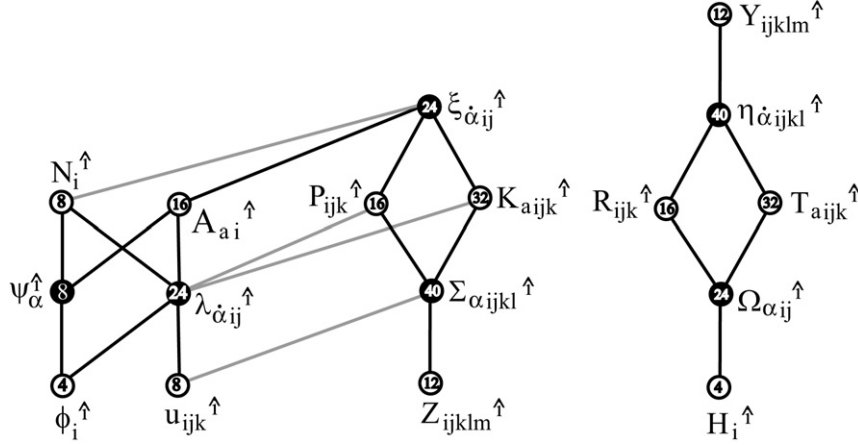
Fig. 1. Adinkraic depiction of the transformation rules given in (5) and (6).

er" nodes correspond to fields with lower engineering dimension while "higher" nodes have greater engineering dimension. Black edges correspond to terms which act both "upward", by transforming fields with a lower engineering dimension into fields with higher engineering dimension, and also "downward" by transforming fields with a higher engineering dimension into derivatives of fields with a lower engineering dimension. Grey edges correspond to terms which only act "upward". It is a noteworthy fact about the Relaxed Extended Tensor multiplet that its Adinkra contains grey edges.

The supersymmetric action corresponding to the Relaxed Extended Tensor multiplet (RETM) coupled to a Dual Sextet multiplet is given by

$$
\begin{aligned}
S = \int d^4x \Bigg(&-\frac{3}{4}\phi_i{}^{\hat{\imath}}\Box\phi^i{}_{\hat{\imath}} - \frac{9}{2}u_{ijk}{}^{\hat{\imath}}\Box u^{ijk}{}_{\hat{\imath}} + \frac{32}{3}Z^{ijklm}{}_{\hat{\imath}}\Box Z_{ijklm}{}^{\hat{\imath}} \\
&+ N_i{}^{\hat{\imath}}N^i{}_{\hat{\imath}} - A_{ai}{}^{\hat{\imath}}A^{ai}{}_{\hat{\imath}} - \frac{16}{3}P^{ijk}{}_{\hat{\imath}}P_{ijk}{}^{\hat{\imath}} + \frac{16}{3}K_a^{ijk}{}_{\hat{\imath}}K_{aijk}{}^{\hat{\imath}} \\
&- i\bar{\psi}^{\hat{\imath}}\slashed{\partial}\psi_{\hat{\imath}} - \frac{9}{2}i\bar{\lambda}_{ij}{}^{\hat{\imath}}\slashed{\partial}\lambda^{ij}{}_{\hat{\imath}} + \frac{32}{3}i\bar{\Sigma}^{ijkl}{}_{\hat{\imath}}\slashed{\partial}\Sigma_{ijkl}{}^{\hat{\imath}} \\
&+ 4i\varepsilon^{im}\varepsilon^{jn}\varepsilon_{\hat{\imath}\hat{\jmath}}\bar{\xi}_{ij}{}^{\hat{\imath}}\lambda_{mn}{}^{\hat{\jmath}} + 4i\varepsilon_{im}\varepsilon_{jn}\varepsilon^{\hat{\imath}\hat{\jmath}}\bar{\xi}^{ij}{}_{\hat{\imath}}\lambda^{mn}{}_{\hat{\jmath}} \\
&+ A_{ai}{}^{\hat{\imath}}\partial^a\phi^i{}_{\hat{\imath}} + 16K_a^{ijk}{}_{\hat{\imath}}\partial^a u_{ijk}{}^{\hat{\imath}} - Z_{ijklm}{}^{\hat{\imath}}Y^{ijklm}{}_{\hat{\imath}} \\
&+ H^i{}_{\hat{\imath}}\partial^a A_{ai}{}^{\hat{\imath}} - 2K_a^{ijk}{}_{\hat{\imath}}T_{ijk}^a{}^{\hat{\imath}} + P_{ijk}{}^{\hat{\imath}}R^{ijk}{}_{\hat{\imath}} + P^{ijk}{}_{\hat{\imath}}R_{ijk}{}^{\hat{\imath}} \\
&- i\bar{\eta}^{ijkl}{}_{\hat{\imath}}\Sigma_{ijkl}{}^{\hat{\imath}} - i\bar{\eta}_{ijkl}{}^{\hat{\imath}}\Sigma^{ijkl}{}_{\hat{\imath}} \\
&- i\bar{\xi}^{ij}{}_{\hat{\imath}}\Omega_{ij}{}^{\hat{\imath}} - i\bar{\xi}_{ij}{}^{\hat{\imath}}\Omega^{ij}{}_{\hat{\imath}} \Bigg).
\end{aligned}
\tag{7}
$$

