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Picard–Fuchs Uniformization and Modularity of the Mirror Map

Charles F. Doran[⋆]

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA.
E-mail: doran@math.harvard.edu

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Abstract: Arithmetic properties of mirror symmetry (type IIA-IIB string duality) are studied. We give criteria for the mirror map q-series of certain families of Calabi–Yau manifolds to be automorphic functions. For families of elliptic curves and lattice polarized K3 surfaces with surjective period mappings, global Torelli theorems allow one to present these criteria in terms of the ramification behavior of natural algebraic invariants – the functional and generalized functional invariants respectively. In particular, when applied to one parameter families of rank 19 lattice polarized K3 surfaces, our criterion demystifies the Mirror-Moonshine phenomenon of Lian and Yau and highlights its non-monstrous nature. The lack of global Torelli theorems and presence of instanton corrections makes Calabi–Yau threefold families more complicated. Via the constraints of special geometry, the Picard–Fuchs equations for one parameter families of Calabi–Yau threefolds imply a differential equation criterion for automorphicity of the mirror map in terms of the Yukawa coupling. In the absence of instanton corrections, the projective periods map to a twisted cubic space curve. A hierarchy of "algebraic" instanton corrections correlated with the differential Galois group of the Picard–Fuchs equation is proposed.

1. Introduction

Numerous remarkable properties of the type IIA-IIB string duality better known as mirror symmetry have been revealed since its discovery a decade ago. Mathematically this symmetry entails a correspondence between complex moduli in one family of Calabi–Yau manifolds and Kähler moduli of a mirror family. In the neighborhood of a large complex structure/large radius limit point mirror symmetry is described by the mirror map q-series. The mirror map is a locally holomorphic function determined by the behavior of fundamental solutions to the Picard–Fuchs equation for periods of a Calabi–Yau

[⋆] *Present address:* Center for Geometry and Mathematical Physics, Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA family about a point of maximal unipotent monodromy. For a family of Calabi–Yau threefolds the mirror map q-series and the Yukawa couplings determine a generating function for the Gromov–Witten invariants. These invariants (conjecturally) count the number of rational curves on a generic member of the family. In fact, the original predictions of Candelas [2] for the one parameter family of Fermat-type quintic Calabi–Yau hypersurfaces have now been proven mathematically [22].

In a series of papers, Lian and Yau [23–26] investigate arithmetic properties of the mirror maps of several "torically defined" families of elliptic curves, K3 surfaces, and Calabi–Yau threefolds constructed in their work with Hosono, Klemm, Roan and Theisen [14,15,19,16,18]. Each of the one parameter families of elliptic curves and K3 surfaces they study has a globally defined mirror map, automorphic with respect to the global monodromy group of the family. The mirror maps of these elliptic curve families are classical modular functions for finite index subgroups of $\mathrm{PSL}(2, \mathbf{Z})$, while the mirror maps of the K3 surface families are, up to an additive integer correction, always reciprocals of some McKay–Thompson series associated to the monster in the "Monstrous Moonsine" lists of Conway and Norton [4]. In particular, the mirror maps of their examples are always automorphic functions for genus zero subgroups of $\mathrm{PSL}(2, \mathbf{R})$, a phenomenon Lian and Yau dub "Mirror-Moonshine".

When such modularity properties are possessed by a mirror map, other properties of potential physical interest can be derived: e.g., integrality of the mirror map and prepotential, congruences satisfied by the mirror map coefficients, the effect on instanton corrections, etc. Thus a question of mathematical interest and physical relevance is:

Question 1. When is the mirror map an automorphic function?

Unlike other questions regarding the mirror map studied in the literature, this is an inherently global question. We are asking for which families of Calabi–Yau manifolds does the mirror map admit an extension to a map from the whole period domain to the entire base of the family.

Our question is related to the classical problem of characterizing modular relations between automorphic functions and the elliptic modular function. In fact, for families of elliptic curves we will see in Sect. 2.1 that this is all that is involved: we recover the classical criterion for just such modular relations from [11,1,41]. In [7,8] we answer our question for families over $\mathbb{P}^1$. For elliptic curve families we use Kodaira's *functional invariant* $\mathcal{J}$ to pull back the uniformizing differential equation for the elliptic modular function from the coarse moduli space of elliptic curves (the J-line). The existence of the functional invariant $\mathcal{J}$ can be interpreted as a consequence of the (trivial) classical analogue of the global Torelli theorem. In the case of lattice polarized K3 surface families, we apply the global Torelli theorem of Nikulin (see the lists of related works in Dolgachev [6]) to define a *generalized functional invariant* mapping again from the base of a family to the associated coarse moduli space. We use this generalized functional invariant to explain the Mirror-Moonshine phenomenon for families of K3 surfaces over $\mathbb{P}^1$ with third order Picard–Fuchs differential equations – the setting in which the Mirror-Moonshine Conjecture of Lian and Yau was originally formuated.

The basic idea behind our approach to answering the modularity question for one parameter families is quite simple: The mirror map of a family of elliptic curves (resp. rank 19 lattice polarized K3 surfaces) is classically modular (resp. automorphic) if and only if the Picard–Fuchs differential equation is a classical uniformizing differential equation (resp. the symmetric square of one). We call this *Picard–Fuchs uniformization*.

