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A "periodic table" for supersymmetric M-theory compactifications

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We develop a systematic method for classifying supersymmetric orbifold compactifications of M-theory. By restricting our attention to Abelian orbifolds with low order, in the special cases where elements do not include coordinate shifts, we construct a "periodic table" of such compactifications, organized according to the orbifolding group (order $\leqslant 12$) and dimension (up to 7). An intriguing connection between supersymmetric orbifolds and G_2 structures is explored. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1581972]

I. INTRODUCTION

Manifolds with SU(N) holonomy have been a source of significant interest for mathematicians and physicists alike. Indeed, the importance of $K3$ manifolds and Calabi–Yau threefolds in the arena of consistent superstring background geometries could hardly be overstated. Aside from their undisputed beauty, compactification on these spaces allows for important control of supersymmetry which, in turn, permits ready access to potentially testable phenomenological consequences of string theory itself. It is for such reasons that understanding the geometry of these objects, including the classification of associated gauge bundles,[1–3] has been such a relevant and fruitful endeavor. It is nowadays accepted, however, that there exists a more fundamental 11-dimensional underpinning, code-named M-theory, which appropriately describes nonperturbative aspects of fundamental physics. In contrast to the situation in perturbative string theory, within the context of M-theory the most important geometric compactification spaces have special holonomy.[4–6]

The connection between M-theory and four-dimensional $N=1$ supersymmetric models of particle physics is provided by 11-dimensional supergravity on compact seven manifolds with G_2 holonomy. Considerably less is known about these objects as compared to the case of Calabi–Yau manifolds. In light of the above discussion, however, it is important to develop a useful classification of the relevant supersymmetric M-theory models. The rudiments of a mathematical classification scheme for G_2 holonomy seven manifolds, each a resolution of an orbifold of a seven torus, has been provided by Joyce.[7] The purpose of this paper is to describe a complementary scheme, based on physics, of a class of seven-dimensional orbifold constructions which meet the criterion of $N=1$ supersymmetry preservation.

In previous papers[8–11] we described various technical aspects of the extraction of effective physics from M-theory. Generally, our techniques apply to global orbifold compactifications, and rely on significant constraints which follow from the requirement of chiral anomaly cancellation pointwise in 11 dimensions, most notably on distinguished even-dimensional submanifolds. Recently,[12] we have described how to obtain a pair of particular four-dimensional $N=1$ super Yang–Mills theories with chiral matter content from an M-theoretic intersecting brane-world scenario. In that paper we included a scan of multiplicities of supersymmetric M-theory orbifold models of a particular class. In this paper we derive this scan, explaining in more detail the physical and mathematical criteria involved in finding such models.

Presupposing an ultimate connection between M-theory and standard model four-dimensional physics, a seven-dimensional compactification space must be Ricci-flat and admit singularities.[13] For these reasons, the class of toroidal orbifolds T^7/Γ, for a finite group Γ, holds special interest. The necessary geometrical singularities are of finite quotient type, and hence readily permit mathematical analysis. Each is modelled on $(M^1)^{7-n}\times\mathbb{R}^n/G$ for some subgroup $G\subset\Gamma$, where $M^1=S^1$ or the unit interval $I^1=S^1/\mathbb{Z}_2$. Moreover, as we shall see in this paper, under the right conditions one can explicitly describe a well-defined lift of the action of Γ to the 11-dimensional spinorial supercharge. This allows us to determine how much supersymmetry is preserved on the various fixed-point loci (''fixed-planes'') of spacetime $T^7/\Gamma\times\mathbb{R}^{3,1}$.

The mathematical problem of identifying candidate compactification spaces with supersymmetric fixed-planes is quite elegant, and divides neatly into four parts. First of all, we must decide on a class of tori to orbifold. A torus T^7 is determined by a choice of a rank seven lattice $\Lambda\subset\mathbb{R}^7$. Throughout this paper we will assume that the lattice has the form $\Lambda:=A_1\oplus A_2^3$, i.e., the direct sum of three copies of the usual hexagonal lattice in the complex plane with one copy of $\mathbb{Z}$. More generally, our analysis applies to compactification on tori T^n modelled on lattices $A_1^a\oplus A_2^b$ with $a+2b=n$, $1\leqslant n\leqslant 7$. These are by no means the only lattices from which we could construct our tori. In fact, a particularly interesting case, especially as regards the discussion in Sec. V of this paper, is that of the irreducible lattice $A_7\subset\mathbb{R}^7$, whose automorphism group is one of the maximal finite subgroups of the group G_2. We choose to restrict attention to our particular class of decomposable lattices simply because it is both sufficiently general to subsume the orbifolds studied previously, and easy in this setting to describe the action on the 11-dimensional supercharge in the Clifford algebra.

The second step in identifying the desired supersymmetric orbifolds is to choose a particular class of groups Γ acting on T^7. An action on the torus is an action on $\mathbb{R}^7$ that preserves the lattice Λ. We will consider only group actions which respect the decomposition of Λ into direct summands. Thus, an element $g\in\Gamma$ acts as $\exp(2\pi i\vec{f})$, where $\vec{f}=(f_1,f_2,f_3,f_4)$, with $f_{1,2,3}\in\mathbb{Z}/6$ and $f_4\in\mathbb{Z}/2$. In this way we define a class of representations in which each element acts by rotations in two-dimensional subplanes plus the possibility of a parity reversal on one real coordinate. We call such actions ''pseudoplanar representations,'' and groups which admit such representations ''pseudoplanar groups.'' This eliminates, for the time being, orbifolds constructed from non-Abelian orbifolding groups, an omission we hope to rectify in future work.

Having restricted ourselves to pseudoplanar groups, next we need to enumerate all possible actions. For organizational purposes we wish to index the candidate orbifolds of T^n by their orbifolding groups Γ. For reasons of bounding complexity, in this paper we restrict attention to finite Abelian groups Γ of order $\leqslant 12$. The method we use, explained in detail in Sec. II, consists of classifying the Λ-compatible representations of Γ on $\mathbb{R}^7$, using the decomposition into irreducible characters to determine equivalence classes of group actions. Properly taking into account the automorphisms of the group Γ allows us to distinguish inequivalent group actions, and a matrix formalism makes quick work of computing the dimensions of the fixed-planes corresponding to each element of Γ, and provides for a concise accounting of various geometric data associated with each orbifold.

Finally, given this data, we must have a criterion for determining the exact amount of supersymmetry preserved on each fixed-plane. This fourth and final ingredient is provided by a systematic analysis of lifts of Γ to actions on 11-dimensional spinors. The necessary properties of Clifford algebra are reviewed in Sec. III A, and the criterion, our supersymmetric restriction theorem, is summarized in Table VI. We apply this criterion to the full class of groups Γ acting compatibly on our lattice Λ. In this way, we identify and classify the relatively small number of orbifolds T^n/Γ which maintain some supersymmetry at all points, constructions we refer to as *supersymmetric orbifolds*.

In the language of Ref. 12, the orbifolds considered here are all *hard* orbifolds, i.e., there are no fixed-point-free coordinate ''shifts'' in the Γ action, since we act directly through a represen-

TABLE I. The "periodic table" listing all supersymmetric, hard, pseudoplanar Abelian orbifolds T^n/Γ of M-theory, for cases $|\Gamma| \leqslant 12$. The labeling system is explained in Sec. II.

Γ	1	2	3	4	5	6	7
$\mathbb{Z}_2$	(1)*			(4)	(5)		
$\mathbb{Z}_3$				(4)			
$\mathbb{Z}_4$				(04)	(14)	(24)	(34)
$\mathbb{Z}_2 \times \mathbb{Z}_2$					(014)*	(222)	(223)
$\mathbb{Z}_2 \times \mathbb{Z}_3$				(004)	(140)*	(222)	(322)
					(104)		
$\mathbb{Z}_2 \times \mathbb{Z}_4$					(00104)*	(10104)*	(20104)*
						(00222)	(00322)
						(01122)	(01222)
							(10222)
							(11122)
$(\mathbb{Z}_2)^3$							(2220001)*
							(1111111)
$(\mathbb{Z}_3)^2$							
$(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$					(1000004)*	(0020220)	(0122022)*
							(0122020)
							(0030220)
$\mathbb{Z}_3 \times \mathbb{Z}_4$							

tation on the space $\mathbb{R}^7$ over $T^7 = \mathbb{R}^7/\Lambda$. By contrast, Joyce's examples of orbifolds of T^7 admitting a resolution as a G_2 manifold[7] are all *soft* orbifolds. In Sec. V we use the first class of G_2 resolvable examples studied by Joyce[14,15] as a launching point for a discussion of the relationship between supersymmetric orbifolds of T^7 and the notion of a G_2-structure (a weaker, necessary condition for the orbifold to admit a G_2 holonomy resolution).

We have collected the results of our search, accounting for all pseudoplanar orbifold groups with low order, into a so-called periodic table of orbifolds, which we include in this introduction as Table I. In Table I we exhibit each supersymmetric orbifold T^n/Γ, with rows corresponding to distinct pseudoplanar groups Γ, listed by increasing group order, and columns corresponding to the representation dimensions $1 \leqslant n \leqslant 7$. In each block of this table are listed the complete set of hard supersymmetric orbifolds corresponding to associated n-dimensional representations of Γ compatible with our lattices. Each orbifold is indicated by a particular *label*, which codifies the group action of Γ on T^n in a manner explained in detail in Sec. II. For a subset of the supersymmetric orbifolds, the corresponding orbifold label has an asterix appended. These models are those which split off a separate $S^1/\mathbb{Z}_2$ factor. Such models are the only ones which have 10-dimensional fixed-planes. Owing to this distinction, there is a more direct connection between this class of supersymmetric orbifolds and perturbative heterotic string models than is the case for the orbifolds listed without stars.

There are two natural extensions of our work in this paper which we plan to investigate in the near future. First, we would like to remove the pseudoplanar and abelian restrictions on the Γ action, allowing instead any $\Gamma \subset \mathrm{Aut}(\Lambda)$. In particular this will allow many non-Abelian group actions, which in turn will require a generalization of the supersymmetric restriction proof of Sec. III C. An important step towards such a formulation is described in Sec. IV. It would also be quite valuable to reformulate both the analysis herein and the anomaly cancellation compatibility checks in Refs. 11 and 12 using the theory of principal bundles on orbifolds. In this setting both the supersymmetric restriction criterion and anomaly cancellation mechanism should find expression in the language of characteristic classes of such bundles.

