

Mirror Symmetry and Integral Variations of Hodge Structure Underlying One-parameter Families of Calabi-Yau Threefolds

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ABSTRACT. This proceedings note introduces aspects of the authors' work relating mirror symmetry and integral variations of Hodge structure. The emphasis is on their classification of the integral variations of Hodge structure which can underlie families of Calabi-Yau threefolds over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with $b^3 = 4$, or equivalently $h^{2,1} = 1$, and the related issues of geometric realization of these variations. The presentation parallels that of the first author's talk at the BIRS workshop.

1. Integral structures in mirror symmetry

1.1. The first examples. Since its introduction to the mathematical community through the seminal papers of Greene-Plesser [GP] and Candelas-de la Ossa-Green-Parkes [CdOGP], Mirror Symmetry has been the source of the most persistently rich and subtle novel mathematics yet to emerge from the study of string dualities. One of its key features is its resistance to rigorous mathematical definition and even more to being described within any single traditional mathematical setting. Research in mathematics related to mirror symmetry is thus driven by the goal of discerning deeper, more complete, and purely mathematical avatars of this very physical duality.

For example, one of the earliest mirror symmetric mathematical predictions was that a “mirror pair” of Calabi-Yau threefolds X and \tilde{X} should have the property that their Hodge numbers “mirror” one another, i.e., that $h^{1,1}(X) = h^{2,1}(\tilde{X})$ and $h^{2,1}(X) = h^{1,1}(\tilde{X})$.

The first proposal for such a pair was introduced in [GP] where it was applied to Calabi-Yau threefolds presented as hypersurfaces in a weighted projective space. The construction involves the operation of quotienting by a finite group (or “orbifolding”).

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The simplest of Calabi-Yau threefold hypersurfaces, the generic quintic hypersurface in \mathbb{P}^4 , often denoted by $\mathbb{P}^4[5]$, has $h^{1,1} = 1$ (coming from the polarization class) and $h^{2,1} = 101$ (corresponding to 101 complex structure moduli). This last follows from Kodaira-Spencer theory [**Kod**] and the theorem of Bogomolov-Tian-Todorov [**Tian**] that any Calabi-Yau manifold admits a locally universal deformation over a smooth base. In this case all of the complex structure deformations arise from varying the coefficients of the defining equations for quintic hypersurfaces in \mathbb{P}^4 . Let G denote the finite group $(\mathbb{Z}/5\mathbb{Z})^3$ presented as

$$\left\{ (a_1, a_2, a_3, a_4, a_5) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum_{i=1}^5 a_i = 0 \right\} / (\mathbb{Z}/5\mathbb{Z}),$$

where the action of $\mathbb{Z}/5\mathbb{Z}$ is the diagonal one. Upon quotienting the projective space \mathbb{P}^4 by G , acting as

$$(x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (\mu^{a_1} x_1 : \mu^{a_2} x_2 : \mu^{a_3} x_3 : \mu^{a_4} x_4 : \mu^{a_5} x_5)$$

(where $\mu = \exp(2\pi i/5)$), resolving the resulting orbifold singularities, and keeping track of the hypersurface throughout the process, another family of Calabi-Yau threefolds is constructed — no longer as hypersurfaces in projective space, but now with the property that generically $h^{1,1} = 101$ and $h^{2,1} = 1$. It follows that these threefolds sit in a one-parameter family of complex structures. The specific presentation of the mirror family \tilde{X}_z in \mathbb{P}^4/G is given by

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5z^{-1/5} x_1 x_2 x_3 x_4 x_5 = 0.$$

Here the complex deformation parameter is identified with a coordinate on the base of the family $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

REMARK 1.1. In [**GP**, Table 1] this construction is extended to include intermediate quotients by subgroups of G . In particular, instead of the quintic itself one can consider the quotient of the quintic by the free $\mathbb{Z}/5\mathbb{Z}$ -action given by

$$(x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_1 : \mu x_2 : \mu^2 x_3 : \mu^3 x_4 : \mu^4 x_5),$$

resulting in a family of Calabi-Yau threefolds with $h^{1,1} = 1$ and $h^{2,1} = 21$. These “quintic twin” Calabi-Yau threefolds have fundamental group $\mathbb{Z}/5\mathbb{Z}$. The mirror family is constructed in [**GP**] as another intermediate quotient, and consists (as expected) of simply connected Calabi-Yau threefolds with $h^{1,1} = 21$ and $h^{2,1} = 1$. Moreover the parameter space of this quintic twin mirror family \tilde{X}_z^{twin} is again $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ as for the original quintic mirror family.

Since the mirror quintic family $\tilde{\mathfrak{X}} = \{\tilde{X}_z\}$ depends on just one parameter, an old method of Griffiths-Dwork [**CoKa**, §5.3] can be readily applied to determine the ordinary differential equation satisfied by the periods of the (unique up to complex scaling) holomorphic three-form on the Calabi-Yau threefold. More formally, consider the local system denoted $R^3\varphi_*\mathbb{C}$ arising from the third cohomology (with complex coefficients) of the fibers \tilde{X}_z of the algebraic family $\varphi: \tilde{\mathfrak{X}} \rightarrow B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Integration over 3-cycles defines a pairing, making this local system dual to the local system whose fiber over z is $H_3(\tilde{X}_z, \mathbb{C})$. Every class $\gamma \in H_3(\tilde{X}_z, \mathbb{C})$ extends uniquely as a multi-valued flat section of this latter local system. Consider the holomorphic bundle $R^3\varphi_*\mathbb{C} \otimes \mathcal{O}_B$; the local system $R^3\varphi_*\mathbb{C}$ determines a flat structure on this vector bundle and hence a flat connection, ∇_z . The sub-line bundle $\mathcal{H}^{3,0}$ of classes of relative holomorphic 3-forms is of special interest. A holomorphic section $\varpi(z)$

of $\mathcal{H}^{3,0}$ amounts to a holomorphic family of holomorphic 3-forms. Generically $\varpi(z)$ and its first three covariant derivatives, $\nabla_z^i(\varpi)$ ($1 \leq i \leq 3$) are linearly independent, and $\nabla_z^4(\varpi)$ is expressible as a linear combination of these with coefficients meromorphic functions on the base B . This relationship is the differential equation associated to the family of cohomology classes $[\varpi(z)]$ — the *Picard-Fuchs equation*. There is also the *Picard-Fuchs ODE* obtained by replacing Θ by $z \frac{d}{dz}$. An integral lattice in the space of all solutions of this ODE is given by the periods $\int_{\gamma} \varpi(z)$ as γ varies over all (integral) three-cycles. There is a holomorphic map of the dual of the vector bundle of solutions of the Picard-Fuchs ODE to $R^3\varphi_*\mathbb{C} \otimes \mathcal{O}_B$. This map is an isomorphism near any regular singular point of the ODE.

For the quintic mirror family \mathfrak{X} , letting $\Theta = z\nabla_z$, for an appropriate holomorphic family $\varpi(z)$ of holomorphic 3-forms the Picard-Fuchs equation is the generalized hypergeometric differential equation

$$\left[\Theta^4 - z \left(\Theta + \frac{1}{5} \right) \left(\Theta + \frac{2}{5} \right) \left(\Theta + \frac{3}{5} \right) \left(\Theta + \frac{4}{5} \right) \right] \varpi(z) = 0 .$$

The Picard-Fuchs ODE has 3 singular points $0, 1, \infty$, all of which are regular singular points. Let γ_0 be an indivisible integral cycle invariant under monodromy about $z = 0$. A holomorphic solution to the Picard-Fuchs ODE is given by the hypergeometric-type series

$$\int_{\gamma_0} \varpi(z) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} \left(\frac{z}{5^5} \right)^n$$

defined on the punctured unit disk. This function extends across the origin, reflecting the fact that it is integration of a family of holomorphic 3-forms over a flat family of 3-cycles invariant under monodromy around $z = 0$. This power series is absolutely convergent inside the unit disk. Of course, it also admits a multi-valued analytic continuation to the whole thrice-punctured sphere given explicitly in terms of Meijer functions (Mellin-Barnes type integrals) on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [HTF, GL].