In this paper, instead of deriving the modularity criterion "from scratch" from the local behavior of uniformizing differential equations on $\mathbb{P}^1$, we use the theory of branched covers of orbifolds as described by Namba [31]. This approach gives us directly the modularity criterion in the neatest possible form, and applies to both

1. one parameter families of elliptic curves (Sect. 2.1) and rank 19 lattice polarized K3 surfaces (Sect. 3.1) over a base curve of arbitrary genus, and
2. multiparameter families of lattice polarized K3 surfaces with surjective period mapping (Sect. 3.2).

We replace the uniformizing differential equations for the elliptic curve and lattice polarized K3 surface families with holomorphic projective connections and holomorphic conformal connections respectively. Picard–Fuchs uniformization occurs when the Gauss-Manin connection of such a family of elliptic curves (resp. lattice polarized K3 surfaces) is a holomorphic projective (resp. holomorphic conformal) connection.

The lack of a global Torelli theorem for Calabi–Yau threefolds (in particular no presentation of the coarse moduli space as a locally symmetric space) prevents one from algebraically defining generalized functional invariants or mimicing the previous arguments for elliptic curves and K3 surfaces. Instead of an algebraic criterion for modularity of the mirror map, we must settle for a differential algebraic one in general.

The Picard–Fuchs equation for a one parameter family of Calabi- Yau manifolds with $h^{2,1} = 1$ has order four. Moreover the constraints imposed by special geometry imply that about a point of maximal unipotent monodromy there is a set of fundamental solutions of the form

$$u, u \cdot t, u \cdot \dot{F}, u \cdot (t\dot{F} - 2F) ,$$

where $u(z)$ is the fundamental solution locally holomorphic at the point of maximal unipotent monodromy, $t(z)$ is the mirror map, and $F(z)$ is the prepotential (the derivative $\dot{F}$ is taken with respect to the mirror map coordinate t). Following Lian and Yau, one can derive a “quantum Schwarzian equation” relating the second order coefficient of the Picard–Fuchs equation, the mirror map, and the Yukawa couplings (Sect. 4.1).

In the absence of instanton corrections, this quantum Schwarzian reduces to a classical one, and the Picard–Fuchs equation takes the special form of a symmetric cube of a second order equation. We give first a criterion for modularity of such mirror maps in the beginning of Sect. 4.2.

Suppose on the other hand that there are instanton corrections, so the quantum Schwarzian is not classical. If we assume that the mirror map is an automorphic function, it will satisfy another classical Schwarzian equation. By subtracting the two to eliminate the Schwarzian derivative terms, and applying a reduction of order argument to the original Picard–Fuchs equation, we obtain a nonlinear differential equation in the Yukawa coupling and coefficients of the Picard–Fuchs and classical Schwarzian equations (the “modularity equation” in Theorem 9). The mirror map does not appear in this expression, yet the equation will hold if and only if the mirror map is automorphic. This is our general criterion for modularity of the mirror map for Calabi–Yau threefolds.

The absence of instanton corrections in a one parameter family of Calabi–Yau threefolds corresponds to the existence of a homogeneous third order relation among the four periods, i.e., the image of the period mapping lies on a twisted cubic space curve. It is natural to ask what other homogeneous algebraic relations can occur between periods of one parameter families of Calabi–Yau threefolds. We call these “algebraic” instanton corrections. In Sect. 4.3 we apply a century old theorem of Fano to give a rough classification, paralleling the structure of the differential Galois group of the Picard–Fuchs equation.

Most of the results on the mirror map for Calabi–Yau manifolds which appear in the literature depend on the hypothesis that the families of Calabi–Yau threefolds arise "torically", i.e., as particular parametrized families of hypersurfaces or complete intersections in Fano toric varieties. By working in the setting of transcendental algebraic geometry, we obtain general results about whole classes of families of Calabi–Yau manifolds. There has been a major effort in the literature to produce examples, first of mirror maps in general [19, 14–16] and then to test the (generalized) Mirror-Moonshine phenomenon in particular [23–26, 42]. Since the point of this paper is to explain general tools and results, we refer the reader interested in examples to the papers cited above.

2. Elliptic Curve Families

In this section we derive the modularity criterion for mirror maps of one parameter families of elliptic curves with section (Theorem 2), and make some comments on the case of multiparameter families of elliptic curves (Sect. 2.2). It does not really make sense to ask our question in this latter case, but we will use it to motivate some aspects of the problem for multiparameter families of K3 surfaces.

2.1. One parameter families of elliptic curves with section. In the early 1960's Kodaira developed a general theory of elliptic surfaces, i.e., compact complex surfaces fibered over curves, with generic fiber an elliptic curve. In particular he showed that every elliptic surface with section is determined by a pair of natural invariants. The first of these, the *functional invariant*, is a meromorphic function on the base of the family which keeps track of the J-value of each elliptic curve fiber. The second, the *homological invariant*, is nothing more than the monodromy representation associated with the second order Fuchsian ordinary differential equation satisfied by the periods, i.e., the monodromy of the Picard–Fuchs equation.