II. HARD ORBIFOLDS, SUPERSYMMETRY, AND CHARACTERS

Each distinct representation R, with real dimension $n \leqslant 7$, of any finite group $\Gamma \subset \mathrm{Aut}(T^n)$, can be used to define an orbifold T^n/Γ, and a corresponding compactification scheme in M-theory.

In this section we describe some useful tools for efficiently accounting for large numbers of such constructions, and explain how these feed naturally into an algorithm for selecting those which satisfy a particular criterion: that the 11-dimensional supercharge Q have nonvanishing components at all points in M^{11}. This is done in two steps. First we review some standard results pertaining to representations of finite groups. Then we explain some original technology which adapts these results to the special purpose of sifting through all possible representations and finding those which satisfy our criterion.

A. Representations of finite Abelian groups

Let Γ be an Abelian (commutative) group, and $\rho:\Gamma\to GL_n(\mathbb{C})$ an n-dimensional complex matrix representation of Γ, i.e., ρ is a homomophism of groups. Since Γ is Abelian, $g_1 g_2 = g_2 g_1$ for all $g_i \in \Gamma$, and each element $g\in\Gamma$ equals its own conjugacy class. A basic result in the theory of representations of finite groups states that for a group of order q, with s conjugacy classes, there are, up to equivalence, s distinct irreducible representations $R_1,\ldots,R_s$ over $\mathbb{C}$. Moreover, if R_i has dimension n_i, then

$$q := \sum_{i=1}^{s} (n_i)^2 \tag{2.1}$$

(Ref. 16, Theorem 2.3). When applied to an Abelian group Γ, this shows that each $n_i=1$, i.e., that each of the irreducible representations of Γ is itself a *character* (one-dimensional representation) of the group.

In fact, it is a simple matter to describe all the characters of a finite Abelian group. Any finite Abelian group Γ can be written as a direct product of m cyclic groups of orders $q_1,\ldots,q_m$, respectively, so that

$$q = |\Gamma| = q_1 q_2 \cdots q_m . \tag{2.2}$$

A typical element of Γ can be represented by the m-tuple $\vec{a}:=(a_1,a_2,\ldots,a_m)$, where $0\leqslant a_i < q_i$, with composition of elements given by componentwise addition followed by reduction of the ith component to its least non-negative remainder modulo r_i, $i=1,\ldots,m$. Then corresponding to each m-tuple $\vec{c}:=[c_1,c_2,\ldots,c_m]$, where $0\leqslant c_i < r_i$, there exists a character

$$\Gamma_{\vec{c}}(\vec{a}) := \exp\left(2\pi\iota\sum_{i=1}^{m}\left(\frac{a_i c_i}{r_i}\right)\right) \tag{2.3}$$

of Γ, and all q characters arise in this way (Ref. 16, Theorem 2.4). The identity element of Γ corresponds to the trivial character.

The obvious correspondence between the characters $\vec{c}$ and elements $\vec{a}$ of Γ is not canonical. Even though they are each composed of m-tuples of integers modulo q_i, $i=1,\ldots,m$, the isomorphism between these is only well-defined up to an automorphism of the group Γ.

Any n dimensional representation of an Abelian group Γ can be written as a direct sum of n of its characters $\Gamma_{\vec{c}}$, i.e., by the data of an n-tuple $\{\vec{c}^{\,1},\ldots,\vec{c}^{\,n}\}$ of m-tuples $\vec{c}^{\,j}:=[c_1^j,\ldots,c_m^j]$, for $j=1,\ldots,n$. Two such representations are considered *equivalent* if these n-tuples agree as unordered lists.

B. Character tables and *C*-matrices

The set of hard orbifolds T^n/Γ is equivalent to the set of distinct n-dimensional representations of Γ consistent with the lattice that defines T^n. If Γ is a finite Abelian group, then these, in turn, are equivalent to the possible ways to order sets with elements chosen freely from among the characters of Γ, allowing for repetition. It is, therefore, a straightforward exercise, in principle, to construct comprehensive lists of hard orbifolds T^n/Γ. This is so because it is also straightforward

TABLE II. The "elemental" character tables for the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$.

$\mathbb{Z}_2$	Ω	Γ	$\mathbb{Z}_3$	Ω	Σ	$\bar{\Sigma}$	$\mathbb{Z}_4$	Ω	Ψ	Σ	$\bar{\Psi}$
1	+	+	**1**	+	+	+	**1**	+	+	+	+
α	+	−	β	+	1/3	−1/3	γ	+	1/4	1/2	−1/4
			β^2	+	−1/3	1/3	γ^2	+	1/2	+	1/2
							γ^3	+	−1/4	1/2	1/4

to determine the characters for any finite Abelian group, using the following simple algorithm. (As described above, we shall limit our discussion to the case of pseudoplanar groups.)

Each pseudoplanar group is given by the direct product of some number each of $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ factors. For each of these three "elemental" groups, the list of characters are contained in the character tables exhibited in Table II. In these tables, the elements of the group are enumerated rowwise, while each column corresponds to a distinct character. (Our convention differs from that used in the mathematical literature, wherein character tables typically list group elements as columns and characters as rows. Our choice of convention is more suited to the particular application to physics described in this paper.) For the case of $\mathbb{Z}_2$ the characters are real, i.e., each describes a group action on one real coordinate; in our case this corresponds to an action on an A_1 lattice; a plus sign in the table indicates a trivial action, while a minus sign indicates a sign change $x \to -x$ on the associated coordinate. For the groups $\mathbb{Z}_3$ and $\mathbb{Z}_4$ the nontrivial characters are complex; i.e., each describes a group action on a pair of real coordinates; in our case this corresponds to an action on an A_2 lattice; a plus sign indicates a trivial action, other rational numbers indicate the fraction of a complete counterclockwise rotation in the plane spanned by the relevant A_2 lattice. Such entries are defined modulo 1.

It is useful to assemble the entries of a given character table for a group Γ into a "character matrix" $\sigma(\Gamma)$. The data in Table II can be written as

$$\sigma(\mathbb{Z}_2)=\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \sigma(\mathbb{Z}_3)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/3 & -1/3 \\ 0 & -1/3 & 1/3 \end{pmatrix}, \quad \sigma(\mathbb{Z}_4)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & -1/4 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & -1/4 & 1/2 & 1/4 \end{pmatrix}.$$

We adopt the convention that trivial actions (plus signs in the character tables) are represented in the character matrix with zeros, and parity reversals (minus signs in the character tables) are represented with the fraction 1/2. The character matrix for a generic pseudoplanar group with m elemental factors $\Gamma = G_1 \times \cdots \times G_m$, is obtained by combining the character matrices $\sigma(G_i)$ as an outer sum. For instance, the character matrix $\sigma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ can be written as a a two-by-two array of three-by-three block matrices, wherein the upper left block is computed by adding the upper left entry in $\sigma(\mathbb{Z}_2)$ to the entire matrix $\sigma(\mathbb{Z}_3)$, the second block in the first row of blocks in $\sigma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is given by adding the entry $\sigma(\mathbb{Z}_2)_{12}$ to the entire matrix $\sigma(\mathbb{Z}_3)$, and so forth. The matrix $\sigma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ formed in this way can be usefully re-expressed in terms of a character table, with the result shown in Table III. In Table III, all rational entries are defined modulo 1. Furthermore, a trivial action, denoted by a zero in the corresponding character matrix, is represented in the character table by a plus sign. Finally, on complex characters an entry 1/2, describing a 180 degree rotation, is represented in the table by a minus sign. Upon reconstituting the character matrix $\sigma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ into Table III we have inserted a useful naming convention for the group elements and characters; we have named the order-two generating element α and the order-three generating element β. Similarly, we have named the trivial character Ω, the order two character Λ, the order-three characters Σ and $\bar{\Sigma}$ and the order-six characters Ψ and $\bar{\Psi}$.

By repeating the operation of combining character matrices as outer sums, in the manner described above, the character matrix and, equivalently, the character table for any pseudoplanar

TABLE III. The character table for the group $\mathbb{Z}_2\times\mathbb{Z}_3$.

$\mathbb{Z}_2\times\mathbb{Z}_3$	Ω	Σ	$\bar{\Sigma}$	Λ	Ψ	$\bar{\Psi}$
1	+	+	+	+	+	+
β	+	1/3	$-1/3$	+	1/3	$-1/3$
β^2	+	$-1/3$	1/3	+	$-1/3$	1/3
α	+	+	+	$-$	$-$	$-$
$\alpha\beta$	+	1/3	$-1/3$	$-$	$-1/6$	1/6
$\alpha\beta^2$	+	$-1/3$	1/3	$-$	1/6	$-1/6$

group can be generated readily from the three elemental character matrices $\sigma(\mathbb{Z}_2)$, $\sigma(\mathbb{Z}_3)$, and $\sigma(\mathbb{Z}_4)$. For an illustration in the case $\Gamma=(\mathbb{Z}_2)^3$, see Table VII. Here $\Omega:=\Gamma_{[0,0,0]}$ is the trivial character, and the group elements down the left-hand side are indexed in the usual binary ordering:

$$1:=(0,0,0)\,,\quad \gamma:=(0,0,1)\,,\quad \beta:=(0,1,0)\,,\ldots,\alpha\beta\gamma:=(1,1,1).$$

In general, however, there is quite a lot of physically irrelevant redundancy in the full character table for a given orbifolding group Γ. For instance, in the case of $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_3$, the characters which we have named Σ and Ψ describe group actions on a complex coordinate z. However, if we describe these same characters in terms of their actions on the complex conjugate $\bar{z}$, these same characters would appear to act precisely as do $\bar{\Sigma}$ and $\bar{\Psi}$ on the original coordinate z. Thus, complex characters are physically indistinguishable from their conjugates. Since conjugate pairs of characters can be can be mapped into each other by a merely semantical renaming of the coordinates, we can more efficiently describe the relevant representation theory of this group by considering a restricted set of essential nontrivial characters. For the case of $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_3$, these would be Λ, Σ and Ψ. At the same time, elements with order greater than two have nontrivial inverses. These inverse elements have precisely the same locus of fixed-points in the physical space T^n/Γ as do the original elements. So we can characterize the geometry of a given orbifold in terms of a representative set of essential nontrivial elements, thereby removing this second, physical, redundancy.