REMARK 1.2. The story for the quintic twin mirror family is similar, with everything working just as above. In fact, for appropriate choices of the family of holomorphic three-forms, the Picard-Fuchs ODE defined as above and satisfied by the periods of the two families of holomorphic forms on \tilde{X}_z and \tilde{X}_z^{twin} are exactly the same. Thus, the difference between the two families is not reflected in the Picard-Fuchs differential equations. It turns out that it is reflected in the lattices spanned by the periods of the holomorphic three-form over a basis of integral cycles. Of course, the difference is also seen from the fact that the ranks of the even-dimensional cohomologies of the two families are different.

Mirror symmetry predicts that the periods over a suitable integral basis of three-cycles on the mirror quintic contain extremely subtle information about the geometry of the original quintic hypersurface in \mathbb{P}^4 . In particular, mirror symmetry predicts the number of rational curves of a given degree lying on the hypersurface [CdOGP, CoKa]. These curve counts can be read off from an appropriate generating function built from the “mirror map” q -series — the single-valued local inverse to the projectivized period mapping about a regular singular point of maximal unipotent monodromy (or, in terms of the geometry of the family of Calabi-Yau threefolds, of maximal degeneracy). In many cases these curve counts, and their

descriptions in terms of hypergeometric functions, have now been rigorously established [LLY1, LLY2, Giv]. We will not be pursuing this approach here; our focus is the integral lattice structure and monodromy rather than enumerative geometry.

1.2. Classical Hodge theory. In order to study more general families of Calabi-Yau threefolds than just the mirror quintic or mirror quintic twin hypersurfaces we introduce here Hodge structures and their variations. Consider a Calabi-Yau threefold \tilde{X} with $h^{2,1} = 1$. The cohomology $H^3(\tilde{X}, \mathbb{R})$ is polarized by the intersection form, denoted $\langle \cdot, \cdot \rangle$, which is unimodular and skew, and has a Hodge decomposition

$$H^3(\tilde{X}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^3(\tilde{X}, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

with associated Hodge filtration

$$F^3 = H^{3,0}, \quad F^2 = H^{3,0} \oplus H^{2,1}, \quad F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}.$$

Furthermore, for all i , $0 \leq i \leq 3$ these filtrants satisfy the conditions

$$(1.1) \quad (F^i)^{\perp} = F^{4-i}$$

and

$$(1.2) \quad F^i \oplus \overline{F^{4-i}} = H^3(\tilde{X}, \mathbb{C}).$$

Notice that we recover the Hodge decomposition from the filtration since $F^i \cap \overline{F^{3-i}} = H^{i,3-i}$.

For us a Calabi-Yau threefold has holonomy all of $SU(3)$ and hence $H^1(\tilde{X}, \mathbb{C}) = 0$, implying that all of H^3 is primitive. Thus, the Hermitian form

$$h(\omega_1, \omega_2) = i \int_M \omega_1 \wedge \bar{\omega}_2 = i \langle \omega_1, \bar{\omega}_2 \rangle$$

is positive definite on $H^{3,0}$ and negative definite on $H^{2,1}$, i.e., for $\omega \neq 0$,

$$(1.3) \quad h(\omega, \omega) \text{ is } \begin{cases} > 0 & \text{when } \omega \in H^{3,0} \\ < 0 & \text{when } \omega \in H^{2,1}. \end{cases}$$

Fix a rank 4 real vector space V with a nondegenerate skew form. Weight-three Hodge structures with $h^{2,1} = 1$ on V (and by this we shall always mean primitive and polarized weight-three Hodge structures) are classified by the period domain

$$\mathcal{D} = \mathrm{Sp}(4, \mathbb{R})/\mathrm{U}(1) \times \mathrm{U}(1).$$

\mathcal{D} is embedded as a domain inside $\mathrm{Sp}(4, \mathbb{C})/P$, where P is the maximal parabolic subgroup. A holomorphic family of such structures on V parameterized by a complex curve U consists of a holomorphic filtration by subbundles

$$\mathcal{F}^3 \subset \mathcal{F}^2 \subset \mathcal{F}^1 \subset \mathcal{F}^0 = V \otimes_{\mathbb{R}} \mathcal{O}_U,$$

with Conditions (1.1), (1.2), and (1.3) holding on every fiber, resulting in a family of (polarized) Hodge structures. Given such a holomorphic family, the *period map* is the (holomorphic) classifying map $\Pi: U \rightarrow \mathcal{D}$.

There is an additional geometric condition on the period map $\Pi: U \rightarrow \mathcal{D}$ if the family of Hodge structures is geometric, i.e., arises from a family of smooth varieties. This is the *Griffiths transversality condition*, also called horizontality, [Grif, Condition (2.8) on page 143] which says that $(\mathcal{F}^i)' \subset \mathcal{F}^{i-1}$ (where the derivative is taken with respect to a local coordinate on U). Horizontal families of Hodge structures are called *variations of Hodge structure (VHS)*, or \mathbb{Z} -VHS, to

emphasize the underlying flat integral structure. Horizontality can be reformulated in terms of a (non-integrable) distribution, called the *horizontal distribution*, on \mathcal{D} . Any period map $\Pi: U \rightarrow \mathcal{D}$ from a horizontal family is tangent to this distribution.

For the Hodge structures we are considering, there is the natural map

$$p_3: \mathcal{D} \rightarrow \mathbb{P}(V_{\mathbb{C}})$$

associating to each Hodge filtration the line F^3 in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. The horizontality condition for a curve $C \subset \mathcal{D}$ is equivalent to

$$T(p_3(C))_p \subset p^{\perp}/\langle p \rangle \subset (V_{\mathbb{C}})/\langle p \rangle = T(\mathbb{P}(V_{\mathbb{C}}))_p.$$

Conversely, given an analytic curve $C \subset \mathbb{P}(V_{\mathbb{C}})$ satisfying this condition, there is a unique lifting $\mathcal{C} \subset \mathcal{D}$ of C that is horizontal: Given $p \in C$, the Hodge filtration is given as follows: $F^3 = \langle p \rangle$, F^2 is the preimage in $V_{\mathbb{C}}$ of $T(C)_p \subset (V_{\mathbb{C}})/\langle p \rangle$, and $F^1 = (F^3)^{\perp}$. For a fuller explanation of this see [BG].