The elliptic surfaces with a section, the *basic elliptic surfaces*, play a distinguished role in Kodaira's theory. There is a *canonical form* for such a family of elliptic curves $\pi : X \to C$ with section, exhibiting X as a divisor in a $\mathbb{P}^2$-bundle over the base curve C:

Theorem 1 ([29], Theorem (2.1)). *Let Σ denote the given section of π, i.e., $\Sigma = s(C)$, a divisor on X which is taken isomorphically onto C by π. Let $\mathcal{L} = \pi_*[\mathcal{O}_X(\Sigma)/\mathcal{O}_X]$. Suppose that the general fiber of π is smooth. Then $\mathcal{L}$ is invertible and X is isomorphic to the closed subscheme of $\mathbb{P} = \mathbb{P}(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_Y)$ defined by*

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 \ ,$$

where

$$g_2 \in \Gamma(C, \mathcal{L}^{\otimes -4}) \ , \ g_3 \in \Gamma(C, \mathcal{L}^{\otimes -6}) \ ,$$

and $[x, y, z]$ is the global coordinate system of $\mathbb{P}$ relative to $(\mathcal{L}^{\otimes 2}, \mathcal{L}^{\otimes 3}, \mathcal{O}_C)$. Moreover the pair (g_2, g_3) is unique up to isomorphism, and the discriminant

$$g_2^3 - 27 g_3^2 \in \Gamma(C, \mathcal{L}^{\otimes -12})$$

vanishes at a point $s \in C$ precisely when the fiber X_s is singular. □

For a family of elliptic curves in Weierstrass form, the functional invariant takes the form

$$\mathcal{J} = g_2^3/\Delta : C \to \mathbb{P}^1_J . \tag{1}$$

The fact that the functional invariant takes the special form of Eq. (1) is evidence of the coarseness of the "J- line" moduli space of elliptic curves. By contrast, if we were to mark the two torsion on each elliptic curve, i.e., use the Legendre family $y^2 = x(x-1)(x-\lambda(s))$ "λ-line" moduli space, then *any* rational function on the base curve C would be a "λ-functional invariant" for a family of elliptic curves with level two structure over C.

Kodaira has classified the singular fiber types which can arise in Weierstrass fibered elliptic surfaces. The singular fibers which appear in a smooth minimal elliptic surface fall into "types":

$$\mathrm{I}_n\ (n \geq 0), \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_n^*\ (n \geq 0), \mathrm{IV}^*, \mathrm{III}^*, \text{ and } \mathrm{II}^*.$$

Denote a smooth elliptic fiber by I_0. The fiber of type I_1 is a rational curve with a single node. More generally, fibers of type I_n consist of an n-cycle of intersecting rational curves for $n \geq 1$. A fiber of type II is just a rational curve with a single cusp. Type III fibers consist of two rational curves with a single point of tangency. Fibers of type IV consist of three rational components intersecting at a single point. There are also fibers of types I_n^*, $n \geq 0$, IV^*, III^*, and II^*, whose dual intersection graphs, minus in each case a multiplicity one component, correspond to those graphs of Dynkin types D_{n+4}, E_6, E_7, and E_8 respectively.

We now recall how the Kodaira fiber types correlate with the ramification behavior of the $\mathcal{J}$-map.

Lemma 1 ([30], Lemma IV.4.1). *Let $F = X_s$ be the fiber of π over $s \in C$, and let $\nu_s(\mathcal{J})$ be the multiplicity of the functional invariant at s.*

1. *If F has type* II, IV, IV^*, *or* II^*, *then $\mathcal{J}(s) = 0$. Conversely, suppose that $\mathcal{J}(s) = 0$. Then*
   – *F has type I_0 or I_0^* if and only if $\nu_s(\mathcal{J}) \equiv 0 \bmod 3$,*
   – *F has type* II *or* IV^* *if and only if $\nu_s(\mathcal{J}) \equiv 1 \bmod 3$,*
   – *F has type* IV *or* II^* *if and only if $\nu_s(\mathcal{J}) \equiv 2 \bmod 3$.*
2. *If F has type* III *or* III^*, *then $\mathcal{J}(s) = 1$. Conversely, suppose that $\mathcal{J}(s) = 1$. Then*
   – *F has type I_0 or I_0^* if and only if $\nu_s(\mathcal{J}) \equiv 0 \bmod 2$,*
   – *F has type* III *or* III^* *if and only if $\nu_s(\mathcal{J}) \equiv 1 \bmod 2$.*
3. *F has type I_n or I_n^* with $n \geq 1$ if and only if $\mathcal{J}$ has a pole at s of order n.* □

Following [35, p. 304], one can apply the Griffiths-Dwork approach to computing the Picard–Fuchs equation of a Weierstrass elliptic surface as a Fuchsian system:

$$\frac{d}{dz}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{bmatrix} \frac{-1}{12} d\log\Delta & \frac{3\delta}{2\Delta} \\ \frac{-g_2\delta}{8\Delta} & \frac{1}{12} d\log\Delta \end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},$$

where

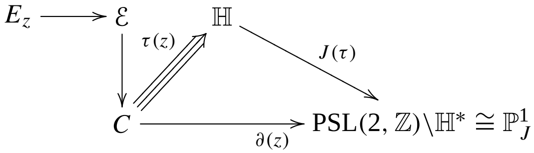
$$\Delta = g_2^3 - 27g_3^2\ ,\ \delta = 3g_3g_2{}' - 2g_2g_3{}'$$

and

$$\eta_1 = \int_\gamma \frac{dx}{y}\ ,\ \eta_2 = \int_\gamma \frac{xdx}{y}.$$

From this, for a one parameter family of elliptic curves in Weierstrass form it is not difficult to write down the Picard–Fuchs second order ordinary differential equation satisfied by the periods of the holomorphic one form $\omega = dx/y$ over the one cycles on the fibers. Picking a basis of one cycles γ_i, $i = 0, 1$, we denote by $f_i = \int_{\gamma_i} \omega$ the basis of solutions to the Picard–Fuchs equation.