The number of nontrivial representative elements of any finite Abelian group is equivalent to the number of essential nontrivial characters. This number provides a "physical rank" r of the group. By including only the essential nontrivial elements and characters, we can replace the full character table with an "abbreviated character table." For the case of $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_3$, we would thereby replace Table III with the abbreviated character table shown in Table IV. Notice that we may choose at will the ordering of the elements (rows) and, independently, the ordering of the characters (columns) when we construct a character table; there is no *a priori* canonical ordering. Notice, as well, that no information is sacrificed by replacing a full character table with an abbreviated character table.

By multiplying the abbreviated character table by the order of the group we define an integer-valued, square $r\times r$ matrix, which we denote $C(\Gamma)$. For the case of $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_3$, this is easily obtained from Table IV by multiplying the entries by 6, which is the order of $\mathbb{Z}_2\times\mathbb{Z}_3$. In this way we determine

TABLE IV. The abbreviated character table for the group $\mathbb{Z}_2\times\mathbb{Z}_3$.

$\mathbb{Z}_2\times\mathbb{Z}_3$	Γ	Σ	Ψ
α	1/2	0	1/2
β	0	1/3	1/3
$\alpha\beta$	1/2	1/3	$-1/6$

TABLE V. The representation $R=(322)$ of the group $\mathbb{Z}_2\times\mathbb{Z}_3$.

$\mathbb{Z}_2\times\mathbb{Z}_3$	x_1	x_2	x_3	z_1	z_2
α	$-$	$-$	$-$	$+$	$-$
β	$+$	$+$	$+$	1/3	1/3
$\alpha\beta$	$-$	$-$	$-$	1/3	$-1/6$

$$C(\mathbb{Z}_2\times\mathbb{Z}_3)=\begin{pmatrix} 3 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & -1 \end{pmatrix}. \tag{2.4}$$

Of course, owing to the freedom to independently rearrange the rows and the columns of the abbreviated character matrix, the corresponding matrix $C(\Gamma)$ is defined only up to similar reorderings. However, such flexibility can always be used to render the C-matrix symmetrical. We deem this canonical. It is also possible, while keeping $C(\Gamma)$ symmetrical, to arrange the rows and columns so that the corresponding elements and characters have monotonically increasing order. This too, we deem canonical.

All the information described by Table III is also contained in the matrix $C(\mathbb{Z}_2\times\mathbb{Z}_3)$ shown in (2.4). It is interesting that quite a lot of information pertaining to properties of any finite Abelian group, including the complete representation theory can be codified in a symmetric matrix $C\in \mathrm{GL}(r,\mathbb{Z})$. As it turns out, the matrices $C(\Gamma)$ are valuable tools in the search for supersymmetric orbifolds. The matrices $C(\Gamma)$ for each of the pseudoplanar groups with group order $\leqslant 12$ are listed in the Appendix.

C. The enumeration of distinct orbifolds

A representation of Γ is designated by choosing a set of real and complex characters, including the possibility of degeneracy, from the list of essential nontrivial characters. Generally, order-two characters are real, while characters with higher order are complex. Therefore, if we select a order-two characters and b higher-order characters, the corresponding representation will act on $n=a+2b$ real dimensions, $2b$ of which are complexified. Since the set of essential nontrivial characters correlates with the columns of the matrix $C(\Gamma)$, we can unambiguously designate a representation by an ordered list of r multiplicities, each indicating the number of real coordinates transforming according to a corresponding character. The ordering of the multiplicities corresponds to the ordering of the characters described by the rows of the C matrix. Of course, the multiplicities corresponding to complex characters are necessarily even, while those corresponding to order-two characters may be even or odd.

As an example, in the case of the group $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_3$, the physical rank is 3, and the corresponding C-matrix is given by (2.4). In this case, each representation is given by a 3-tuple, $R=(a_1, a_2, a_3)$, where a_1, a_2 and a_3 are the number of real coordinates transforming according to the characters Λ, Σ, and Ψ, respectively. In this case, $a_1\in\mathbb{N}$ since the character Γ is real, and $a_{2,3}\in 2\mathbb{N}$ since the characters Σ and Ψ are complex. To be quite specific, the representation of $\mathbb{Z}_2\times\mathbb{Z}_3$ described by the three-tuple $R=(3\;2\;2)$ is a representation which acts on $3+2+2=7$ real coordinates. In this case, however, four of the real coordinates are complexified as two complex coordinates. The first multiplicity (3) in the label (322) indicates that three real coordinates, say $x_{1,2,3}$ transform according to the character Λ, the second multiplicity (2) indicates that two real coordinates, combined into one complex coordinate, say z_1, transform according to the character Σ, and the third multiplicity (2) indicates that one more complex coordinate, say z_2, transforms according to the character Ψ. The corresponding group actions are shown in Table V.

The particular orbifold T^n/Γ which corresponds to a given representation R generically includes a locus of special points which remain invariant under elements of Γ. These generically constitute hyperplanes of various dimensionalities which intersect, forming an intricate network.

One of our primary concerns is to decide what sorts of physics, in the form of localized states, are described by these planes and their intersections. In the next section we will describe in detail how one studies the issue of how many supercharges are retained on these. A primary consideration in this regard is, of course, to describe the number of dimensions which are spanned by the fixed-planes associated with each group element.

There is a simple formula which allows one to compute the set of dimensions corresponding to the r representative nontrivial elements. This formula is most easily described in terms of another useful matrix, which we call $M(\Gamma)$, obtained from $C(\Gamma)$ by replacing all zero entries with ones, and all nonzero entries by zeros. By way of illustration, we focus again on the example $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_3$. In this case, this prescription, applied to $C(\mathbb{Z}_2 \times \mathbb{Z}_3)$, as given in (2.4), yields

$$M(\mathbb{Z}_2 \times \mathbb{Z}_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

For a given orbifold, described by the representation R of a group Γ, the dimensionality of the fixed-planes of each representative element is described by another r-tuple, $d(R)$, given by

$$d(R) = (\,11 - n\,)\,\mathbf{1} + R\,M, \tag{2.6}$$

where $\mathbf{1}$ is the row vector with ones in each entry and $n = R \cdot \mathbf{1} = \Sigma_i a_i$. As an example, for the particular orbifold described by $R = (322)$, we compute $n = 7$ and

$$d(122) = (11-7)\,(1 \quad 1 \quad 1) + (3 \quad 2 \quad 2)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (6 \quad 7 \quad 4)\,. \tag{2.7}$$

Thus, the respective fixed-planes associated with the elements α, β, and $\alpha\beta$ have dimensionality 6, 7, and 4. This result can be verified from the precise group actions in Table V.

Now we have all the information we need to form comprehensive lists of all pseudoplanar orbifolds, including all the data pertaining to the group actions. First we choose a group Γ. Then, we form lists of orbifolds T^n/Γ, for each value of $1 \leqslant n \leqslant 7$ by sequencing through the ordered partitions of n into r non-negative integers, in the manner described above. For the case of $T^7/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ orbifolds, for example, we create the sequence of 3-tuples $(\,a_1\,, a_2\,, a_3\,)$ which describe ordered partitions of $n=7$ into sums of $r=3$ nonnegative integers, subject to the constraints that $a_1 \in \mathbb{N}$ and $a_{2,3} \in 2\,\mathbb{N}$. The complete list of such 3-tuples is given in the usual ascending order in mod 7 arithmetic, as (106), (124), (142), (160), (304), (322), (340), (502), (520), (700). We describe these ordered sets of multiplicities as orbifold *labels*. In each case the corresponding group actions can be determined by dividing the C-matrix by the group order and then selecting rows from this divided C-matrix with the appropriate multiplicity indicated by the corresponding label. The group actions for the orbifold $T^7/(\mathbb{Z}_2 \times \mathbb{Z}_3)_{(322)}$, described above, were obtained in precisely this way, and stand as an example of this methodology.

D. The periodic table

We use the algorithm described in the previous paragraph to systematically cycle through each orbifold label of each pseudoplanar group, obtaining all of the relevant group actions in each case. In each instance, each element of Γ can be represented, by arranging the coordinates judiciously, by a set of three fractional rotations $(\,f_1\,, f_2\,, f_3\,)$ describing counterclockwise rotations in respective planes spanned by three A_2 lattices, plus possibility of a parity reversal in one real coordinate, which we codify as the binary choice $P \in \{0,1\}$, describing the respective absence or presence of a parity reversal.

As it turns out, the values of $(f_1, f_2, f_3 \mid P)$ for each element of Γ corresponding to a given orbifold label provide all the data necessary to resolve the amount of supersymmetry on the corresponding orbifold plane. The precise corrrespondence is derived in the next section, where it is presented as a supersymmetric restriction theorem. This result says that an orbifold plane is supersymmetric if and only if there is at least one way to add or subtract the three corresponding fractions f_i to obtain, in the case $P=0$, an even integer, or, in the case $P=1$, any integer (even or odd).

For our restricted class of lattices, a given element is compatible only if the three f_i are each elements of the set $\{0, 1/2, \pm 1/3, \pm 1/4, \pm 1/6\}$ mod 1. (There are, therefore $7^3\times 2=686$ possibilities for each element.) It is possible to have compatible elements which do not have compatible products. For example, in the case $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_4$, there are several models which pass the criteria of the supersymmetric restriction theorem, except that the representations in question involve order-twelve rotations in at least one plane. These would be acceptable as supersymmetric orbifolds of $\mathbb{R}^n$, but not of T^n, because there is no lattice in $\mathbb{C}$ compatible with such a rotation. The number of global orbifolds T^n/Γ is therefore much smaller than the number of orbifold singularities which can be modelled locally as $\mathbb{R}^n/\Gamma$.