Now let $V = H^3(\tilde{X}, \mathbb{R})$ and let U be a local simply connected open subset in the (one-dimensional) moduli space of complex structures on \tilde{X} . Over U , the family of holomorphic 3-forms $\varpi(z)$ is a holomorphic section of $V_{\mathbb{C}}$. It satisfies a differential equation of the form

$$\nabla_z^4(\varpi(z)) + \sum_{i=0}^3 P_{4-i}(z) \nabla^i(\varpi(z)) = 0.$$

Using a local C^∞ -trivialization of the family, we extend any 3-cycle $\gamma \in H_3(\tilde{X}_{z_0}, \mathbb{C})$ to a family of 3-cycles in all neighboring fibers. The periods $\int_{\gamma} \varpi(z)$ form a (local) holomorphic function on the base. Since the family of cycles γ is a parallel section, the function

$$\varphi_{\gamma}(z) = \int_{\gamma} \varpi(z)$$

satisfies the Picard-Fuchs ODE (i.e., the ODE associated to the original Picard-Fuchs equation for ϖ):

$$\left(\frac{d}{dz} \right)^4 \varphi_{\gamma} + \sum_{i=1}^4 P_{4-i}(z) \left(\frac{d}{dz} \right)^i \varphi_{\gamma} = 0.$$

The 4-dimensional space of periods as γ ranges over all of $H_3(\tilde{X}_b; \mathbb{C})$ is the space of all solutions of the Picard-Fuchs ODE. This gives an identification (over a contractible neighborhood of $b \in B$) between the space of solutions to the Picard-Fuchs ODE and the vector bundle $(R^3\varphi_* \mathbb{C})^* \otimes \mathcal{O}_B$. Fixing a basis $\gamma_1^*, \dots, \gamma_4^*$ of the lattice $H^3(\tilde{X}, \mathbb{Z})/\{\text{Torsion}\}$, we have

$$\varpi(z) = \sum_{i=1}^4 f^i(z) \cdot \gamma_i^*.$$

Of course the $f^i(z)$ are simply the periods

$$\int_{\gamma_i} \varpi(z)$$

where $\gamma_1, \dots, \gamma_4$ is the basis of $H_3(\tilde{X}_z, \mathbb{Z})/\{\text{Torsion}\}$ dual to $\gamma_1^*, \dots, \gamma_4^*$. The periods $f^i(z)$ are a basis of solutions of the Picard-Fuchs ODE.

Let $\varphi: \tilde{\mathfrak{X}} \rightarrow B$ be a family of complex structures on \tilde{X} with a not necessarily simply connected base. The fiberwise sublattice $R^3\varphi_*\mathbb{Z}$ produces a flat connection ∇ , the Gauss-Manin connection, on $R^3\varphi_*\mathbb{C} \otimes \mathcal{O}_B$ which cannot (usually) be globally trivialized. A C^∞ trivialization of the family along paths in B , with basepoint b , defines a monodromy representation

$$\rho: \pi_1(B, b) \rightarrow \text{Aut}\left(H^3(\tilde{X}_b, \mathbb{Z}), \langle \cdot, \cdot \rangle\right) =: \Gamma.$$

Of course Γ acts naturally on \mathcal{D} , and the distribution associated to the transversality condition is Γ -invariant. The local period maps globalize to a horizontal holomorphic map

$$\Pi: B \rightarrow \mathcal{D}/\Gamma.$$

These maps are classifying Integral Variations of Hodge Structure (\mathbb{Z} -VHS), i.e., locally liftable horizontal variations of Hodge structure together with the integral local system, see [Grif, page 165]. The image of such maps is contained in the smooth locus of \mathcal{D}/Γ . These objects were abstracted and axiomatized by Deligne [Del1]. Deligne even established a (non-effective) finiteness theorem for \mathbb{Z} -VHS over a base with a fixed discriminant (e.g., over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$) [Del3]. In particular Deligne's result implies that there are finitely many complete horizontal locally liftable integral curves isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in the smooth locus of the period domain. (It is elementary to see [Grif, Theorem (9.8) on page 167] that there are no such curves isomorphic to $\mathbb{P}^1 \setminus A$ where A has cardinality at most two.) Given a holomorphic section $\varpi(z)$ of the line bundle \mathcal{F}^3 one still has the Picard-Fuchs equation satisfied by $\varpi(z)$ and the $\nabla^i \varpi(z)$, $1 \leq i \leq 4$. There is a meromorphic map from the trivial bundle of solutions of the corresponding Picard-Fuchs ODE to the vector bundle given by the dual local system. This map is a holomorphic isomorphism at every regular point for the Picard-Fuchs ODE, and it identifies the monodromy of the Picard-Fuchs ODE with the monodromy of the Gauss-Manin connection on the dual local system.

1.3. Toric examples. The proposals for just what constitutes a mirror pair of Calabi-Yau threefolds have evolved significantly since [GP]. The most commonly used is the Batyrev-Borisov construction [Bat1, Bat2, Bor, BB1, BB2] in the toric setting, which reduces the description of mirror pairs of Calabi-Yau hypersurfaces and complete-intersections in Gorenstein Fano toric varieties to the classical “polar duality” of reflexive polytopes (with nef partitions in the complete intersection case). In this setting Batyrev-Borisov [BB2] establish the basic mirror prediction for Hodge numbers of the proposed mirror pairs, and as a consequence these pairs are widely believed to be examples of mirror pairs [MSCMM].

Let N be a lattice and M the dual lattice. As is well known [Ful], convex lattice polytopes $\Delta \subset N \otimes \mathbb{R}$ define toric varieties $\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$, where S_Δ is the polytope ring of monomials indexed by $n \in k\Delta \cap N$ and graded by total degree k , and where the torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ acts. Recall that a lattice polytope $\Delta \subset N \otimes \mathbb{R}$ is *reflexive* if the *polar* polytope

$$\Delta^\circ = \{x \in M \otimes \mathbb{R} \mid \langle x, \Delta \rangle \geq -1\}$$

is a lattice polytope with respect to M . When the polytope Δ is reflexive [CoKa, §3.5], the toric variety is Fano, and the “divisor at infinity” D_∞ , i.e., the union of the lower-dimensional T_N -orbits, is an anti-canonical divisor. Generic sections of $\mathcal{O}(D_\infty)$ are then Calabi-Yau threefolds. In general \mathbb{P}_Δ is sufficiently singular so that

all sections of $\mathcal{O}(D_\infty)$ produce singular threefolds. To remedy this defect one takes a simplicial decomposition Σ of $\partial\Delta$ with vertices the set of lattice points. Then the fan consisting of the cones on the simplices of Σ determines a toric variety with isolated singularities¹, providing a crepant resolution of \mathbb{P}_Δ . Because the singularities are isolated and the resolution is crepant, the generic section of the pullback of $\mathcal{O}(D_\infty)$ is a smooth Calabi-Yau 3-fold. The proposal of Batyrev [Bat2] is that the Calabi-Yau X obtained from a maximal triangulation of a reflexive polytope Δ is mirror to the Calabi-Yau \tilde{X} obtained from a maximal triangulation of the polar Δ° .

EXAMPLE 1.3. The Fano toric variety $\mathbb{P}_\Delta = \mathbb{P}^4$ is specified by the reflexive polytope Δ with vertices

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-1, -1, -1, -1)\}.$$

Here the “divisor at infinity” is a union of the five coordinate hyperplanes in \mathbb{P}^4 . The quintic family $\mathbb{P}^4[5]$ is interpreted in this context as the family of hypersurfaces in this (ample) anticanonical divisor class. The ambient toric variety $\mathbb{P}_{\Delta^\circ}$ in which the quintic mirror family of hypersurfaces is constructed is given by the polar reflexive polytope Δ° with vertices

$$\{(4, -1, -1, -1), (-1, 4, -1, -1), (-1, -1, 4, -1), (-1, -1, -1, 4), (-1, -1, -1, -1)\}.$$

This time, however, there are many more integral points in the polytope besides the vertices and origin and hence one must take a simplicial decomposition to obtain smooth hypersurfaces. A combinatorial formula of Batyrev allows one to determine the Hodge numbers of both the quintic and quintic mirror from the arrangement of these integral points on the facets of Δ and Δ° .

The quintic twin family also has such a toric hypersurface representation, this time with reflexive polytope the simplex with vertices

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 2, 3, 5), (-2, -3, -4, -5)\}$$

and its polar polytope is

$$\{(-1, -1, 4, -2), (4, -1, -1, 0), (-1, 4, -1, -1), (-1, -1, -1, 2), (-1, -1, -1, 1)\}.$$

As before, the families of both the quintic twin and quintic twin mirror hypersurfaces are the families of generic sections of the anti-canonical line bundle over the toric varieties (appropriately resolved). Notice that the sublattice spanned by the integral points of the quintic twin lattice polytope has index 5 in the integral lattice.