We can now reinterpret Kodaira's functional invariant $\mathcal{J}$ as the composition of the projective period morphism $\tau := f_1/f_0 : C \to \mathbb{H} \subset \mathbb{P}^1$ and the morphism $J : \mathbb{H} \to \mathbb{P}^1$ extending the classical modular function, i.e., $\mathcal{J} = J \circ \omega_1/\omega_0$:



Recall that a regular singular point of a Fuchsian ordinary differential equation of order k

$$\frac{d^k f}{ds^k} + P_1(s)\frac{d^{k-1} f}{ds^{k-1}} + \ldots + P_k(s) f = 0 \ , \ \ P_i(s) \in \mathbb{C}(s) \ , \tag{2}$$

is called a *point of maximal unipotent monodromy* if the local monodromy matrix G is such that $G - \mathrm{I}_k$ is nilpotent with exact order k. In a neighborhood of a point of maximal unipotent monodromy, Frobenius' method tells us that there is a basis of solutions such that the first is holomorphic at the point, the second has logarithmic behavior, the next behaves like $\log^2, \ldots,$ up to $\log^{k-1}$.

An easy consequence of Lemma 1 is

Corollary 1. *The points of maximal unipotent monodromy in the base curve C of an elliptic surface $\mathcal{E}$ are the points $s \in C$ over which there is a singular fiber of type I_n or I_n^*, $n \geq 1$ (i.e., the support of the semistable elliptic fibers).* □

Moreover, the presence of a point of maximal unipotent monodromy has global effects:

Corollary 2. *The Picard–Fuchs differential equation of an elliptic surface has a point of maximal unipotent monodromy if and only if the global monodromy group has infinite order if and only if the family of elliptic curves is not isotrivial.* □

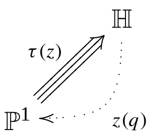
Consider more generally a one parameter family of Calabi–Yau manifolds $\pi : X \to C$, whose Picard–Fuchs equation has a point of maximal unipotent monodromy. In a neighborhood of such a point consider the multivalued truncated period vector consisting only of the holomorphic solution and the logarithmic solution

$$[p_{\mathrm{hol}}(s) : p_{\log}(s)] : C \Rrightarrow \mathbb{P}^1.$$

If the image lies in the upper half plane $\mathbb{H} \subset \mathbb{P}^1$, then, possibly after composition with projective linear transformations so that the singular point lies at $0 \in \mathbb{P}^1$ and maps to $\imath\infty \in \mathbb{H}^* \subset \mathbb{P}^1$, we can consider the q-series for the local inverse mapping

$$z(q(\tau)) : \mathbb{H} \dashrightarrow C \ , \ q(\tau) = e^{2\pi\imath\tau}.$$

This q-series $z(q)$ is called the *mirror map* of the family $\pi : X \to \mathbb{P}^1$ about the point of maximal unipotent monodromy:



Example 1. Consider the family $\mathcal{E}_J$ of elliptic curves over $\mathbb{P}^1$ defined by the equation

$$\mathcal{E}_J : \ y^2 = 4x^3 - \frac{27s}{s-1}x - \frac{27s}{s-1}.$$

The periods of the form dx/y may be given in terms of the hypergeometric function ${}_2F_1$ (see [39, pp. 232–233] for explicit expressions). The Picard–Fuchs equation is

$$\frac{d^2f}{ds^2} + \frac{1}{s}\frac{df}{ds} + \frac{(31/144)s - 1/36}{s^2(s-1)^2}f = 0.$$

There is a basis of solutions with local monodromies G_0, G_1, G_∞ about the regular singular points $\{0, 1, \infty\}$ respectively, where

$$G_0 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \ G_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ G_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The unique point of maximal unipotent monodromy lies at $s = \infty \in \mathbb{P}^1$. The mirror map about this point is quite familiar. Since the maximal unipotent monodromy point is at ∞, we change variables first to $z = 1/s$ so the mirror map q-series will be locally holomorphic. The single-valued local inverse to the period mapping is then the reciprocal of the q-series for the elliptic modular function $J(q)$,

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O(q^3),$$

$$z(q) = \frac{1}{J(q)} = q - 744q^2 + 356652q^3 - 140361152q^4 + O(q^5).$$

The period mapping is defined as a map to projective space. If one is interested in the mirror map it is often preferable to consider the Picard–Fuchs differential equation only up to "projective equivalence". The *projective normal form* of a Fuchsian ordinary differential equation (e.g., that in Eq. (2) above) is the unique Fuchsian ordinary differential equation without a $(k-1)^{\text{st}}$ order derivative

$$\frac{d^kg}{ds^k} + R_2(s)\frac{d^{k-2}g}{ds^{k-2}} + \ldots + R_k(s)g = 0, \ R_i(s) \in \mathbb{C}(s)$$

whose fundamental solutions define the same projective period map as that of Eq. (2). It is always possible to pass to the projective normal form differential equation by rescaling each fundamental solution of the original equation by the k^{th} root of the Wronskian.

Example 2. Suppose now that $k = 2$, i.e., the initial differential equation is

$$\frac{d^2f}{ds^2} + P_1(s)\frac{df}{ds} + P_2(s)f = 0,$$

then the projective normal form of this differential equation takes the particularly simple form

$$\frac{d^2g}{ds^2} + \left(P_2(s) - \frac{1}{2}P_1{}'(s) - \frac{1}{4}P_1(s)^2\right)g = 0.$$

Let Λ_J denote the projective normal form of the Picard–Fuchs equation of the family $\mathcal{E}_J$ from Example 1,

$$\Lambda_J : \ \frac{d^2}{ds^2} + \frac{36s^2 - 41s + 32}{144s^2(s-1)^2}.$$

As the process of taking the projective normal form does not alter the position or type of a maximal unipotent monodromy point, and as the projective solution determines the mirror map there, we see that

the mirror map $z(t)$ of a family of Calabi–Yau manifolds about a point of maximal unipotent monodromy of the Picard–Fuchs equation is determined by the projective normal form of this differential equation.