The search for supersymmetric orbifolds consists of four steps. First, for a given choice of Γ and n, we generate the complete list of compatible orbifold labels. Second, for each orbifold label, we use the matrix $C(\Gamma)$ to determine the data $(f_1, f_2, f_3 \mid P)$ for each of the r repressentative elements. Third, we apply the supersymmetric restriction theorem to remove each orbifold which has any element whose data does not meet the restriction criterion. Fourth, we examine the list of orbifolds which satisfy these restrictions, and we remove cases which are redundant. We have created a number of Mathematica functions which fully automate this process, and have used this to generate Table I which appears in the introduction. In this way, we can easily generalize our periodic table to arbitrary group order.

Our periodic table includes all of the hard global *M*-theory orbifolds described previously by other authors as well as by ourselves. For instance the $S^1/\mathbb{Z}_2$ model corresponds to the original *M*-theory model described in Refs. 17 and 18. The four T^4/Γ models correspond to the four global orbifold limits of $K3$. The $T^5/\mathbb{Z}_2$ model was discussed in Refs. 19 and 20. (It was the study of that simple model which first implicated wandering five-branes as a means of unifying ostensibly unique vacua into classes linked by phase transitions.) The four starred models T^5/Γ correspond to the four global orbifold limits of $K3\times S^1/\mathbb{Z}_2$ and were studied in Refs. 8–10 and 21. The $T^7/(\mathbb{Z}_2)^3$ model with label (2220001) was described in Ref. 11 and the $T^7/(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$ with label (0122022) was described in Ref. 12. Finally, resolutions of the softenings of the $T^7/(\mathbb{Z}_2)^3$ model with label (1111111) were presented by Joyce in Refs. 14, 15, and 7 as prototype G_2 manifolds, and studied by Acharya as candidate *M*-theory compactification spaces[22,23] (see also Sec. V).

III. A SUPERSYMMETRIC RESTRICTION THEOREM

In the bulk of M^{11} (i.e., at all points not within the locus of orbifold fixed-planes) the 11-dimensional supercharge is completely preserved. It is within the locus of fixed-planes, therefore, that the issue of supercharge preservation becomes important. Since the supercharge transforms nontrivially under elements of Γ, only those components of Q which remain invariant are not projected to zero on those spacetime points inert under those same elements. The invariant components of Q typically resolve as a d-dimensional spinor, where d is the total dimension of the invariant locus associated with that element near a given point. Precisely how many irreducible SO$(d-1,1)$-spinors are included in this set determines the amount of local sypersymmetry preserved on that fixed plane. For the sorts of pseudoplanar orbifolds defined above, it is possible to delineate a concise criterion for selecting those which, in this way, retain supersymmetry. In this section, which is relatively technical, we derive this supersymmetric restriction theorem. We start by establishing notational conventions and stating the result, and then prove it by analyzing the question of how the spinorial supercharge is influenced by lifts of various elements of finite subgroups of SO(10,1).

A. Notations

We use space–time coordinates $x^I \equiv \{x^0, x^i\}$, where $i=1,\ldots,3,5,\ldots 11$, and gamma matrices Γ_I, which satisfy the Clifford algebra $\{\Gamma_I,\Gamma_J\}=2\,\eta_{IJ}$, where $\eta_{IJ}=\mathrm{diag}(-+\cdots+)$ is the flat metric. The gamma matrices are chosen such that Γ_0 is anti-Hermitian, $\Gamma_0^\dagger=-\Gamma_0$, and Γ_i are Hermitian, $\Gamma_i^\dagger=\Gamma_i$. It is sometimes useful to define $\Gamma_4=i\,\Gamma_0$. Another useful identity is $\Gamma_{11}=i\,\Gamma_1\cdots\Gamma_{10}$. The matrices $\Gamma_{IJ}=\frac{1}{2}[\,\Gamma_I,\Gamma_J\,]$ are the generators of spin (11). We define a complex structure by writing the six real coordinates $x_{5,\ldots,10}$ in terms of three complex coordinates, according to $z_{1,2,3}\equiv x_{5,7,9}+i\,x_{6,8,10}$.

Consider an element which acts as simultaneous rotations in three complex planes, with coordinates z_1, z_2, and z_3, and possibly a parity flip on one real coordinate x^{11},

$$\alpha:(z_1,z_2,z_3;x^{11})\rightarrow(e^{i\,\theta_1}z_1,e^{i\,\theta_2}z_2,e^{i\,\theta_3}z_3;(-)^P x^{11}), \tag{3.1}$$

with $\theta_i=2\,\pi f_i$. The three rational numbers f_i describe the fraction of a complete rotation imparted, respectively, on the three complex planes. The parameter $P\in\{0,1\}$ indicates whether or not the element includes a parity flip. The order of α is given by the least common positive integer multiple of the denominators of the reduced form of the three f_i's and also $(1/2)^P$.

For a given choice of $(f_1,f_2,f_3\,|\,P)$, Eq. (3.1) describes a particular global SO(10,1) transformation plus the possibility of a parity transformation. On spinors this induces $\psi\rightarrow\Omega\,\psi$, where

$$\Omega=\exp(\tfrac{1}{4}\theta^{IJ}\Gamma_{IJ})\,(\Gamma_{11})^P \tag{3.2}$$

and where θ^{IJ} are the parameters of the SO(10,1) transformation. From (3.1) we read off the nonvanishing parameters as $\theta^{5,6}=-\theta^{6,5}\equiv\theta_1$, $\theta^{7,8}=-\theta^{8,7}\equiv\theta_2$, and $\theta^{9,10}=-\theta^{10,9}\equiv\theta_3$. Thus we can rewrite (3.2) as

$$\Omega=\exp(\tfrac{1}{2}\theta_1\Gamma_{5,6})\exp(\tfrac{1}{2}\theta_2\Gamma_{7,8})\exp(\tfrac{1}{2}\theta_3\Gamma_{9,10})(\Gamma_{11})^P. \tag{3.3}$$

The supercharge Q is an 11-dimensional Majorana spinor, which transforms precisely as $Q\rightarrow\Omega\,Q$. It is useful to append subscripts to spinors, and to spinorial operators, to indicate dimensionality. We thus write the 11-dimensional supercharge Q as $Q_{(11)}$ to indicate that this field takes its values in spin (11). Similarly, we write the operator Ω, defined in (3.3), as $\Omega_{(11)}$ to indicate that this object operates on 11-dimensional spinors.

If the fixed-point locus associated with an element (3.1) has dimensionality d, then, in the neighborhood of this locus, the structure group is broken from SO(10,1) down to SO($d-1$,1) $\times$SO($11-d$). Accordingly, we write the supercharge as a tensor product of an SO($d-1$,1) fixed-plane spinor and an SO($11-d$) "normal" spinor, as $Q_{(11)}=Q_{(d)}\otimes Q_{(11-d)}$. Similarly, the operator Ω decomposes as $\Omega_{(11)}=\Omega_{(d)}\otimes\Omega_{(11-d)}$. In order to resolve the amount of unbroken fixed-plane supersymmetry, we solve the equation $Q_{(11)}=\Omega_{(11)}\,Q_{(11)}$, and then count the degrees of freedom which describe the most general solution.

The analysis described below is completely general, modulo irrelevant reordering of coordinates. Thus, in this section we consider the element $(0,0,f\,|\,0)$ equivalent to $(0,f,0\,|\,0)$, for instance.

B. Statement of the theorem

The conclusions which we draw in each of the seven cases discussed in Sec. III C can be easily summarized as follows. The condition for supersymmetry on a fixed-plane associated with any element $\alpha\in\Gamma$, of the special sort characterized by (3.1), is

$$f_1\pm f_2\pm f_3\in\begin{cases}2\,\mathbb{Z}, & P=0\\ \mathbb{Z}, & P=1\end{cases} \tag{3.4}$$

TABLE VI. The seven sorts of elements, listed along with the associated fixed-plane dimensionality and the generic amount of supersymmetry retained when the conditions listed in (3.4) are satisfied. The $d=6$ and the $d=4$ cases have exceptions. In the $d=6$ case, if $f=1/2$ then the supersymmetry is merely halved, not quartered. In the $d=4$ case if $|f_1|$, $|f_2|$, and $|f_3|$ are drawn, one each, from either set (1/2, 1/3, 1/6) or (1/2, 1/4, 1/4) then supersymmetry is merely quartered, not eighthed.

$(f_1, f_2, f_3 \mid (-)^P)$	d_{fixed}	SUSY
$(0,0,0 \mid -)$	10	1/2
$(0, 0, f \mid +)$	9	NONE
$(0, 0, f \mid -)$	8	NONE
$(0, f, \pm f \mid +)$	7	1/2
$(0, f, \pm f \mid -)$	6	1/4*
$(f_1, f_2, f_3 \mid +)$	5	1/4
$(f_1, f_2, f_3 \mid -)$	4	1/8*

for any one of the four possible choices of unspecified signs. If condition (3.4) is satisfied, then supersymmetry is generically reduced to the minimal amount possible in the dimensionality of the fixed plane. Exceptions occur in cases where $P=1$, when one of the sums in (3.4) gives an even integer and another gives an odd integer. In such cases, the fixed-plane supersymmetry is twice the minimal amount. These results are reflected in Table VI.

Note that this result applies to orbifolds $\mathbb{R}^7/\Gamma$ as well as to orbifolds T^7/Γ. In the former case, there is no restriction on the choices of the f_i's other than that they should be rational. In the latter case, there are additional constraints which follow from the requirement that $\Gamma \subset \text{Aut}(T^7)$. For the cases described in Sec. III C, this requirement amounts to the restriction that $f_{1,2,3}$ must be chosen from the set $\{0, 1/2, \pm 1/3, \pm 1/4, \pm 1/6\}$ mod 1.

C. Proof of the theorem

The result is established by explicitly analyzing how the spinorial supercharge is influenced by lifts of various elements of finite subgroups of SO(10,1).