What of the associated Picard-Fuchs equations for the family of holomorphic 3-forms? In this toric setting the machinery of Gel'fand-Graev-Zelevinsky [GGZ] and Gel'fand-Kapranov-Zelevinsky [GKZ] applies to construct, from the reflexive polytope corresponding to a given Calabi-Yau family, a differential equation, or more generally a system of such equations in the case of multiparameter families, satisfied by the periods of the mirror family (GKZ-system). This is a system of generalized hypergeometric equations. This system may possess more solutions than the genuine Picard-Fuchs system, but at least in the hypersurface case there are methods for describing the reduction to the actual Picard-Fuchs system [HLY1, HLY2].

¹In higher dimension these varieties are smooth through codimension 3.

The situation is easier to describe for one-dimensional families coming from toric geometry. When the family \tilde{X} of Calabi-Yau threefolds that are complete intersections in a Gorenstein Fano toric variety have $h^{2,1}(\tilde{X}_z) = 1$, then the corresponding Picard-Fuchs ODE is always a generalized hypergeometric ordinary differential equation [CoKa, §5.5]. These equations have base \mathbb{P}^1 and have exactly three regular singular points. The parameter space B of the variation can always be taken to be $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The local monodromy about the point at infinity in B (conventionally taken to be $z = 0$) corresponding to the toric divisor at infinity D_∞ is maximally unipotent. Let γ_0 be an invariant cycle for this monodromy. The period on γ_0 of a family of holomorphic 3-forms (defined over all of B) is always represented in the unit disk as a GKZ-generalized hypergeometric series. A second regular singular point (which we locate at $z = 1$) has local monodromy unipotent of rank one and is called a *conifold singularity* in the physics literature. Of course, by general principles [BN], the latter has quasi-unipotent monodromy.

For the quintic mirror and the quintic twin mirror this generalized hypergeometric series is the one given in Section 1.1.

1.4. Homological Mirror Symmetry. Much more than a correspondence of Hodge numbers of mirror pairs is predicted by mirror symmetry. Naïvely, for a mirror pair (X, \tilde{X}) deformation of the complex structure on \tilde{X} is mirror to deformation of the symplectic (i.e., Kähler) structure on X . So far there has been no general construction of a moduli space of symplectic structure deformations mirror to the usual moduli space of complex structure deformations. (For such a proposed construction in the case of $K3$ surfaces, see [Bri].) In particular, there is no known analogue for X of the \mathbb{Z} -VHS for \tilde{X} . Nevertheless, there is a proposal due to Kontsevich which significantly refines the statement that $b^{\text{even}}(X) = b^{\text{odd}}(\tilde{X})$. This is his Homological Mirror Symmetry (HMS) Conjecture. It posits that mirror pairs (X, \tilde{X}) should have the property that the derived category of bounded complexes of coherent sheaves on X (clearly a generalization of the even cohomology) should be equivalent to a category built from the Fukaya symplectic category of the mirror \tilde{X} , taking into account certain flat line bundles on the special Lagrangian cycles (which is a generalization of the third cohomology of \tilde{X}). It is not yet a precise conjecture, and much work in algebraic and symplectic geometry is underway to make it so. The conjecture was inspired by the earliest inklings of what eventually became D -brane physics, and, through the introduction of the derived category, has itself provided a language for physicists to express that theory [MSCMM]. In the context of the Batyrev-Borisov mirror pairs, illustrative examples have been studied by many (see for example [Asp1, Asp2, BD, BDM, DJP, Doug, Hor, Hos1, Hos2, Mor] and references therein).

As we noted before, there is more structure underlying the family \tilde{X} than just the Gauss-Manin connection, or equivalently the Picard-Fuchs differential equation satisfied by the family of holomorphic 3-forms. There is the \mathbb{Z} -VHS where the integral local system arises by taking periods over integral 3-cycles. This is one of the main objects of our interest in this investigation. There is a general result, due to Griffiths, Deligne, and Schmid (see the end of this section) that tells us in this case that the integral monodromy representation determines the \mathbb{Z} -VHS. Now by considering “ D -brane charges”, i.e., by taking cohomology in the categories on both sides of the HMS correspondence, and retaining the information about

categorical automorphisms on both sides, we obtain a mathematical prediction from mirror symmetry. By this reasoning, mirror symmetry predicts that there is an equivalence between the monodromy representation of the \mathbb{Z} -VHS for $\tilde{\mathfrak{X}}$ and automorphisms of integral even topological K -theory $K^0(X)$ obtained from the categorical autoequivalences. There is a precise conjecture [Kon] for the K -theory automorphism mirror to the maximally unipotent monodromy for any Calabi-Yau family. There is also a conjecture [BD, BDM, DJP] (generalizing a proposal of Kontsevich in the case of the quintic) for the K -theory automorphism mirror to the monodromy about $z = 1$ for one-parameter families, or more generally for the monodromy about the conifold locus in the compactification of the moduli space.

Let us give more details about these proposed K -theory automorphisms. Let (X, \tilde{X}) be a mirror pair with $h^{1,1}(X) = h^{2,1}(\tilde{X}) = 1$. The maximally unipotent monodromy for the one-parameter \mathbb{Z} -VHS for $\tilde{\mathfrak{X}}$ is proposed to be mirror to the K -theory automorphism $K(X) \rightarrow K(X)$ given by

$$(1.4) \quad \xi \mapsto \xi \otimes \mathcal{L}$$

where $c_1(\mathcal{L})$ is the positive generator of $H^{1,1}(X, \mathbb{Z})$. The conifold monodromy (conventionally taken about $z = 1$) is proposed to be mirror to the Fourier-Mukai “push-pull” transform

$$(1.5) \quad \xi \mapsto \sum_{\lambda} (p_1)_* (\{(\lambda^{-1} \boxtimes \lambda) \rightarrow \mathcal{O}_{\Delta}\} \otimes p_2^*(\xi))$$

where $\Delta \subset X \times X$ is the diagonal, p_i are the projections onto the factors, and λ runs over flat line bundles on X . This proposal implies that the monodromy automorphism of $H^3(\tilde{X}, \mathbb{Z})$ around the conifold locus is then a sum of terms each of which is a Picard-Lefschetz transformation on a vanishing cycle. The vanishing cycles are all equivalent modulo torsion, so that this monodromy is divisible by the number of terms in the sum, which is $|\pi_1(X)|$. The remaining monodromy (conventionally taken at ∞), whether of finite order (as in the case of the quintic family) or not, is determined as the product of the monodromies at 0 and 1.