Since the projective normalized Picard–Fuchs equation determines the mirror map, it is natural to ask if there is a simpler expression for this differential equation. In fact, by direct computation one can check that

Proposition 1. *The projective normalized Picard–Fuchs equation of a one parameter family of elliptic curves with section equals the projective normal form of the pullback $\mathcal{J}^*(\Lambda_J)$ of Λ_J from $\mathbb{P}^1_J$ to C by the functional invariant.* □

Thus the mirror map of a one parameter family of elliptic curves is determined by the functional invariant $\mathcal{J}$. This suggests that the answer to our modularity question should be expressed purely in terms of properties of the functional invariant itself.

We now discuss three approaches to characterize modular mirror maps, each yielding the same criterion stated in terms of properties of the functional invariant. The three methods amount to the characterization of modular functions on the upper half plane $\mathbb{H}$ in terms of

1. modular relations between modular hauptmoduls and the elliptic modular function J,
2. uniformizing differential equations (genus $g = 0$) and holomorphic projective connections ($g \geq 1$) on modular curves, and
3. branched covers of the J-line elliptic modular orbifold, respectively.

The first of these is the most classical, implicit in fact in the early works of Fricke and Klein [11]. They introduce the notion of a single valued local uniformizer, or *hauptmodul*, $H(\tau)$ for a genus zero modular curve. They compute several classical examples of *modular relations* between hauptmoduls $H(\tau)$ and the elliptic modular function $J(\tau)$, i.e., rational functions $\mathcal{R}(z) \in \mathbb{C}(z)$ with the property that $\mathcal{R}(H(\tau)) = J(\tau)$. In [1] Atkin and Swinnerton-Dyer state the following characterization of modular relations:

Proposition 2. *A function $f(\tau)$ is a hauptmodul for a finite index subgroup of the classical elliptic modular group* $\mathrm{PSL}(2, \mathbb{Z})$ *if and only if there is a rational function $\mathcal{R}(z) \in \mathbb{C}(z)$ such that*

1. $\mathcal{R}(f(\tau)) = J(\tau)$,
2. $\mathcal{R}(z)$ *ramifies only over* $\{0, 1, \infty\} \subset \mathbb{P}^1_J$, *and*
3. *the orders of ramification are* $= 1$ *or* 3 *over* 0, *and* $= 1$ *or* 2 *over* 1. □

They comment further that this divisibility criterion extends to automorphic functions for subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ of arbitrary genus. Their proof was extended by Venkov [41] to genus zero Fuchsian groups of the first kind more general than the classical elliptic modular group. The mirror map of a one parameter family of elliptic curves is modular when the functional invariant satisfies the three conditions of the proposition.

The second approach, the one used to characterize modular mirror maps for families of elliptic curves over $\mathbb{P}^1$ in [8], focuses on the local properties of Fuchsian second order ordinary differential equations in projective normal form which characterize *uniformizing differential equations*. The uniformization theory for Riemann surfaces can be reformulated after Gunning [13] in terms of *holomorphic projective connections* on the

Riemann surface. On a local chart, or over a genus zero Riemann surface, this projective connection takes the form of a second order Fuchsian ordinary differential equation in projective normal form, i.e.,

$$\frac{d^2 f}{dz^2} + Q(z) f = 0. \tag{3}$$

A fundamental set of solutions $\{f_1, f_2\}$ to a uniformizing differential equation (3) has the property that $Q(z)$ is the Schwarzian derivative of the projective solution $\tau(z) = f_1(z)/f_2(z)$ with respect to z, i.e., $Q(z) = \{\tau(z); z\}$. The local behavior of the Schwarzian at poles then characterizes the class of $Q(z)$ corresponding to uniformizing differential equations. Our criterion for modularity of the mirror map becomes a constraint on the functional invariant $\mathcal{J}$ so that the projective normalization of the pullback of the projective normalized Λ_J (uniformizing differential equation for the J-line) is again a uniformizing differential equation. The "no excess ramification" condition (i.e., no ramification except over $\{0, 1, \infty\} \subset \mathbb{P}^1_J$) means that the projective normal form of the Picard–Fuchs equation must be free of apparent singularities. For a detailed discussion see [8, §3,4].