Ten-dimensional fixed-planes: The only transformation (3.1) which has 10-dimensional fixed-planes is an order-two element acting as $(f_1, f_2, f_3 \mid P) = (0,0,0 \mid 1)$. In this case, the supercharge transforms as $Q_{(11)} \to \Gamma_{11} Q_{(11)}$. We decompose $Q_{(11)}$ according to $Q_{(11)} = Q_{(10)R} \oplus Q_{(10)L}$, where $Q_{(10)R,L} = \pm \Gamma_{11} Q_{(10)R,L}$ are 10-dimensional Majorana–Weyl projections of $Q_{(11)}$. The fixed-plane condition, $Q_{(11)} = \Omega_{(11)} Q_{(11)}$ can now be written

$$Q_{(10)\,R,L} = \pm Q_{(10)\,R,L}. \tag{3.5}$$

This equation is easy to solve. On the fixed-plane we have $Q_{(11)} \to Q_{(10)R}$. Thus, the bulk supersymmetry is *halved* on the fixed-planes.

Nine-dimensional fixed-planes: Nine-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 \mid P) = (0,0,f \mid 0)$, where $f \neq 0$ mod 1. In this case, the supercharge transforms as $Q_{(11)} \to \Omega_{(11)} Q_{(11)}$, where

$$\Omega_{(11)} = \exp(\pi f \Gamma_9 \Gamma_{10}). \tag{3.6}$$

We write $Q_{(11)}$ as a tensor product of a nine-dimensional fixed-plane spinor with an SO(2) "normal" spinor, according to $Q_{(11)} = Q_{(9)} \otimes Q_{(2)}$. Similarly, we use the following representation for the gamma matrices,

$$\Gamma_\mu = \hat{\Gamma}_\mu \otimes \mathbf{1}, \quad \mu = 1, \ldots, 8,$$
$$\Gamma_{8+i} = \hat{\Gamma}_9 \otimes \sigma_i, \quad i = 1,2, \tag{3.7}$$
$$\Gamma_{11} = \hat{\Gamma}_9 \otimes \sigma_3,$$

where $\{ \hat{\Gamma}_\mu, \hat{\Gamma}_9 \}$ are nine-dimensional gamma matrices and σ_i are the Pauli matrices. In addition, we decompose the normal spinors according to $Q_{(2)} = Q_{(2)R} + Q_{(2)L}$ where $Q_{(2)R,L} = \pm\, \sigma_3 Q_{(2)R,L}$. Using these conventions, and also the identity $\sigma_1 \sigma_2 = i\, \sigma_3$, we can write

$$Q_{(11)} = Q_{(9)} \otimes (Q_{(2)R} + Q_{(2)L}),$$
$$\Omega_{(11)} = \exp(i\, \pi f \mathbf{1} \otimes \sigma_3). \tag{3.8}$$

The fixed-plane condition $Q_{(11)} = \Omega_{(11)} Q_{(11)}$ can now be written as

$$Q_{(9)} \otimes Q_{(2)R,L} = e^{\pm i \pi f} Q_{(9)} \otimes Q_{(2)R,L}. \tag{3.9}$$

Because there are no nontrivial solutions to either of these equations, subject to the restriction that $f \neq 0 \bmod 1$, we conclude that, as we approach a fixed-nine-plane, $Q_{(11)} \to 0$. Thus supersymmetry is broken completely on any nine-dimensional fixed-plane associated with an element (3.1).

Eight-dimensional fixed-planes: Eight-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 \,|\, P) = (0, 0, f \,|\, 1)$, where $f \neq 0 \bmod 1$. In this case, the supercharge transforms as $Q_{(11)} \to \Omega_{(11)} Q_{(11)}$, where

$$\Omega_{(11)} = \exp(\pi f \Gamma_9 \Gamma_{10}) \Gamma_{11}. \tag{3.10}$$

We write $Q_{(11)}$ as a tensor product of an eight-dimensional fixed-plane spinor with an SO(3) "normal" spinor, according to $Q_{(11)} = Q_{(8)} \otimes Q_{(3)}$, and we represent the gamma matrices precisely as in (3.7). In addition we decompose the SO(7,1) spinors via $Q_{(8)} = Q_{(8)\,R} + Q_{(8)\,L}$ where $Q_{(8)R,L} = \pm\, \hat{\Gamma}_9 Q_{(8)R,L}$, and write the normal spinors as $Q_{(3)} = Q_{(2)\,R} + Q_{(2)\,L}$, where $Q_{(2)\,R,L} = \pm\, \sigma_3 Q_{(2)\,R,L}$. We introduce a useful shorthand notation whereby $L \equiv Q_{(2)\,L}$ and $R \equiv Q_{(2)\,R}$. Using these conventions, and also the identity $\sigma_1 \sigma_2 = i\, \sigma_3$, we can write

$$Q_{(11)} = (Q_{(8)\,R} + Q_{(8)\,L}) \otimes (R + L),$$
$$\Omega_{(11)} = \exp(i\, \pi f \mathbf{1} \otimes \sigma_3) \hat{\Gamma}_9 \otimes \sigma_3. \tag{3.11}$$

The fixed-plane condition $Q_{(11)} = \Omega_{(11)} Q_{(11)}$ can now be written

$$Q_{(8)\,R,L} \otimes R = \pm e^{i \pi f} Q_{(8)\,R,L} \otimes R,$$
$$Q_{(8)\,R,L} \otimes L = \mp e^{-i \pi f} Q_{(8)\,R,L} \otimes L. \tag{3.12}$$

Because there are no nontrivial solutions to any of these four equations, subject to the restriction that $f \neq 0 \bmod 1$, we conclude that, as we approach an eight-dimensional fixed-plane, $Q_{(11)} \to 0$. Thus, supersymmetry is broken completely on any eight-dimensional fixed-plane associated with an element (3.1).

Seven-dimensional fixed-planes: Seven-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 \,|\, P) = (0, f, f' \,|\, 0)$, where $f \neq 0 \bmod 1$ and $f' \neq 0 \bmod 1$. In this case, the supercharge transforms as $Q_{(11)} \to \Omega_{(11)} Q_{(11)}$, where

$$\Omega_{(11)} = \exp(\pi f \Gamma_7 \Gamma_8) \exp(\pi f' \Gamma_9 \Gamma_{10}). \tag{3.13}$$

We write $Q_{(11)}$ as a tensor product of a seven-dimensional fixed-plane spinor with an SO(4) "normal" spinor according to $Q_{(11)}=Q_{(7)}\otimes Q_{(4)}$. Similarly, we use the following representation for the gamma matrices,

$$\begin{aligned}\Gamma_\mu&=\hat{\Gamma}_\mu\otimes\mathbf{1}, \quad \mu=1,\ldots,6,\\ \Gamma_{6+i}&=\hat{\Gamma}_7\otimes\gamma_i, \quad i=1,\ldots,4,\\ \Gamma_{11}&=\hat{\Gamma}_7\otimes\gamma_5,\end{aligned} \tag{3.14}$$

where $\{\hat{\Gamma}_\mu,\hat{\Gamma}_7\}$ are the seven-dimensional Gamma matrices and γ_i are four-dimensional gamma matrices. In addition, we decompose the normal SO(4) $\mathrm{SO}(4)\rightarrow\mathrm{SO}(2)\times\mathrm{SO}(2)$, where $\mathrm{SO}(2)\times\mathrm{SO}(2)$ is a convenient maximal subgroup of SO(4). Thus, we write the "normal" spinor as $Q_{(4)}=Q_{(2)}\otimes Q_{(2)}$, and the four-dimensional gamma matrices as $\gamma_{1,2}=\sigma_{1,2}\otimes\mathbf{1}$ and $\gamma_{3,4}=\sigma_3\otimes\sigma_{1,2}$, where σ_i are Pauli matrices. Also, using the identities $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ and $\sigma_1\sigma_2=i\,\sigma_3$, we derive $\gamma_5=-\sigma_3\otimes\sigma_3$. This allows us to rewrite (3.14) as

$$\begin{aligned}\Gamma_{1,\ldots,6}&=\hat{\Gamma}_{1,\ldots,6}\otimes\mathbf{1}\otimes\mathbf{1},\\ \Gamma_{7,8}&=\hat{\Gamma}_7\otimes\sigma_{1,2}\otimes\mathbf{1},\\ \Gamma_{9,10}&=\hat{\Gamma}_7\otimes\sigma_3\otimes\sigma_{1,2},\\ \Gamma_{11}&=-\hat{\Gamma}_7\otimes\sigma_3\otimes\sigma_3.\end{aligned} \tag{3.15}$$

Note that the normal spinor can be decomposed as $Q_{(4)}=Q_{(4)\,R}+Q_{(4)\,L}$ where $Q_{(4)\,R,L}=\pm\gamma_5\,Q_{(4)\,R,L}$. In terms of $\mathrm{SO}(2)\times\mathrm{SO}(2)\subset\mathrm{SO}(4)$, however, these same objects can be written as

$$\begin{aligned}Q_{(4)R}&=Q_{(2)R}\otimes Q_{(2)L}+Q_{(2)L}\otimes Q_{(2)R},\\ Q_{(4)L}&=Q_{(2)R}\otimes Q_{(2)R}+Q_{(2)L}\otimes Q_{(2)L},\end{aligned} \tag{3.16}$$

where $Q_{(2)\,R,L}=\pm\sigma_3\,Q_{(2)\,R,L}$. Using these results we can decompose the 11-dimensional supercharge in two steps. First, we write $Q_{(11)}=Q_{(7)}\otimes(Q_{(4)\,R}+Q_{(4)\,L})$ and then we rewrite the terms $Q_{(4)\,R,L}$ using (3.16). We now define $LL\equiv Q_{(2)L}\otimes Q_{(2)L}$, and similar expressions for LR, RL, and RR. Using these definitions, and also the conventions introduced above, we can write

$$\begin{aligned}Q_{(11)}&=Q_{(7)}\otimes(LL+LR+RL+RR),\\ \Omega_{(11)}&=\exp(i\,\pi f\,\mathbf{1}\otimes\sigma_3\otimes\mathbf{1}+i\,\pi f'\,\mathbf{1}\otimes\mathbf{1}\otimes\sigma_3).\end{aligned} \tag{3.17}$$

The fixed-plane condition $Q_{(11)}=\Omega_{(11)}\,Q_{(11)}$ can now be written as

$$\begin{aligned}Q_{(7)}\otimes LL&=e^{-i\,\pi(f+f')}\,Q_{(7)}\otimes LL,\\ Q_{(7)}\otimes LR&=e^{-i\,\pi(f-f')}\,Q_{(7)}\otimes LR,\\ Q_{(7)}\otimes RL&=e^{+i\,\pi(f-f')}\,Q_{(7)}\otimes RL,\\ Q_{(7)}\otimes RR&=e^{+i\,\pi(f+f')}\,Q_{(7)}\otimes RR.\end{aligned} \tag{3.18}$$

Therefore, as we approach a fixed-seven-plane, the bulk supercharge is projected according to

$$Q_{(11)} \rightarrow \begin{cases} Q_{(7)} \otimes (\, LL + RR \,), & f + f' \in 2\,\mathbb{Z}, \\ Q_{(7)} \otimes (\, LR + RL \,), & f - f' \in 2\,\mathbb{Z}, \end{cases} \tag{3.19}$$

and is projected to zero if neither of these conditions are met. Thus we retain some supersymmetry on a fixed-seven-plane if and only if one of the two sums $f \pm f'$ is an even integer. This is the case only if $f = \pm f' \bmod 1$. In such cases, the bulk supersymmetry is *halved* on the fixed-plane in question.