2. Classification of \mathbb{Z} -VHS

For the purpose of investigating this proposed Hodge-theoretic/ K -theoretic mirror relationship it is very useful to have a complete description of the possible integral variations of Hodge structure that can underlie a family of Calabi-Yau threefolds over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with $h^{2,1} = 1$, subject to the assumption of the existence of a point of maximally unipotent monodromy (at $z = 0$), a point of unipotent monodromy of rank one (at $z = 1$), and quasi-unipotency of their product (necessary at $z = \infty$ by the Monodromy Theorem [BN]). In this case the monodromy representation of the \mathbb{Z} -VHS determines that variation up to isomorphism. To see this let \mathcal{H} and \mathcal{H}' be \mathbb{Z} -VHS over the same connected algebraic base B with $h^{3,0} = h^{2,1} = 1$ and with isomorphic irreducible monodromy ((2) of Theorem 2.1 establishes irreducibility for our specific case). The bundle $\text{Hom}(\mathcal{H}, \mathcal{H}')$ with its natural induced flat connection, integral structure and Hodge filtration is a \mathbb{Z} -VHS of weight zero over B . By [Sch, Theorem (7.22)], the $\pi_1(B)$ -invariant subbundle is a sub \mathbb{Z} -VHS. Of course, the invariant subbundle is the flat bundle whose fiber over any point $x \in B$ is the $\pi_1(B, x)$ -invariant subspace of $\text{Hom}_{\mathbb{C}}(\mathcal{H}_x, \mathcal{H}'_x)$. Since the monodromies of \mathcal{H} and \mathcal{H}' are irreducible, by Schur’s lemma, the invariant subbundle is one-dimensional. But a weight zero Hodge structure on a one-dimensional

space is of type $(0, 0)$. This means that any non-trivial $\pi_1(B, x)$ -invariant homomorphism $\mathcal{H}_x \rightarrow \mathcal{H}'_x$ preserves the Hodge filtration and hence is an isomorphism of Hodge structures, showing that \mathcal{H} and \mathcal{H}' are isomorphic as \mathbb{Z} -VHS over B .

In this section we classify \mathbb{Z} -VHS over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with $h^{3,0} = h^{2,1} = 1$ subject to the conditions that the local monodromy about $z = 0$ is maximally unipotent and the local monodromy about $z = 1$ is unipotent of rank one. As noted above, the key to this classification is the classification of possible monodromy representations underlying such variations.

2.1. Classification results for $\rho: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \rightarrow Sp(4, \mathbb{Z})$. We begin by classifying real representations. Let V be a four-dimensional real vector space. Let $T: V \rightarrow V$ be a unipotent automorphism of V and let N be the nilpotent endomorphism of V defined by $N = T - \text{Id}$. Then $N^{\dim V} = 0$. T is said to be *maximally unipotent* if $N^{\dim V - 1} \neq 0$. Also, the *rank* of T is defined to be the dimension of the image of N , so that T is maximally unipotent if and only if its rank is one less than the dimension of V .

Throughout we fix $x \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and a representation

$$\rho: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x) \rightarrow GL(V).$$

We have geometric generators γ_0, γ_1 , and γ_∞ for $\pi_1(X, x)$ which are represented by loops in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ encircling $0, 1, \infty$ respectively. These are oriented positively so that they satisfy the relation $\gamma_0 \gamma_1 \gamma_\infty = 1$. We denote the images of these generators under ρ by T_0, T_1, T_∞ , respectively, and in view of the eventual application to flat bundles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ call them the *monodromy* around $0, 1, \infty$.

THEOREM 2.1. *Suppose that the monodromy T_0 is maximally unipotent and that the monodromy T_1 is unipotent of rank 1. Set $N_i = T_i - \text{Id}$ for $i = 0, 1$. Define an invariant $m \in \mathbb{R}$ by choosing a non-zero vector $v \in \text{Ker } N_0$ and setting $N_0^3(N_1(v)) = -mv$. Then:*

- (1) *m is independent of the choice of v .*
- (2) *The representation ρ is irreducible if and only if $m \neq 0$.*
- (3) *If the monodromy at infinity has an invariant vector, then $m = 0$.*
- (4) *In the case when $m \neq 0$, the representation ρ is determined, up to conjugation by an element of $GL(V)$, by the characteristic polynomial of the monodromy transformation around infinity. Direct computation shows that the characteristic polynomial of $T_\infty^{-1} = T_0 T_1$ is*

$$(2.1) \quad x^4 + (a - 4)x^3 + (6 - 2a + bm)x^2 + (a - 4 + m - bm)x + 1 .$$

Notice that a and b are determined by this polynomial. The real numbers a, b, m are complete invariants of the conjugacy class of ρ as a representation into $GL(V)$.

- (5) *For $m \neq 0$ take as basis for V the vectors*

$$(2.2) \quad N_1(-v), N_0(N_1(-v)), \frac{N_0^2(N_1(-v))}{m}, v .$$

Then in this basis matrices for T_0 and T_1 are

$$(2.3) \quad T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$(2.4) \quad T_1 = \begin{pmatrix} 1 & -a & -b & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the rest of this section we assume that T_0 is maximally unipotent, that T_1 is a unipotent of rank one, and that ρ is irreducible, i.e., $m \neq 0$, and we let m, a, b be as in the statement of the previous lemma.

COROLLARY 2.2. *There is a non-degenerate symplectic form on V invariant under ρ if and only if $b = 1$. In this case, this form is unique up to non-zero scalar multiplication and the matrix for this form in the basis (2.2) is (up to a positive scalar)*

$$(2.5) \quad \langle \cdot, \cdot \rangle = \pm \begin{pmatrix} 0 & -a & -1 & -1 \\ a & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We now fix a non-degenerate skew pairing $\langle \cdot, \cdot \rangle$ on V .

COROLLARY 2.3. *There are exactly 14 conjugacy classes of representations $\rho: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \rightarrow Sp(V)$ with T_0 and T_1 as given in Theorem 2.1 and with T_∞ being quasi-unipotent. These are classified by the pairs of nonzero integers (m, a) (with $b = 1$) and a complete set of such pairs is given in the first two columns of Table 1.*

We are actually interested in representations to $Sp(4, \mathbb{Z})$, or equivalently in lattices $L \subset V$ invariant under ρ and on which the symplectic form is a perfect integral pairing. Here is a first result along these lines.

COROLLARY 2.4. *Suppose that $L \subset V$ is an integral lattice invariant under ρ such that the restriction of $\langle \cdot, \cdot \rangle$ to L is a perfect integral pairing. Then $b = 1$ and $a, m \in \mathbb{Z}$. Conversely, if $b = 1$ and $a, m \in \mathbb{Z}$, then there is an integral lattice L_0 in V invariant under ρ on which the pairing is a perfect integral pairing. Furthermore, there is a basis for L_0 in which T_0 , T_1 and the pairing are given by the matrices in Equations (2.3), and (2.4), respectively, and the symplectic form is given, up to sign, by Equation (2.5).*

Notice that we are not claiming that, given a lattice $L \subset V$ invariant under ρ and on which the symplectic form is a perfect integral pairing, the lattice L_0 as in Corollary 2.4 is related in any way to L . In fact, they are closely related (though not necessarily equal), but that relationship is somewhat delicate and depends on the arithmetic properties of m and a .

Now suppose that $b = 1$ and that a and $m \neq 0$ are integers. Let e_1, e_2, e_3, e_4 be a basis of V in which the symplectic pairing $\langle \cdot, \cdot \rangle$ is given by Equation (2.5), and T_0 and T_1 are given by Equations (2.3) and (2.4), respectively. Our goal is to study all

possible integral lattices invariant under ρ and on which the symplectic pairing is a perfect integral pairing. By Corollary 2.4 the lattice L_0 spanned by $\{e_1, \dots, e_4\}$ is such a lattice.

We introduce the (increasing) weight filtration $W_*(V)$ associated with T_0 by setting $W_{2i} = W_{2i+1} = \text{Ker}(N_0)^{i+1}$ for $i = 0, \dots, 3$. This is the monodromy weight filtration in the limiting mixed Hodge structure as defined by Deligne [Del1]. This filtration has the property that $N_0: W_j \rightarrow W_{j-2}$ and N_0 induces an isomorphism on the associated gradeds: $N_0: W_j/W_{j-1} \rightarrow W_{j-2}/W_{j-3}$. Fix an integral lattice L on which the pairing is a perfect integral pairing and which is invariant under ρ . Then $W_{2i}(L) = L \cap W_{2i}$ has rank equal to $i + 1$, and hence the quotients $W_{2i}(L)/W_{2i-2}(L) \cong \mathbb{Z}$ for $i = 0, \dots, 3$. Notice in particular that for the lattice L_0

$$N_1: W_0(L_0) \rightarrow L_0/W_4(L_0)$$

and

$$N_0: L_0/W_4(L_0) \rightarrow W_4(L_0)/W_2(L_0)$$

are isomorphisms.