The third method, characterizing branched covers of orbifolds, is the most easily generalized of these three, and hence is our method of choice. We sketch here the theory of branched covers of orbifolds due to Kato, following Yoshida [44, §5.1]. Let X be a compact Riemann surface of genus g, equipped with $m \geq 1$ marked "orbifold points" $a_j \in X$ and associated "orbifold weights" $b_j \in \mathbb{Z}$ $(2 \leq b_j \leq \infty)$. Suppose that $g = 0$ and $m \geq 3$. Fix the following data: $X_0 := X \setminus \{a_1, \ldots, a_m\}$; $\tilde{X}_0$ the universal covering of X_0; H the fundamental group of X_0, which we also view as the transformation group of $\tilde{X}_0$; μ_j the element of H represented by a simple loop about a_j; $H[\mu^b]$ the smallest normal subgroup of H containing $\mu_j^{b_j}$ $(j = 1, \ldots, m)$ (determined uniquely independent of choice of μ_j or basepoint for H). Let K be an arbitrary subgroup of H containing $H[\mu^b]$, X'_0 the covering of X_0 corresponding to K, and X' the completion of X'_0, i.e., the space obtained by adding to X'_0 all points over the a_j with finite b_j. Then we have a sort of "galois correspondence" of branched covers: The space X' is a branched cover of X branching at a_j with a ramification index dividing b_j; we say that X' is *branched at most over the divisor* $D = \sum_{j=1}^m b_j \cdot (a_j) \in \mathrm{Pic}(X)$. Conversely, to such a branched covering of X there corresponds a subgroup K, $H[\mu^b] \subset K \subset H$. The covering M corresponding to $K = H[\mu^b]$ is called the *universal branched covering* of X. In other words we have the following diagram of correspondences:

$$\begin{array}{ccccccc}
 & & & & \tilde{X}_0 & \leftrightarrow & 1 \\
 & & & & \downarrow & & | \\
1 & \leftrightarrow & M & \supset & M_0 & \leftrightarrow & H[\mu^b] \\
| & & \downarrow & & \downarrow & & | \\
K/H[\mu^b] & \leftrightarrow & X' & \supset & X'_0 & \leftrightarrow & K \\
| & & \downarrow & & \downarrow & & | \\
H/H[\mu^b] & \leftrightarrow & X & \supset & X_0 & \leftrightarrow & H
\end{array}$$

In this language we can most cleanly state our modularity criterion for the mirror map:

Theorem 2. *The mirror map of a one parameter family of elliptic curves with section* $\pi : \mathcal{E} \to C$ *is an automorphic function for a finite index subgroup of* $\mathrm{PSL}(2, \mathbb{Z})$ *if and only if the functional invariant* $\mathcal{J}(z)$ *is branched at most over* $3 \cdot (0) + 2 \cdot (1) \in Pic(\mathbb{P}^1_J)$.

Proof. Apply the galois correspondence above to the J-line orbifold. The Riemann surface $X \cong \mathbb{P}^1_J$ (genus $g = 0$), $m = 3$, $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, $b_1 = 3$, $b_2 = 2$, $b_3 = \infty$, $D = 3 \cdot (0) + 2 \cdot (1) \in \mathrm{Pic}(\mathbb{P}^1_J)$. There is a correspondence between Riemann surfaces uniformized by subgroups of $H/H[\mu^b] \cong \mathrm{PSL}(2, \mathbb{Z})$ and covers of the J-line branched at most over $D = 3 \cdot (0) + 2 \cdot (1)$. The mirror map is an automorphic function for a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ if and only if the base C of the family is so uniformized. But the branched covering $C \to \mathbb{P}^1_J$ is given by $\mathcal{J}$. Hence the modularity criterion is just that the natural cover of the J-line defined by the functional invariant branch at most over D. □

This is not the end of the story in the elliptic curve case. By Lemma 1 we know the correspondence between local ramification behavior of the functional invariant and the type of Kodaira singular fiber to appear in the elliptic surface. In particular, if the mirror map of a basic elliptic surface is modular, then there are no singular fibers of types IV or II*. Moreover, if one restricts to the case of rational elliptic surfaces where all combinations of singular fiber types are known, one can list all rational elliptic modular surfaces with section. See [8, Theorem 4.11].

2.2. Multi-parameter families of elliptic curves. The definition of Weierstrass fibrations in the one parameter case extends naturally to multiparameter families of elliptic curves with section. It is natural to ask if the modularity characterization extends in any way to families $\pi : \mathcal{E} \to S$ of elliptic curves with section where $\dim(S) \geq 2$. This isn't possible, but the obstruction is of interest in itself, and suggests an important hypothesis to make in the case of multiparameter families of K3 surfaces (Sect. 3.2).

To begin with, the Gauss-Manin system for a multiparameter family of elliptic curves consists of a rank two system of linear partial differential equations. With a slight modification, we can construct a family of varieties for which the Gauss–Manin system takes a recognizable projective normal form. Replace an n parameter family of elliptic curves fiberwise with their nth power. The resulting Gauss-Manin system (essentially the nth symmetric power of the original) is a rank $n+1$ system of linear partial differential equations in n independent variables. A (projective) normal form exists for such differential equations [28]:

$$\frac{\partial^2 w}{\partial z^i \partial z^j} = \sum_{k=1}^{n} P^k_{ij} \frac{\partial w}{\partial z^k} + P^0_{ij} w \quad (i, j = 1, \ldots, n).$$

In the one parameter setting ($n = 1$) these equations reduce to projective normalized second order Fuchsian ordinary differential equations. Local conditions coming from the Schwarzian derivative define a natural subclass consisting of uniformizing differential equations for Riemann surfaces with respect to subgroups of $\mathrm{PSL}(2, \mathbb{R})$ (projective connections if $g \geq 1$). In the multivariable case, the analogous subclass consists of the multiparameter *holomorphic projective connections* (connections modelled after projective space) much studied by Kobayashi [20] in a program established by Cartan. Holomorphic projective connections generalize the Schwarzian derivative, and uniformize quotients of the n-ball

$$\mathbb{B}_n := \{[z_0 : \ldots : z_n] \in \mathbb{P}^n \;\mid\; |z_0|^2 - |z_1|^2 - \ldots - |z_n|^2 > 0\}$$

by a discrete subgroup of the analytic automorphisms.