Six-dimensional fixed-planes: Six-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 \,|\, P) = (0, f, f' \,|\, 1)$, where $f \neq 0 \bmod 1$ and $f' \neq 0 \bmod 1$. In this case the supercharge transforms as $Q_{(11)} \rightarrow \Omega_{(11)}\, Q_{(11)}$, where

$$\Omega_{(11)} = \exp(\, \pi f\, \Gamma_7\, \Gamma_8 \,) \exp(\, \pi f'\, \Gamma_9\, \Gamma_{10} \,)\, \Gamma_{11}\,. \tag{3.20}$$

We write $Q_{(11)}$ as a tensor product of a six-dimensional fixed-plane spinor with an SO(5) "normal" spinor according to $Q_{(11)} = Q_{(6)} \otimes Q_{(5)}$, and we represent the Gamma matrices precisely as in (3.15). In addition, we decompose the SO(5,1) spinors via $Q_{(6)} = Q_{(6)\,R} + Q_{(6)\,L}$, where $= Q_{(6)\,R,L} = \pm \hat{\Gamma}_7\, Q_{(6)\,R,L}$, and write the normal SO(5) spinors as $Q_{(5)} = Q_{(4)\,R} \oplus Q_{(4)\,L}$, where $Q_{(4)\,R,L}$ are given by (3.16). Now, using the notation introduced above, we can write

$$\begin{aligned} Q_{(11)} &= (\, Q_{(6)R} + Q_{(6)L} \,) \otimes (\, LL + LR + RL + RR \,), \\ \Omega_{(11)} &= -\exp(\, i\,\pi f\, \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} + i\,\pi f'\, \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \,)\, \hat{\Gamma}_7 \otimes \sigma_3 \otimes \sigma_3\,. \end{aligned} \tag{3.21}$$

The fixed-plane condition $Q_{(11)} = \Omega_{(11)}\, Q_{(11)}$ can now be written as

$$\begin{aligned} rQ_{(6)\,R,L} \otimes LL &= \mp e^{-i\,\pi\,(f+f')}\, Q_{(7)} \otimes LL, \\ Q_{(6)\,R,L} \otimes LR &= \pm e^{-i\,\pi\,(f-f')}\, Q_{(7)} \otimes LR, \\ Q_{(6)\,R,L} \otimes RL &= \pm e^{+i\,\pi\,(f-f')}\, Q_{(7)} \otimes RL, \\ Q_{(6)\,R,L} \otimes RR &= \mp e^{+i\,\pi\,(f+f')}\, Q_{(7)} \otimes RR\,. \end{aligned} \tag{3.22}$$

Therefore, as we approach a fixed-six-plane, the bulk supercharge is projected according to

$$Q_{(11)} \rightarrow \begin{cases} Q_{(6)L} \otimes (\, LL + RR \,), & f + f' \in 2\,\mathbb{Z}, \\ Q_{(6)R} \otimes (\, LR + RL \,), & f - f' \in 2\,\mathbb{Z}, \\ Q_{(6)R} \otimes (\, LL + RR \,), & f + f' \in 2\,\mathbb{Z} + 1, \\ Q_{(6)L} \otimes (\, LR + RL \,), & f - f' \in 2\,\mathbb{Z} + 1\,. \end{cases} \tag{3.23}$$

and is projected to zero if none of these conditions is met. Thus, we retain some supersymmetry on a fixed-six-plane if and only if one of the two sums $f \pm f'$ is integer (even or odd). In generic cases of this sort, the bulk supersymmetry is *quartered* on the fixed-plane in question. [A special case is when $(\, f, f' \,) = (\, 1/2\,,\, 1/2 \,)$, in which case supersymmetry is merely *halved*. In that special case the difference $f - f' = 0$ is an even integer, while the sum $f + f' = 1$ is an odd integer; therefore, on the fixed-plane, $Q_{(11)}$ retains nonvanishing components of both the first and also the third case in (3.23).] Note that in the cases where the supersymmetry is quartered the element α necessarily has even order N. In those cases there is necessarily an order-two element in Γ, namely $\alpha^{N/2}$, for which $(f_1, f_2, f_3 \,|\, P) = (0,0,0 \,|\, 1)$. The fixed-plane associated with $\alpha^{N/2}$ is 10-dimensional, while the six-plane associated with α is a submanifold of this ten-plane.

Five-dimensional fixed-planes: Five-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 \,|\, P) = (f_1, f_2, f_3 \,|\, 0\,)$, where $f_{1,2,3} \neq 0 \bmod 1$. In this case, the supercharge transforms as $Q_{(11)} \rightarrow \Omega_{(11)}\, Q_{(11)}$, where

$$\Omega_{(11)}=\exp(\,\pi f_1\,\Gamma_5\Gamma_6\,)\exp(\,\pi f_2\,\Gamma_7\Gamma_8\,)\exp(\,\pi f_3\,\Gamma_9\Gamma_{10}\,). \tag{3.24}$$

We write $Q_{(11)}$ as a tensor product of a five-dimensional fixed-plane spinor with a normal spinor according to $Q_{(11)}=Q_{(5)}\otimes Q_{(6)}$. Similarly, we use the following representation for the gamma matrices,

$$\begin{aligned}\Gamma_\mu&=\gamma_\mu\otimes\mathbf{1},\quad \mu=1,\ldots,4,\\ \Gamma_{4+i}&=\gamma_5\otimes\hat\Gamma_i,\quad i=1,\ldots,6,\\ \Gamma_{11}&=\gamma_5\otimes\hat\Gamma_7.\end{aligned} \tag{3.25}$$

where $\{\gamma_\mu,\gamma_5\}$ are five-dimensional gamma matrices and $\hat\Gamma_i$ are six-dimensional gamma matrices. In addition, we decompose the normal SO(6) spinors using the decomposition SO(6) $\to\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$, where $\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$ is a convenient maximal subgroup of SO(6). Thus, we write the normal spinor as $Q_{(6)}=Q_{(2)}\otimes Q_{(2)}\otimes Q_{(2)}$, and the six-dimensional gamma matrices as $\hat\Gamma_{1,2}=\sigma_{1,2}\otimes\mathbf{1}\otimes\mathbf{1}$ and $\hat\Gamma_{3,4}=\sigma_3\otimes\sigma_{1,2}\otimes\mathbf{1}$, and $\hat\Gamma_{5,6}=\sigma_3\otimes\sigma_3\otimes\sigma_{1,2}$, where $\sigma_{1,2,3}$ are Pauli matrices. Now, using the relation $\hat\Gamma_7=i\,\hat\Gamma_1\cdots\hat\Gamma_6$ we then derive $\hat\Gamma_7=\sigma_3\otimes\sigma_3\otimes\sigma_3$. Using these expressions, it is possible to rewrite (3.25) as

$$\begin{aligned}\Gamma_{1,\ldots,4}&=\gamma_{1,\ldots,4}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1},\\ \Gamma_{5,6}&=\gamma_5\otimes\sigma_{1,2}\otimes\mathbf{1}\otimes\mathbf{1},\\ \Gamma_{7,8}&=\gamma_5\otimes\sigma_3\otimes\sigma_{1,2}\otimes\mathbf{1},\\ \Gamma_{9,10}&=\gamma_5\otimes\sigma_3\otimes\sigma_3\otimes\sigma_{1,2},\\ \Gamma_{11}&=\gamma_5\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3.\end{aligned} \tag{3.26}$$

We now define $LLL\equiv Q_{(2)\,L}\otimes Q_{(2)\,L}\otimes Q_{(2)\,L}$, and similar expressions for LLR, LRL, LRR, RRR, RRL, RLR, and RLL, where $Q_{(2)\,R,L}=\pm\sigma_3\,Q_{(2)\,R,L}$. In terms of these definitions and conventions, we can write

$$\begin{aligned}Q_{(11)}&=Q_{(5)}\otimes(\,RRR+RLL+LRL+LLR+LLL+LRR+RLR+RRL\,),\\ \Omega_{(11)}&=\exp(\,i\,\pi f_1\;\mathbf{1}\otimes\sigma_3\otimes\mathbf{1}\otimes\mathbf{1}+i\,\pi f_2\;\mathbf{1}\otimes\mathbf{1}\otimes\sigma_3\otimes\mathbf{1}+i\,\pi f_3\;\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\sigma_3\,).\end{aligned} \tag{3.27}$$

The fixed-plane condition $Q_{(11)}=\Omega_{(11)}\,Q_{(11)}$ can now be written as

$$\begin{aligned}&Q_{(5)}\otimes LLL=e^{-i\pi(f_1+f_2+f_3)}\,LLL,\quad Q_{(5)}\otimes RRR=e^{+i\pi(f_1+f_2+f_3)}\,RRR\\ &Q_{(5)}\otimes LLR=e^{-i\pi(f_1+f_2-f_3)}\,LLR,\quad Q_{(5)}\otimes RRL=e^{+i\pi(f_1+f_2-f_3)}\,RRL\\ &Q_{(5)}\otimes LRL=e^{-i\pi(f_1-f_2+f_3)}\,LRL,\quad Q_{(5)}\otimes RLR=e^{+i\pi(f_1-f_2+f_3)}\,RLR\\ &Q_{(5)}\otimes LRR=e^{-i\pi(f_1-f_2-f_3)}\,RLL,\quad Q_{(5)}\otimes RLL=e^{+i\pi(f_1-f_2-f_3)}\,LRR.\end{aligned}$$

Therefore, as we approach a fixed-five-plane, the bulk supercharge is projected according to

$$Q_{(11)}\to\begin{cases}Q_{(5)}\otimes(\,RRR+LLL\,), & f_1+f_2+f_3\in 2\,\mathbb{Z},\\ Q_{(5)}\otimes(\,RLL+LRR\,), & f_1-f_2-f_3\in 2\,\mathbb{Z},\\ Q_{(5)}\otimes(\,LRL+RLR\,), & f_1-f_2+f_3\in 2\,\mathbb{Z},\\ Q_{(5)}\otimes(\,LLR+RRL\,), & f_1+f_2-f_3\in 2\,\mathbb{Z},\end{cases} \tag{3.28}$$

and is projected to zero if none of these conditions are met. Thus, we retain some supersymmetry on a fixed-five-plane if and only if at least one of the four sums $f_1 \pm f_2 \pm f_3$ is an even integer. In generic cases of this sort, the bulk supersymmetry is *quartered* on the fixed-plane in question.