LEMMA 2.5. *Suppose that $\ell_1, \ell_2, \ell_3, \ell_4$ is a basis for L such that for each i the subset $\ell_{4-i}, \dots, \ell_4$ is a basis for $W_{2i}(L)$. Then the matrix for the symplectic form is*

$$\begin{pmatrix} 0 & \alpha & \beta & \pm 1 \\ -\alpha & 0 & \pm 1 & 0 \\ -\beta & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix}$$

for appropriate integers α and β .

REMARK 2.6. We now fix our conventions regarding integral bases for L pertaining to the above isomorphisms on associated graded. From now on we shall consider bases for L satisfying:

- (1) For every $i \leq 4$ the set $\{\ell_{4-i}, \dots, \ell_4\}$ forms a basis for $W_{2i}(L)$;
- (2) $\langle \ell_1, \ell_4 \rangle = -1$;
- (3) $\langle \ell_2, \ell_3 \rangle = 1$.
- (4) Duality tells us that there is a non-zero integer r such that $N_0(\ell_1) \equiv r\ell_2 \pmod{\ell_3, \ell_4}$ and $N_0(\ell_3) = r\ell_4$. By changing the signs of ℓ_2 and ℓ_3 if necessary, we arrange that $r > 0$.
- (5) We define s by requiring that $N_0(\ell_2) \equiv s\ell_3 \pmod{\ell_4}$.
- (6) We define t such that $N_1(\ell_4) = -t\ell_1 \pmod{\ell_2, \ell_3, \ell_4}$.

Of course, r is the divisibility of N_0 as a map $W_6/W_4 \rightarrow W_4/W_2$ and also $W_2/W_0 \rightarrow W_0$ while s is the divisibility of $N_0: W_4/W_2 \rightarrow W_2/W_0$ and t is the divisibility of $N_1: W_0 \rightarrow W_6/W_4$.

In particular, we have:

THEOREM 2.7. *The integers r, s and t are invariants of the isomorphism class of $(L, \rho, \langle \cdot, \cdot \rangle)$.*

Notice that these integral invariants are related to the integer invariant m of the real conjugacy class of ρ by

$$(2.6) \quad m = r^2 st \quad \text{and} \quad t|a.$$

For each i , $1 \leq i \leq 4$, the real subspace of V spanned by $\{\ell_i, \dots, \ell_4\}$ is equal to that spanned by $\{e_i, \dots, e_4\}$, which means the matrix whose columns express the ℓ_i in terms of the e_j is weakly lower triangular. Since the image of the representation ρ is Zariski dense in $Sp(V)$ general considerations imply that the coefficients in this matrix are contained in $\mathbb{Q}[\sqrt{(1/d)}]$ for some integer d . In fact, a much more explicit result holds.

As an indication of how the basis $\{\ell_i\}$ of the new lattice L is expressed in terms of the original basis $\{e_i\}$ of the original lattice L_0 , consider the following lemma.

LEMMA 2.8. *There exists a basis $\ell_1, \ell_2, \ell_3, \ell_4$ for L as described above such that*

$$\begin{aligned}\ell_1 &= \frac{e_1}{\sqrt{t}} + \frac{\alpha e_2}{rt\sqrt{t}} + \frac{\beta e_3}{\sqrt{t}} + \frac{\gamma e_4}{rt\sqrt{t}} \\ \ell_2 &= \frac{e_2}{r\sqrt{t}} + \frac{\delta e_3}{\sqrt{t}} + \frac{\mu e_4}{r\sqrt{t}} \\ \ell_3 &= r\sqrt{t}e_3 + \frac{\alpha e_4}{\sqrt{t}} \\ \ell_4 &= \sqrt{t}e_4\end{aligned}$$

for appropriate $\alpha, \beta, \gamma, \delta, \mu \in \mathbb{Z}$.

Once again let us impose the condition that T_∞ is quasi-unipotent, i.e., that the pair (m, a) is one of the 14 pairs listed in Table 1. Case-by-case analysis results in a complete classification of the possible $\alpha, \beta, \gamma, \delta, \mu$ corresponding to each allowable r, s, t . Adding the numbers in Column 4 of Table 1 one sees that there are 112 possibilities.

In Equations (1.4) and (1.5) we gave explicit formulas for the proposed automorphisms of K -theory. Translating these by mirror symmetry gives the following conditions on T_0 and T_1 , which we express in terms of the invariants r and t :

CONJECTURE 2.9. *For any \mathbb{Z} -VHS arising from a family of Calabi-Yau threefolds $\widetilde{\mathfrak{X}}$ with $h^{2,1} = 1$ over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ we have:*

$$(2.7) \quad r = 1 \text{ (mirror to the fact that } c_1(\mathcal{L}) \text{ is a generator of } H^2(X, \mathbb{Z}))$$

$$(2.8) \quad N_1 = t \cdot \overline{N}_1 \text{ for some indivisible integral transformation } \overline{N}_1.$$

Furthermore, t is equal to the order of the fundamental group of the mirror Calabi-Yau threefold X .

REMARK 2.10. A stronger version of the first equation was conjectured by Morrison to hold for the local maximally unipotent monodromy in arbitrary dimensional families of Calabi-Yau threefolds, [CoKa, p. 150]. Condition (2.8) and the last statement are conjectured to hold for the local monodromy around the principal discriminant for arbitrary dimensional families of Calabi-Yau threefolds.

The invariants m and a encode natural geometric information about the Calabi-Yau threefolds X in the last column of Table 1. As observed by Borcea [Borc] for the 13 examples with $t = 1$, the integers we identified as invariants (m, a) above have a mirror interpretation as $m = \mathcal{L}^3$ and $a = \dim(H^0(X, \mathcal{O}(\mathcal{L})))$ respectively. In fact, this interpretation applies to all the geometric examples listed in Table 1.

The classification of representations into $Sp(4, \mathbb{Z})$ satisfying the two conditions in Conjecture 2.9 is simpler to state than the general case:

PROPOSITION 2.11. *An integral lattice $L \subset V$ on which the symplectic pairing is a perfect integral pairing and which is invariant under ρ and which satisfies Conditions (2.7) and (2.8) is determined by t .*

Columns 5 and 6 of Table 1 list the 23 of the 112 representations that satisfy these conditions. Notice that without assuming Conditions (2.7) and (2.8) the invariants m, a, r, s, t do not in general determine the isomorphism class of the lattice.

2.2. From representations to \mathbb{Z} -VHS. As we argued at the end of Section 1.4, for a \mathbb{Z} -VHS over a quasi-projective base with completely irreducible monodromy representation, the isomorphism class of the monodromy representation determines the isomorphism class of the \mathbb{Z} -VHS. According to a theorem of Borel and Narasimhan [BN] the local monodromy of any \mathbb{Z} -VHS is quasi-unipotent.

THEOREM 2.12. *Suppose we have a primitive weight-three \mathbb{Z} -VHS \mathcal{H} over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ satisfying:*

- (1) $h^{3,0} = h^{2,1} = 1$,
- (2) *the underlying monodromy representation ρ into $Sp(4, \mathbb{Z})$ is irreducible,*
- (3) *T_0 is maximally unipotent, and*
- (4) *T_1 is unipotent of rank one.*

Then ρ is conjugate in $Sp(4, \mathbb{Z})$ to one of the 112 representations enumerated in Table 1 and the isomorphism class of \mathcal{H} as a \mathbb{Z} -VHS is determined by the $Sp(4, \mathbb{Z})$ -conjugacy class of ρ .