The difficulty we encounter in generalizing our modularity criterion for the mirror map to multiparameter families of elliptic curves is fundamental. The image of the projective period morphism, even considering the symmetric power family, only lies on a one dimensional submanifold of the period domain $\mathbb{B}_n$! A necessary condition for the Picard–Fuchs equation to uniformize the base S of our family $\mathcal{E}$ is for the period mapping $S \Rrightarrow \mathbb{B}_n$ to be surjective. In fact, as the local inverse to the period mapping, the mirror map itself cannot be defined unless the period mapping is surjective and the dimension of S equals that of the period domain.

This suggests two ingredients which will be needed for the multiparameter K3 surface generalization in Sect. 3.2:

1. a notion of uniformizing differential equation well adapted for Picard–Fuchs equations of K3 surface families, and
2. consideration only of families with surjective period mappings.

3. K3 Surface Families

The results of Sect.2 are extended here to families of lattice polarized K3 surfaces with surjective period mappings, first in the one parameter case (Sect. 3.1) and then for multiparameter families (Sect. 3.2). By applying the resulting criterion for automorphic mirror maps to one parameter families of rank 19 lattice polarized K3 surfaces, we explain the Mirror-Moonshine phenomenon of Lian and Yau.

3.1. One parameter families of K3 surfaces. In their first systematic investigations of mirror symmetry for one parameter families of Calabi–Yau manifolds constructed via the "orbifold construction" [24], Lian and Yau discovered that the reciprocal of the mirror maps for the K3 surfaces they were studying agreed, up to an additive constant, with some of the *McKay–Thompson normalized q-series* in the lists of Conway–Norton [4]. The evidence was sufficiently strong that they formulated

Conjecture 1 (Mirror-Moonshine, [24,23]). If $z(q)$ is the mirror map for a one parameter family of algebraic K3 surfaces from an orbifold construction which has a third order Picard–Fuchs equation, then, for some $c \in \mathbb{Z}$, the q-series

$$\frac{1}{z(q)} + c$$

is a McKay–Thompson series $T_g(q)$ for some element g in the Monster.

In [25,26], Lian and Yau compute many more toric examples (including over a dozen complete intersection examples), and note that the correspondence to monstrous groups persists. This suggested that the hypothesis regarding the "orbifold construction" should perhaps be weakened to the hypothesis "torically constructed".

As noted in the proof of Theorem 5, for a family of lattice polarized K3 surfaces the condition of having a third order Picard–Fuchs equation is equivalent to the generic member possessing a polarization by a lattice of rank 19.

Furthermore, a McKay–Thompson series is in particular a hauptmodul for some "monstrous" genus zero arithmetic group Γ, and the various equivalent hauptmoduls are well-defined as generators of the function field of the rational curve $\Gamma \setminus \mathbb{H}^*$ only up to action of Γ. We see that in Conjecture 1 an equivalent conclusion is that the mirror map is itself a hauptmodul (unnormalized!) for some monstrous Γ.

Before Conjecture 1 was even formulated, Beukers, Peters, and Stienstra had computed the Picard–Fuchs equation of a particular family of rank 19 lattice polarized K3 surfaces [33]. The mirror map was determined by Verrill and Yui [42]. Although it is a hauptmodul, this q-series does not satisfy the conclusion of the Mirror-Moonshine Conjecture. Thus it provides a counterexample to a "monstrous" generalization of the Mirror-Moonshine Conjecture for torically constructed families.

This suggests that we characterize the families of rank 19 lattice polarized K3 surfaces whose mirror maps are hauptmoduls for genus zero groups – a special case of our question from the introduction.

The condition that a one parameter family of K3 surfaces have a third order Picard–Fuchs equation is actually quite natural. The periods obtained by integration of the holomorphic two form $\omega = \omega_{(2,0)}$ over algebraic two cycles all vanish. For a K3 surface X, the intersection form defines on $H_2(X, \mathbb{Z})$ the structure of a lattice, isomorphic to the even unimodular lattice

$$L = U \perp U \perp U \perp -E_8 \perp -E_8 \ ,$$

where U is the standard hyperbolic plane. The sublattice of algebraic cycles in $H_2(X, \mathbb{Z})$ is naturally identified with the Picard group $\mathrm{Pic}(X)$ of divisor classes of X. Thus the rank ρ of the Picard group determines the order of the Picard–Fuchs equation: order of Picard–Fuchs $= 22 - \rho$. In particular, the families considered by Lian and Yau all have Picard rank 19.

Let M be a lattice. An *M-polarized K3 surface* is a pair (X, j) of a K3 surface X and a primitive lattice embedding $j : M \hookrightarrow \mathrm{Pic}(X)$. The examples studied by Lian and Yau relating to Mirror-Moonshine are families of rank 19 lattice polarized K3 surfaces.

A moduli space for lattice polarized K3 surfaces is constructed in [6, §3]. Each isomorphism class of (X, j) is represented by a point of this coarse moduli space $\mathbb{K}_M$. Moreover the global Torelli theorem for lattice polarized K3 surfaces implies, as with the J-line in the case of elliptic curve moduli, that $\mathbb{K}_M$ has the structure of an arithmetic quotient of a symmetric homogeneous space D_M (a bounded symmetric domain of type IV) by an arithmetic group Γ_M. Here

$$D_M \cong O(2, 20 - \rho)/(SO(2) \times O(20 - \rho))$$

and

$$\Gamma_M = \ker\left(O(N) \to \mathrm{Aut}(N^*/N)\right) \ ,$$

where $N := M_L^{\perp}$. In particular, if the rank of M is 19 then $D_M \cong \mathbb{H}$.