This action is invariant under the transformation rules given in (5) and (6). The first fourteen terms are invariant by themselves, and correspond to the unhealthy RETM action referred to above. The remaining nine terms are also invariant by themselves, and correspond to the invariant coupling of the Dual Sextet multiplet to the Sextet portion of the RETM. The addition of these extra terms has two significant effects, one to render the boson $Z_{ijklm}{}^{\hat{\imath}}$ and the fermion $\Sigma_{ijkl}{}^{\hat{\imath}}$ auxiliary, and the other to repair the positivity of the Hamiltonian.

To find a canonical basis, we redefine the component fields in (7) according to

$$
\begin{aligned}
A_{ai}{}^{\hat{\imath}} &\to A_{ai}{}^{\hat{\imath}} - \frac{1}{2}H_i{}^{\hat{\imath}}, \\
H_i{}^{\hat{\imath}} &\to H_i{}^{\hat{\imath}} + \phi_i{}^{\hat{\imath}}, \\
P_{ijk}{}^{\hat{\imath}} &\to P_{ijk}{}^{\hat{\imath}} + \frac{3}{16}R_{ijk}{}^{\hat{\imath}}, \\
K_{aijk}{}^{\hat{\imath}} &\to K_{aijk}{}^{\hat{\imath}} + \frac{3}{16}T_{aijk}{}^{\hat{\imath}}, \\
T_{aijk}{}^{\hat{\imath}} &\to T_{aijk}{}^{\hat{\imath}} + 8\partial_a u_{ijk}{}^{\hat{\imath}}, \\
\Omega_{\alpha ij}{}^{\hat{\imath}} &\to \Omega_{\alpha ij}{}^{\hat{\imath}} + 4\varepsilon_{im}\varepsilon_{jn}\varepsilon^{\hat{\imath}\hat{\jmath}}\lambda^{mn}{}_{\hat{\jmath}}, \\
\eta_{ijkl}{}^{\hat{\imath}} &\to \eta_{ijkl}{}^{\hat{\imath}} + \frac{16}{3}\slashed{\partial}\Sigma_{ijkl}{}^{\hat{\imath}}, \\
Y_{ijklm}{}^{\hat{\imath}} &\to Y_{ijklm}{}^{\hat{\imath}} + \frac{32}{3}\Box Z_{ijklm}{}^{\hat{\imath}}.
\end{aligned}
\tag{8}
$$