The remaining question is one of existence: Which of the 112 representations underlie a \mathbb{Z} -VHS?

Each of the 14 real representations comes from a generalized hypergeometric ODE. That is to say, for each of the fourteen pairs (m, a) in Table 1 we have the following data:

- (1) a flat bundle with a real structure,
- (2) a real symplectic form parallel under the flat connection,
- (3) a holomorphic section ϖ of the complexification of the flat bundle,

such that the Picard-Fuchs equation satisfied by ϖ is the generalized hypergeometric equation

$$(2.9) \quad [\Theta^4 - z(\Theta + a_1)(\Theta + a_2)(\Theta + a_3)(\Theta + a_4)] \varpi(z) = 0 ,$$

where the a_i are the entries in Column 3 of Table 1 corresponding to (m, a) and $\Theta = z\nabla_z$. The putative Hodge filtration on this flat bundle is given by: \mathcal{F}^{3-i} is the span of the holomorphic section and its first i covariant derivatives. This filtration is automatically horizontal and satisfies the first Hodge condition (1.1). It is not clear that the filtration satisfies the second Hodge condition (1.2), but if it does, then the polarization condition (1.3) follows.

By direct geometric constructions we know that all 14 of these representations arise as local systems (or subquotients of local systems) underlying families of Calabi-Yau threefolds. Of these, 13 are local systems of third cohomology for one-parameter families of complete intersection Calabi-Yau threefolds with $h^{2,1} = 1$ in toric varieties (see Section 3). Thus, these thirteen are in fact real representations underlying \mathbb{R} -VHS. Consequently, for these 13 real types there is a Hodge filtration

making this an \mathbb{R} -VHS. In the remaining case for which $(m, a) = (1, 4)$ we do not know whether Conditions (1.2) and (1.3) hold.

Since Conditions (1.1), (1.2), and (1.3) make reference only to the symplectic form and the real structure, a parallel lattice in a \mathbb{R} -VHS on which the symplectic form is a perfect integral pairing determines a \mathbb{Z} -VHS. Hence, all integral representations whose real types satisfy $(m, a) \neq (1, 4)$ underlie (unique isomorphism classes of) \mathbb{Z} -VHS. Since the real representation corresponding to $(m, a) = (1, 4)$ has a unique integral lattice, there is only one of the 112 integral representations enumerated in Table 1 which might not come from a \mathbb{Z} -VHS. Thus, up to this one ambiguity we have completely classified \mathbb{Z} -VHS satisfying the conditions given in Theorem 2.12. This means that, up to the same ambiguity, we have classified the complete curves in the smooth locus of the period space \mathcal{D}/Γ isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with the given local monodromy conditions.

3. Geometric realization of abstract variations by one-parameter families of Calabi-Yau threefolds

The first two columns of Table 1 give the invariants m and a for the 14 \mathbb{R} -VHS in the classification. The third column gives the coefficients of the hypergeometric ODE, see Equation (2.9), or equivalently $(1/2\pi i)$ times the logs of the eigenvalues of T_∞ . The fourth column enumerates the integral lattices within each real representation. The fifth column indicates the number of integral lattices that are “mirror-consistent” in the sense that they satisfy both conditions (2.7) and (2.8). The sixth column shows the t values for each of the mirror consistent representations enumerated in the fifth. For each of these, the final column indicates known geometric examples with integral variations of Hodge structure of this type. Since the examples listed all have $h^{1,1} = 1$, to read each entry preface with “The \mathbb{Z} -VHS of H^3 of the Batyrev-Borisov mirror of ...”.

The two examples with $(m, a) = (2, 3)$ and $(6, 5)$ have an asterisk to indicate that, even though these families of Calabi-Yau complete intersections in weighted projective space are of the correct type, the Newton polytopes of the weighted projective spaces are not reflexive and so these examples don’t quite fit into the Batyrev-Borisov description of mirror pairs (though the Greene-Plesser construction does apply [KT]). Denoting the vertices of the $(2, 3)$ polytope by $\{e_1, \dots, e_5, -e_1 - e_2 - 2e_3 - 2e_4 - 3e_5\}$, and those of the $(6, 5)$ polytope by $\{e_1, \dots, e_5, -e_1 - e_2 - e_3 - e_4 - 2e_5\}$, suitable reflexive polytopes are obtained by adding in each case the new vertex $-e_5$. These have the property that they possess nef partitions of bidegrees $[4, 6]$ and $[3, 4]$ respectively, with Hodge numbers matching those of the original Calabi-Yau hypersurfaces in the two weighted projective spaces.

The example denoted A is the quintic twin, which we already recognized as the free quotient of the quintic by $\mathbb{Z}/5\mathbb{Z}$. The rest of the examples labeled B through H arise in a quite similar fashion, by finding a suitable free action by a group of order $t \neq 1$ on a simply-connected Calabi-Yau of that class (specifically the one described in the corresponding $t = 1$ entry of the fifth column) and taking the quotient. When that action is compatible with the toric structure of the original example, the Batyrev-Borisov mirror construction applies. In the classification of 4D reflexive polytopes by Kreuzer-Skarke [KrSkCY] there are precisely five polytopes which correspond to toric hypersurfaces with $h^{1,1} = 1$, and four of these are among the weighted projective spaces listed in the seventh column of Table 1 — the quintic,

m	a	a_1, a_2, a_3, a_4	$\# L_{\mathbb{Z}}$	$\# L_{MC}$	t	Geometric examples
1	4	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	1	1	1	I
1	3	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	1	1	1	$WP^4_{1,1,1,2,5}[10]$
2	4	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2	2	1	$WP^4_{1,1,1,1,4}[8]$
					2	II
5	5	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	2	2	1	$P^4[5]$
					5	A
1	2	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	1	1	1	$WP^5_{1,1,2,2,3,3}[6,6]$
2	3	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	1	1	1	$WP^5_{1,1,1,2,2,3}[4,6]$ *
3	4	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	1	1	1	$WP^4_{1,1,1,1,2}[6]$
4	5	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	11	1	1	$WP^5_{1,1,1,1,1,3}[2,6]$
4	4	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	8	3	1	$WP^5_{1,1,1,1,2,2}[4,4]$
					2	B
					4	C
6	5	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	1	1	1	$WP^5_{1,1,1,1,1,2}[3,4]$ *
8	6	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	14	2	1	$P^5[2,4]$
					2	D
9	6	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	8	2	1	$P^5[3,3]$
					3	E
12	7	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	11	1	1	$P^6[2,2,3]$
16	8	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	50	4	1	$P^7[2,2,2,2]$
					2	F
					4	G
					8	H

TABLE 1. Table of “Mirror-Consistent” $\rho: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \rightarrow Sp(4)$

the sextic, the octic, and the dectic. The remaining example is the alternative derivation of the quintic twin from Example 1.3 for which the Batyrev mirror pair construction applies. Note that the fact that the “octic twin” does not arise in this way means that if there does exist a free quotient of the octic hypersurface in $\mathbb{WP}_{1,1,1,1,4}^4[8]$ by the group $\mathbb{Z}/2\mathbb{Z}$, then the action cannot be compatible with the toric structure (i.e., arise from a natural involution on the weighted projective ambient space).

The only known construction of example H, a Calabi-Yau whose mirror defines the $t = 8$ integral structure in the same real class as the mirror of the complete intersection of four quadrics in \mathbb{P}^7 , requires a manifestly non-toric group action. It comes from Jae Park’s recent construction [Park] of the mirror of Beauville’s Calabi-Yau threefold with nonabelian fundamental group. Example H is the Beauville manifold, the quotient of $\mathbb{P}^7[2, 2, 2, 2]$ by the group Q_8 of unit quaternions. Examples F and G are then the intermediate quotients obtained through the same action, but restricted to the subgroups \mathbb{Z}_2 and \mathbb{Z}_4 of Q_8 respectively.