Four-dimensional fixed-planes: Four-dimensional fixed-planes correspond to elements $(f_1, f_2, f_3 | P) = (f_1, f_2, f_3 | 1)$, where $f_{1,2,3} \neq 0 \bmod 1$. In this case the supercharge transforms as $Q_{(11)} \to \Omega_{(11)} Q_{(11)}$, where

$$\Omega_{(11)} = \exp(\pi f_1 \Gamma_5 \Gamma_6) \exp(\pi f_2 \Gamma_7 \Gamma_8) \exp(\pi f_3 \Gamma_9 \Gamma_{10}) \Gamma_{11}. \tag{3.29}$$

We write $Q_{(11)}$ as a tensor product of a four-dimensional fixed-plane spinor with a normal spinor according to $Q_{(11)} = Q_{(4)} \otimes Q_{(7)}$, and we represent the gamma matrices precisely as in (3.25). Furthermore, we decompose the four-dimensional spinors as $Q_{(4)} = Q_{(4)\,R} + Q_{(4)\,L}$ where $Q_{(4)\,R,L} = \pm \gamma_5 Q_{(4)\,R,L}$. We also write the SO(7) normal spinor as $Q_{(7)} = Q_{(6)R} \oplus Q_{(6)L}$. Finally, we can decompose the chiral six-dimensional spinors $Q_{(6)R,L}$ into two-dimensional chiral spinors as descried above. Using these conventions, we can write

$$\begin{aligned} Q_{(11)} &= (Q_{(4)\,R} + Q_{(4)\,L}) \otimes (RRR + RLL + LRL + LLR + LLL + LRR + RLR + RRL), \\ \Omega_{(11)} &= \exp(i\pi f_1\, \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} + i\pi f_2\, \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} + i\pi f_3\, \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3)\, \gamma_5 \\ &\quad \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3. \end{aligned} \tag{3.30}$$

The fixed-plane condition $Q_{(11)} = \Omega_{(11)} Q_{(11)}$ can now be written as

$$\begin{aligned} &Q_{(4)\,R,L} \otimes LLL = \mp e^{-i\pi(f_1+f_2+f_3)} LLL, Q_{(4)\,R,L} \otimes RRR = \pm e^{+i\pi(f_1+f_2+f_3)} RRR, \\ &Q_{(4)\,R,L} \otimes LLR = \pm e^{-i\pi(f_1+f_2-f_3)} LLR, \quad Q_{(4)\,R,L} \otimes RRL = \mp e^{+i\pi(f_1+f_2-f_3)} RRL, \\ &Q_{(4)\,R,L} \otimes LRL = \pm e^{-i\pi(f_1-f_2+f_3)} LRL, \quad Q_{(4)\,R,L} \otimes RLR = \mp e^{+i\pi(f_1-f_2+f_3)} RLR, \\ &Q_{(4)\,R,L} \otimes LRR = \mp e^{-i\pi(f_1-f_2-f_3)} RLL, \quad Q_{(4)\,R,L} \otimes RLL = \pm e^{+i\pi(f_1-f_2-f_3)} LRR. \end{aligned}$$

Therefore, as we approach a fixed-four-plane, the bulk supercharge is projected according to

$$Q_{(11)} \to \begin{cases} Q_{(4)L} \otimes LLL + Q_{(4)R} \otimes RRR, & f_1+f_2+f_3 \in 2\,\mathbb{Z}, \\ Q_{(4)R} \otimes LRL + Q_{(4)L} \otimes RLR, & f_1-f_2+f_3 \in 2\,\mathbb{Z}, \\ Q_{(4)R} \otimes RRL + Q_{(4)L} \otimes LLR, & f_1+f_2-f_3 \in 2\,\mathbb{Z}, \\ Q_{(4)L} \otimes RLL + Q_{(4)R} \otimes LRR, & f_1-f_2-f_3 \in 2\,\mathbb{Z}, \\ Q_{(4)R} \otimes LLL + Q_{(4)L} \otimes RRR, & f_1+f_2+f_3 \in 2\,\mathbb{Z}+1, \\ Q_{(4)L} \otimes LRL + Q_{(4)R} \otimes RLR, & f_1-f_2+f_3 \in 2\,\mathbb{Z}+1, \\ Q_{(4)L} \otimes RRL + Q_{(4)R} \otimes LLR, & f_1+f_2-f_3 \in 2\,\mathbb{Z}+1, \\ Q_{(4)R} \otimes RLL + Q_{(4)L} \otimes LRR, & f_1-f_2-f_3 \in 2\,\mathbb{Z}+1, \end{cases} \tag{3.31}$$

and is projected to zero if none of these conditions are met. Thus, we retain some supersymmetry on a fixed-four-plane if and only if any one of the four sums $f_1 \pm f_2 \pm f_3$ is an integer (even or odd). In generic cases of this sort, supersymmetry is *eighthed* on the fixed-plane in question.

There are special cases, however. For instance, for each of the two cases where (f_1, f_2, f_3) are given by $(1/2, 1/3, 1/6)$ and by $(1/2, 1/4, 1/4)$, one sum $f_1 - f_2 - f_3 = 0$ is an even integer, while another sum $f_1 + f_2 + f_3 = 1$ is an odd integer. Therefore, on the fixed-plane, $Q_{(11)}$ retains nonvanishing components of both the fourth and the also the fifth case in (3.31). Thus, supersymmetry is merely quartered to $D=4, N=2$, rather than eighthed to $D=4, N=1$, in these cases. (If the fixed-four-plane in question is a submanifold of the fixed-plane of another element of Γ, one needs to account for the action of this other element as well before drawing conclusions as to the amount of fixed-plane supersymmetry.)

Note that the pairs of summands indicated in (3.31) have correlations owing to the Majorana constraint on $Q_{(11)}$. As a result, the two apparent summands in each of these expressions actually describe the same degrees of freedom. In terms of the choices we have made, this gives rise to a Majorana spinor supercharge in four dimensions. The pairs which appear in (3.31) represent left- and right-handed chiral projections of this four-dimensional Majorana supercharge. As is well known, in four dimensions spinor fields which are Majorana may be equally well represented in terms of either chiral projection.

IV. GENERALIZATION TO NON-ABELIAN ORBIFOLD GROUPS

One obvious extension of our criterion, still for orbifolds of tori of our class, is to consider *all* the automorphisms of the underlying lattices. In addition to the rotations and flips we have already considered, the only new type of "generating" automorphism to deal with is the permutations of the coordinates x^i. In fact, one can easily lift these permutation actions to spinors. Here are the details.

The lattices of our class are all of the form

$$\Lambda=(A_1)^a\oplus(A_2)^b \ , \tag{4.1}$$

with $a+2b=n$, the dimension of the orbifold. Such a lattice has an automorphism group of order $2^a\cdot a!\cdot b!\cdot 12^b$, a product of four factors. Let's check where these two pairs of factors come from.

The first pair, $2^a\cdot a!$, keeps track of the sign flips on each of a possible x^i (the 2^a), and permutations of the same a x^i's (the $a!$). We have already described a way to lift the flips to spinors, via Γ_i, so we only need to lift the permutations. The second pair, $b!\cdot 12^b$, keeps track of the permutations of the b different copies of A_2, i.e., of corresponding ordered pairs (x^i,x^{i+1}) (giving the $b!$), and automorphisms internal to each of the b A_2's (giving the 12^b, as the automorphism group of the A_2 lattice is the dihedral group D_6 of order 12). Consider a single A_2 lattice in the "(x,y)-plane." The dihedral automorphism group D_6 consists entirely of the identity, rotations by $r^k:=2\pi k/6$, $k=1,2,3,4,5$, and flips $(s,sr,sr^2,sr^3,sr^4,sr^5)$ across lines through opposite vertices or opposite edge centers of the period hexagon (we can assume that s is the flip across the x-axis sending $y\mapsto -y$). In particular, D_6 is generated by a rotation r and a flip s, for each of which we have already described a lift to spinors. What remains is again to describe the effect of the permutation automorphisms on spinors.

Since every permutation of the x^i can be expressed as a product of simple transpositions $x^i\leftrightarrow x^j$, $i\neq j$, it suffices to write out the lift of such a transposition. Imagine the (x^i,x^j)-plane. Rotation counterclockwise by $\pi/2$ radians sends $x^i\mapsto x^j$ and $x^j\mapsto -x^i$. This is represented in the Clifford algebra by

$$\exp\left(\frac{\pi}{4}\Gamma_{ij}\right)=\cos(\pi/4)+\sin(\pi/4)\,\Gamma_i\Gamma_j=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\Gamma_i\Gamma_j\right). \tag{4.2}$$

Now compose this with the flip sending $x^i\mapsto -x^i$ given by Γ_i, yielding

$$\Gamma_i\cdot\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\Gamma_i\Gamma_j\right)=\frac{\sqrt{2}}{2}(\Gamma_i+\Gamma_j). \tag{4.3}$$

We can build any permutation we like out of a product of these, and in particular all of the remaining automorphisms come from mixing these with the rotations and flips as above. By adding such permutations, we can obtain a complete supersymmetric restriction criterion for all orbifolds modelled on lattices of type (4.1). In particular, this allows one to check the supersym-

TABLE VII. Character table for the group $(\mathbb{Z}_2)^3$.