When expressed in terms of the re-defined fields, the action (7) becomes

$$
\begin{aligned}
S = \int d^4x \Bigg(&-\frac{3}{4}\phi_i{}^{\hat{\imath}}\Box\phi^i{}_{\hat{\imath}} - \frac{9}{2}u_{ijk}{}^{\hat{\imath}}\Box u^{ijk}{}_{\hat{\imath}} - \frac{1}{4}H^i{}_{\hat{\imath}}\Box H_i{}^{\hat{\imath}} \\
&- i\bar{\psi}^{\hat{\imath}}\slashed{\partial}\psi_{\hat{\imath}} - \frac{9}{2}i\bar{\lambda}_{ij}{}^{\hat{\imath}}\slashed{\partial}\lambda^{ij}{}_{\hat{\imath}} - i\bar{\eta}^{ijkl}{}_{\hat{\imath}}\Sigma_{ijkl}{}^{\hat{\imath}} - i\bar{\eta}_{ijkl}{}^{\hat{\imath}}\Sigma^{ijkl}{}_{\hat{\imath}} \\
&- i\bar{\xi}^{ij}{}_{\hat{\imath}}\Omega_{ij}{}^{\hat{\imath}} - i\bar{\xi}_{ij}{}^{\hat{\imath}}\Omega^{ij}{}_{\hat{\imath}} + \frac{16}{3}K_a^{ijk}{}_{\hat{\imath}}K_{aijk}{}^{\hat{\imath}} - \frac{3}{16}T_a^{ijk}{}_{\hat{\imath}}T_{ijk}^a{}^{\hat{\imath}} \\
&- A_{ai}{}^{\hat{\imath}}A^{ai}{}_{\hat{\imath}} - Z_{ijklm}{}^{\hat{\imath}}Y^{ijklm}{}_{\hat{\imath}} - \frac{16}{3}P^{ijk}{}_{\hat{\imath}}P_{ijk}{}^{\hat{\imath}} \\
&- \frac{3}{16}R^{ijk}{}_{\hat{\imath}}R_{ijk}{}^{\hat{\imath}} + N_i{}^{\hat{\imath}}N^i{}_{\hat{\imath}} \Bigg).
\end{aligned}
\tag{9}
$$

This action describes $96 + 96$ degrees of freedom off-shell, and $16 + 16$ degrees of freedom on-shell. The propagating fields are given by $(\phi_i{}^{\hat{\imath}}, u_{ijk}{}^{\hat{\imath}}, H_i{}^{\hat{\imath}} | \psi^{\hat{\imath}}, \lambda_{ij}{}^{\hat{\imath}})$. Notice that all kinetic terms in (9) have consistent signs.

The model described by (9) has $\mathcal{N} = 2$ supersymmetry with no central charge, finite off-shell degrees of freedom, and a Hypermultiplet sector, the latter comprising a pairing of boson fields and fermion fields transforming under $SU(2)_R \times SU(2)_H$
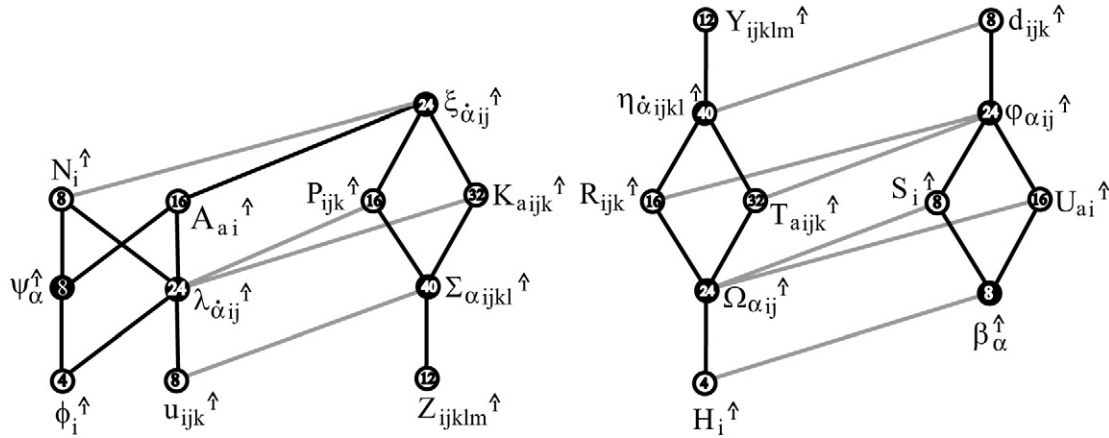
Fig. 2. The Relaxed Extended Tensor multiplet admits, at the classical level, a coupling to a particular amalgamation of the Dual Sextet multiplet with the Dual Quadruplet multiplet, the latter of which is represented by the right-most agglomeration in the above Adinkra.

with the representation content implicit in $\phi_i{}^{\hat{i}}$ and $\psi^{\hat{i}}$. We believe this represents the first appearance of such an action, and this is the principal result motivating this Letter.