For Example II the authors don’t know of a family of Calabi-Yau threefolds with $h^{2,1} = 1$ over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ realizing this \mathbb{Z} -VHS, and for Example I the authors don’t know whether there is a geometric realization of this representation or indeed whether there is a \mathbb{Z} -VHS with this representation.

REMARK 3.1. All of these toric examples satisfy Conjecture 2.9. Notice that geometric families realizing a given (m, a) but different t values give examples of geometric \mathbb{Z} -VHS that are isomorphic as \mathbb{R} -VHS. The quintic mirror and the quintic twin mirror families comprise the simplest example of this phenomenon. More generally, to the best of the authors’ knowledge, all complete intersection Calabi-Yau threefolds arising from a nef partition of the anti-canonical divisor at infinity in Gorenstein Fano toric varieties satisfy the multiparameter generalization of this conjecture.

Some positive results along these lines are:

THEOREM 3.2. *For any Calabi-Yau threefold that is an anti-canonical hypersurface in a Gorenstein Fano toric variety the local monodromy $T_1 = \text{Id} + N_1$ around the principal discriminant satisfies Condition (2.8) with t equal to the order of the fundamental group of the Batyrev mirror Calabi-Yau.*

For any Calabi-Yau threefold \tilde{X} that is a complete intersection arising from a nef partition of the anti-canonical divisor at infinity in a Gorenstein Fano toric variety and with $h^{2,1}(\tilde{X}) = 1$ the parameter space of complex moduli can be identified with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in such a way that the monodromies T_0 and T_1 satisfy the assumptions of Theorem 2.12. Furthermore, T_0 satisfies Condition (2.7).

There are five one-parameter families of Calabi-Yau threefold hypersurfaces in a Gorenstein Fano toric variety with $h^{2,1} = 1$. By this theorem each of their \mathbb{Z} -VHS is one of the 23 listed in column 5 of Table 1, where (m, a) are determined by the local monodromy matrices T_0 and T_1 and t is the order of the fundamental group of the mirror. We conjecture that the same is true for Calabi-Yau threefold complete intersections in Gorenstein Fano toric varieties with $h^{2,1} = 1$.

4. Extensions and final comments

4.1. Extensions. The classification obtained above for \mathbb{Z} -VHS underlying $h^{2,1} = 1$ Calabi-Yau threefold moduli over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ should be extended both

to $h^{2,1} > 1$ multiparameter hypergeometric variations and to cases of $h^{2,1} = 1$ Calabi-Yau threefold moduli over curves of higher genus and with more punctures.

A small but growing number of the latter sorts of Calabi-Yau threefold families are known to exist, mostly via complete intersection constructions in partial flag varieties [BCFKvS], though there is a great deal of indirect evidence for their existence. In particular, recent work of Almkvist-Zudilin [AZ] on ordinary differential equations whose associated formal “Yukawa coupling” series possess integrality and positivity properties that suggest they could be Yukawa couplings for actual Calabi-Yau threefolds — so-called “fake Picard-Fuchs equations” — highlights the importance of classifying compatible integral structures, both as a means of eliminating some cases from consideration and in order to obtain more information about compatible Calabi-Yau manifolds if they do exist (e.g., possible nontrivial fundamental groups).

For a large number of the Almkvist-Zudilin examples (with $t = 1$) Christian van Enckevort and Duco van Straten [vEvS] have used a numerical approach, motivated by HMS and the integrality of formal Gopakumar-Vafa invariants, to determine the integral monodromy representations. One very interesting question is whether their methods can be modified to also detect alternate integral structures with $t > 1$. These would correspond conjecturally to more families of mirrors of non-simply connected Calabi-Yau threefolds.

For each integral structure with $t > 1$ in a class for which a geometric construction of the $t = 1$ representative is known, one can ask whether there is a free action of a finite group of order t on the known example such that the mirror family realizes the $t > 1$ variation. In particular, for the examples in Table 1 the only entry for which this remains an open question is case II. Thus we ask: Is there a free, holomorphic 3-form-preserving action of $\mathbb{Z}/2\mathbb{Z}$ on the “double octic”?

4.2. Final comments on Table 1. As noted already, the authors do not know of any geometric constructions of one-parameter families of Calabi-Yau threefolds with $h^{2,1} = 1$ whose integral monodromy representations, underlying their weight three \mathbb{Z} -VHS, correspond to the representations we found with $(m, a, t) = (1, 4, 1)$ or $(2, 4, 2)$. In fact, we do not even know of candidate mirror families with $h^{1,1} = 1$ and the correct geometric invariants. Nevertheless, these integral monodromy representations do “come from geometry” through restriction from multiparameter, higher rank variations of the same weight.

The task of finding the 4D polytope (in case II) and 5D polytope with nef partition (in case I) with appropriate subloci is not simple, but knowledge of the corresponding integral structures helps to point the way.

The $(m, a) = (1, 4)$ integral structure is unique, so it suffices to merely find a sub-local system of the appropriate hypergeometric type and then to prove that the restricted monodromy is actually integral. The structure of the generalized hypergeometric series ${}_4F_3(\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12})$ suggests looking for a complete intersection of bidegree $[2, 12]$ in $\mathbb{WP}_{1,1,1,1,4,6}^5$. Unfortunately, there is not a well-defined mirror of such a complete intersection (difficulties with the singular locus). This is reflected in the fact that the the Newton polytope for the weighted projective space is not reflexive. We can correct this by considering instead the reflexive polytope (provided by Kreuzer and Schedegger) with vertices given by $\{[e_1, e_3], [e_2, e_4, e_5, -e_1 - e_2 - e_3 - 4e_4 - 6e_5, -2e_4 - 3e_5]\}$, where the brackets denote the two pieces of the nef partition. The complete intersection Calabi-Yau threefold

of bidegree $[2, 12]$ in this Gorenstein Fano toric variety has $h^{1,1} = 3$, but $h_{\text{toric}}^{1,1} = 2$. In terms of the natural (torically defined) complex structure coordinates z_1, z_2 for the polynomial deformations, the restriction locus is just the locus $z_1 = 0$. Following Batyrev, the generalized hypergeometric series for the holomorphic solution in these coordinates is

$$\sum_{k,m} \frac{(2m)!(6k+12m)!}{(3k+6m)!(m!)^4 k!(2k+4m)!} z_1^k z_2^m ,$$

which upon restriction yields the desired series. See [KKRS] for a discussion of the singular geometry and physics of this subfamily.

The 4D polytope for the $(m, a) = (2, 4)$ example is found by a similar method. Here one knows that the desired \mathbb{Z} -VHS has $t = 2$, so one expects if it arises by restriction that it will do so from a family of Calabi-Yau threefolds with fundamental group $\mathbb{Z}/2\mathbb{Z}$. Also, the form of the hypergeometric series ${}_4F_3(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$ suggests looking for a hypersurface of degree 8 in $\mathbb{WP}_{1,1,1,1,4}^4$. Instead of taking this weighted projective space as the starting point, since Kreuzer-Skarke have completely classified the 473,800,776 equivalence classes of 4D reflexive polytopes, one can imagine searching through these for those hypersurfaces with nontrivial fundamental group. In fact the authors have done this, and a unique polytope was found with the property that the hypersurface has fundamental group $\mathbb{Z}/2\mathbb{Z}$ and there is a sublocus of moduli on which the hypergeometric series restricts to the desired form.

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