$(\mathbb{Z}_2)^3$	Ω	$\Gamma_{[0,0,1]}$	$\Gamma_{[0,1,0]}$	$\Gamma_{[0,1,1]}$	$\Gamma_{[1,0,0]}$	$\Gamma_{[1,0,1]}$	$\Gamma_{[1,1,0]}$	$\Gamma_{[1,1,1]}$
1	+	+	+	+	+	+	+	+
γ	+	−	+	−	+	−	+	−
β	+	+	−	−	+	+	−	−
$\beta\gamma$	+	−	−	+	+	−	−	+
α	+	+	+	+	−	−	−	−
$\alpha\gamma$	+	−	+	−	−	+	−	+
$\alpha\beta$	+	+	−	−	−	−	+	+
$\alpha\beta\gamma$	+	−	−	+	−	+	+	−

metry constraints for a large class of *non-Abelian* orbifold groups. The generalized supersymmetric restriction theorem, including the extension to non-Abelian orbifold groups, will be discussed in a forthcoming paper.

V. SOFT ORBIFOLDS AND G_2-STRUCTURES

Consider the supersymmetric orbifold of $\Gamma=(\mathbb{Z}_2)^3$ labeled (1111111) in Table I. The corresponding representation of Γ on $\mathbb{R}^7$ is given by the direct sum of the seven nontrivial characters $\Gamma_{[i,j,k]}$, each with multiplicity one. The character table for $\Gamma=\langle\alpha,\beta,\gamma\rangle$ is given in Table VII, and, by following the prescription given in Sec. II C, we see that the action of these generators on the coordinate 7-tuple $(x_1,\ldots,x_7)$ is given by

$$\alpha: \quad (x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7), \tag{5.1}$$

$$\beta: \quad (x_1,-x_2,-x_3,x_4,x_5,-x_6,-x_7), \tag{5.2}$$

$$\gamma: \quad (-x_1,x_2,-x_3,x_4,-x_5,x_6,-x_7). \tag{5.3}$$

Note that this Γ-action preserves the G_2-invariant differential 3-form φ_0 on $\mathbb{R}^7$,

$$\varphi_0 := \mathrm{d}x_{123}+\mathrm{d}x_{145}+\mathrm{d}x_{167}+\mathrm{d}x_{246}-\mathrm{d}x_{257}-\mathrm{d}x_{347}-\mathrm{d}x_{356},$$

where $\mathrm{d}x_{ijk} := \mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k$. Such an action defines a G_2-*structure* on the orbifold T^7/Γ.

Our supersymmetric hard orbifold (1111111) is one of a family of orbifolds considered by Joyce. [In fact, in the notation of Ref. 15 Eqs. (23)–(25), ours corresponds to $b_1=b_2=c_1=c_3=c_5=0$. We will refer to notation and examples from Ref. 7 instead.] It was his goal to construct compact manifolds with holonomy group G_2. To this end he developed in Refs. 14, 15, and 7 a machinery which, starting with a sufficiently simple orbifold admitting a G_2-structure, establishes the existence of such a metric on a "resolution" of the orbifold. Joyce's method depends on the existence of certain *R-data* ("*R*" for resolution) which, for a given orbifold with G_2-structure, may yield a large number of topologically distinct G_2 holonomy manifolds as resolutions.

In Ref. 7 Secs. 12.2, 3, and 5 three paricular softenings are considered. The actions of the generators of Γ take the form

$$\alpha: \quad (x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7), \tag{5.4}$$

$$\beta: \quad (x_1,-x_2,-x_3,x_4,x_5,b-x_6,-x_7), \tag{5.5}$$

$$\gamma: \quad \left(-x_1,x_2,-x_3,x_4,c-x_5,x_6,\tfrac{1}{2}-x_7\right) , \tag{5.6}$$

with $b,c\in\{0,\frac{1}{2}\}$. In fact, Joyce considers the three cases

$$(b,c)=(0,\tfrac{1}{2}), \quad (\tfrac{1}{2},0), \quad \text{and } (\tfrac{1}{2},\tfrac{1}{2}),$$

and shows that each of these admits a set of R-data defining a G_2 holonomy resolution. In each case the generators α, β, and γ act with fixed-points as before, but some other elements, like

$$\beta\gamma: \quad (-x_1,-x_2,x_3,x_4,c-x_5,b+x_6,\tfrac{1}{2}+x_7) \ , \tag{5.7}$$

act freely. Thus softening an orbifold has the effect of simplifying the singularities. The singularities of the orbifold (1111111) are too complicated to visibly admit a compatible set of R-data, hence his interest in various softenings with simpler singularities. Furthermore, at least for these simplest examples, the set of fixed-plane dimensions associated to a hard orbifold contains as subsets those of all of its softenings. For this reason, one expects that much of the supersymmetric restriction analysis of Sec. III can be usefully adapted to the case of soft orbifolds.

It is clear that having a G_2-structure on a hard orbifold implies such a structure on all of its softenings. This raises the intriguing question of the relationship between our supersymmetry condition and such G_2-structures: Do all supersymmetric hard orbifolds (and hence all of their softenings) carry a G_2-structure? What about the converse: Do all orbifolds with a G_2-structure arise via softening from supersymmetric hard orbifolds? What about the more subtle issue of resolvability as a G_2-manifold (i.e., existence of compatible R-data)? These are all questions to be addressed in future work.

VI. CONCLUSIONS

We have described a systematic method for classifying supersymmetric orbifold compactifications of M-theory, specifically hard orbifolds defined by pseudo-planar representations of Abelian groups of order $\leqslant 12$. Although we stopped our "periodic table" (Table I) at order 12, the methods developed here apply to groups of arbitrary order, and the algorithmics are such that such an extension (using Mathematica) is computationally feasible.

It is physically relevant that we demand orbifold actions be compatible with particular lattices. By doing this we are able to keep control over two separate problems. On the one hand we are studying aspects of supersymmetric singularities. On the other hand, however, we are at the same time learning something about which sets of such singularities, and their neighborhoods, can be assembled together to create a global supersymmetric compactification space. Elucidating the relationship between our own supersymmetric configurations of singularities and those of intersecting branes as studied by other groups[24–35] seems an important direction for further investigation.

Furthermore, we touch upon the interesting mathematical problem of classifying G_2 manifolds. The vast majority of known compact G_2 holonomy manifolds arise by Joyce's resolution of singularities from orbifolds of T^7. Physically one expects that seven-dimensional supersymmetric compactification spaces of the sort we are studying should admit such G_2 resolutions. In order to compare in general Joyce's "resolution data" to the constraints arising from our global supersymmetric restriction, it seems necessary to formulate the restriction for the soft orbifolds of the introduction, as all of Joyce's examples are of this sort.

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APPENDIX: THE *C*-MATRICES FOR SELECTED GROUPS

By applying the simple algorithm described in Sec. II B, we are able to derive the C-matrix for any pseudoplanar group $\Gamma = G_1 \times \cdots \times G_n$, where $G_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4\}$ for $1 \leqslant i \leqslant n$. In Tables VIII and IX we exhibit these for all cases for which $|\Gamma| \leqslant 12$.

TABLE VIII. The C-matrices for pseudoplanar Abelian groups of order $\leqslant 8$.

$$C(\mathbb{Z}_2) = 1$$

$$C(\mathbb{Z}_3) = 1$$

$$C(\mathbb{Z}_4) = \left(\begin{array}{c|c} 0 & 2 \\ \hline 2 & 1 \end{array}\right)$$

$$C(\mathbb{Z}_2\times\mathbb{Z}_2) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$C(\mathbb{Z}_2\times\mathbb{Z}_3) = \left(\begin{array}{c|c|c} 3 & 0 & 3 \\ \hline 0 & 2 & 2 \\ \hline 3 & 2 & -1 \end{array}\right)$$

$$C(\mathbb{Z}_2\times\mathbb{Z}_4) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 4 & 4 \\ 0 & 4 & 4 & 0 & 4 \\ 0 & 4 & 4 & 4 & 0 \\ \hline 4 & 0 & 4 & 2 & 2 \\ 4 & 4 & 0 & 2 & -2 \end{array}\right)$$

$$C(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2) = \begin{pmatrix} 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 0 & 4 & 0 \\ 0 & 4 & 4 & 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 & 4 \end{pmatrix}$$

TABLE IX. The C-matrices for pseudoplanar Abelian groups of orders 9 and 12.

$$C(\mathbb{Z}_3\times\mathbb{Z}_3) = \begin{pmatrix} 0 & 3 & 3 & -3 \\ 3 & 0 & 3 & 3 \\ 3 & 3 & -3 & 0 \\ -3 & 3 & 0 & 3 \end{pmatrix}$$

$$C(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3) = \left(\begin{array}{ccc|c|ccc} 0 & 6 & 6 & 0 & 0 & 6 & 6 \\ 6 & 0 & 6 & 0 & 6 & 0 & 6 \\ 6 & 6 & 0 & 0 & 6 & 6 & 0 \\ \hline 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ \hline 0 & 6 & 6 & 4 & 4 & -2 & -2 \\ 6 & 0 & 6 & 4 & -2 & 4 & -2 \\ 6 & 6 & 0 & 4 & -2 & -2 & 4 \end{array}\right)$$

$$C(\mathbb{Z}_3\times\mathbb{Z}_4) = \left(\begin{array}{c|c|c|c|cc} 0 & 0 & 6 & 0 & 6 & 6 \\ \hline 0 & 4 & 0 & 4 & 4 & 4 \\ \hline 6 & 0 & 3 & 6 & 3 & -3 \\ \hline 0 & 4 & 6 & 4 & -2 & -2 \\ 6 & 4 & 3 & -2 & -5 & 1 \\ 6 & 4 & -3 & -2 & 1 & -5 \end{array}\right)$$

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