We have done further analysis on the structure described above, and have identified a compelling extension to this model. There exists a multiplet, namely the Dual to a constrained Quadruplet multiplet, which has the proper field content to render the fields $u_{ijk}{}^{\hat{i}}$ and $\lambda_{ij}{}^{\hat{i}}$ non-propagating while supplying solely an additional Weyl spinor $SU(2)_H$ doublet, which we call $\beta^{\hat{i}}$, to the set of propagating fields. Thus, in this case the ostensibly propagating fields would have precisely the content of two Hypermultiplets, $(\phi_i{}^{\hat{i}}|\psi^{\hat{i}})$ and $(H_i{}^{\hat{i}}|\beta^{\hat{i}})$. However, the action obtained in this way has the curious feature that the two Hypermultiplet sectors contribute with opposite signs to the Hamiltonian.

The particular way in which the Dual Quadruplet couples to the Relaxed Extended Tensor multiplet is interesting. A supersymmetric invariant action involving these fields exists only if the transformation rules for the Dual Sextet multiplet are extended to admit "one-way" terms in which Dual Sextet fields transform into Dual Quadruplet fields, but not the other way around. These terms are codified by the grey edges in the right-hand connected component of the Adinkra representation of this extended model shown in Fig. 2. These represent a "quasi-relaxation" which closely mimics the one-way terms present in the Relaxed Extended Tensor multiplet. The off-shell states comprising the Dual Quadruplet multiplet are also clarified in Fig. 2.

It is conceivable that by coupling the above model to $\mathcal{N} = 2$ Vector multiplets that the canonical structure of the resulting action would repair the non-unitarity present in the system codified in Fig. 2, thereby producing a perfectly consistent model involving two Hypermultiplets coupled to Vector multiplets, having finite off-shell degrees of freedom and no off-shell central charge. It is also possible that such a structure would split into separable Hypermultiplet-Vector multiplet couplings, one or both of which could be consistently removed, thereby revealing an off-shell representation involving only one Hypermultiplet and only one Vector multiplet. In either case, the elucidation of such a structure would likely shed surprising

light on a possible off-shell extension to the important case of $\mathcal{N} = 4$ super-Yang–Mills theory, an application which has been a prime motivator throughout our work. We are currently engaged in developing these ideas further. But, we find the mere existence of the off-shell extension of the Dual Hypermultiplet pairing exhibited in Fig. 2 surprising and noteworthy in its own right, regardless of what mechanisms might or might not exist to cure its non-unitarity.

As another application of these ideas, the models described in this Letter should generalize to provide non-linear sigma models with various uses. For example, these should allow a new class of c-maps [11]; as known since 1984, by reducing four-dimensional field theories involving vector or tensor fields to two dimensions one obtains new non-linear sigma-models. This follows because the two-dimensional analog of the condition $\partial^a A_{ai}{}^{\hat{i}} = 0$ possesses the solution $\partial^a A_{ai}{}^{\hat{i}} = \epsilon_{ab}\partial^b \varphi_i{}^{\hat{i}}$. In particular, the action (3) reduces in two-dimensions to a model having off-shell 2d, $\mathcal{N} = 4$ supersymmetry. It follows that this must possess non-linear σ-model extensions that lead to a proper c-map. Such non-linear σ-models are characterized by possessing a Kähler-like prepotential that is a doublet–doublet under $SU(2)_R \times SU(2)_H$, or a doublet-$2n$-plet under $SU(2)_H \to Sp(2n)$ for $n > 1$. It is also interesting to note that under a reduction to two dimensions, the dynamical sectors of (3) and (9) become identical.[2]

In conclusion, we have exhibited a concrete example of an off-shell extension of a Hypermultiplet $(\phi_i{}^{\hat{i}}|\psi^{\hat{i}})$ coupled to "Quadruplet" matter $(u_{ijk}{}^{\hat{i}}|\lambda_{ij}{}^{\hat{i}})$, which has finite off-shell degrees of freedom and no off-shell central charge. We have elucidated some novel forms of $\mathcal{N} = 2$ matter, namely the Extended Tensor multiplet, the Sextet multiplet, the Relaxed Extended Tensor multiplet, the Dual Sextet multiplet, the Dual Quadruplet multiplet, and some interesting supersymmetric couplings involving these. We have also described the rudiments of a paradigm based on these constructions which seems to suggest a

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[2] Dr. S. Kuzenko has suggested to us that there may exist a link between some of the multiplets introduced above and the $O(n)$ or $O(2n)$ multiplets discussed in [12].

way to understand features of off-shell extensions to $\mathcal{N} = 4$ super-Yang–Mills theory.

Acknowledgements

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