The *generalized functional invariant* $\mathcal{H}_M : S \to \mathbb{K}_M$ of a family $\pi : \mathcal{X} \to S$ of M-polarized K3 surfaces may now be defined, by analogy with the elliptic curve case, as the composition of the multivalued period morphism $S \Rrightarrow D_M$ and the arithmetic quotient $D_M \to \mathbb{K}_M$.

Since we are particularly interested in the case $\rho = 19$, the Picard–Fuchs equations of such one parameter families must be studied. We begin by examining some preliminary generalities on symmetric powers of second order Fuchsian ordinary differential equations.

Assume that we have a second order Fuchsian ordinary differential equation $L_2 f = 0$, where

$$L_2 = \frac{d^2}{ds^2} + P_1(s)\frac{d}{ds} + P_2(s).$$

The second order equation $L_2 f = 0$ is equivalent to the system of first order differential equations

$$\begin{cases} f' = g \\ g' = -P_2 f - P_1 g \end{cases}$$

with $\{f, g\}$ as fundamental solutions. Observe that

$$\{f^n, f^{n-1}g, \dots, fg^{n-1}, g^n\}$$

forms a set of fundamental solutions for the n^{th} symmetric power $L = L_2^{\circledS n}$. The following result describes a system of first order differential equations for L with these fundamental solutions.

Theorem 3 ([21], Theorem 2). *If* $\{f, g\}$ *satisfy a first order* 2×2 *differential system*

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -P_2 & -P_1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix},$$

then $\{f^n, f^{n-1}g, \dots, fg^{n-1}, g^n\}$ *satisfy the* $(n+1) \times (n+1)$ *system*

$$\frac{d}{dt}\begin{pmatrix} f^n \\ f^{n-1}g \\ \vdots \\ fg^{n-1} \\ g^n \end{pmatrix} = A \begin{pmatrix} f^n \\ f^{n-1}g \\ \vdots \\ fg^{n-1} \\ g^n \end{pmatrix},$$

where $A = (a_{ij})$ *is an* $(n+1) \times (n+1)$ *matrix such that*

$$\begin{aligned} a_{k,k} &= (1-k)P_1, && 1 \le k \le n+1, \\ a_{k,k+1} &= n+1-k, && 1 \le k \le n, \\ a_{k+1,k} &= -kP_2, && 1 \le k \le n, \\ a_{i,j} &= 0, && i > j+1 \text{ or } j > i+1. \qquad \square \end{aligned}$$

Example 3. In particular, when $n = 2$, the case for a symmetric square, one may rewrite the system in terms of a single third order operator

$$\mathrm{Sym}^2(L_2) = \frac{d^3}{ds^3} + 3P_1 \frac{d^2}{ds^2} + (2{P_1}^2 + 4P_2 + {P_1}')\frac{d}{ds} + (4P_1 P_2 + 2{P_2}').$$

Our next task is to show that the Picard–Fuchs equation of a one parameter family of rank 19 lattice polarized K3 surfaces is a symmetric square of a second order equation, and to reduce the modularity question for the mirror map to the second order setting.

Theorem 4 ([38], Lemma 3.1.(b)). *Let* $L_1(y)$ *and* $L_2(y)$ *be homogeneous linear differential polynomials with coefficients in* $\mathbb{C}(t)$. *Then there exists a homogeneous linear differential equation* $L_3(y) = 0$ *with coefficients in* $\mathbb{C}(t)$ *and solution space the* $\mathbb{C}$*-span of*

$$\{\nu_1\nu_2 \mid L_1(\nu_1) = 0 \quad \text{and} \quad L_2(\nu_2) = 0\}. \qquad \square$$

We call the operator $L_3(y)$ constructed above the *symmetric product* of L_1 and L_2, and denote it by $L_1 \circledS L_2$. In fact, the operation is associative, and we may further define $L^{\circledS n}$ for $n \ge 1$ by $L^{\circledS 1} = L$ and $L^{\circledS n} = L^{\circledS n-1} \circledS L$. We call $\mathrm{Sym}^n(L) = L^{\circledS n}$ the n^{th} *symmetric power* of L; conversely, L is the n^{th} *root* of $L^{\circledS n}$.

Lemma 2 ([38], Lemma 4, p. 129). *Let $L(y)$ be a homogeneous linear differential polynomial with coefficients in $\mathbb{C}(t)$. Then $L(y) = L_2^{\circledS n}(y)$ for some second order homogeneous linear differential polynomial $L_2(y)$ with coefficients in $\mathbb{C}(t)$ if and only if there exists a fundamental set of solutions $\{y_1, \dots, y_{n+1}\}$ of $L(y) = 0$ such that*

$$y_i y_{i+2} - y_{i+1}^2 = 0\ ,\ i = 1, \dots, n-1. \qquad \square$$

Corollary 3. *Let $L(y) = 0$ be a third order homogeneous linear equation with coefficients in $\mathbb{C}(t)$. If there exists a nondegenerate homogeneous polynomial P of degree 2 with constant coefficients and a fundamental set of solutions $\{y_1, y_2, y_3\}$ of $L(y) = 0$ such that $P(y_1, y_2, y_3) = 0$, then $L(y)$ is the second symmetric power of a second order homogeneous linear differential equation with coefficients in $\mathbb{C}(t)$.*

Proof. This follows easily from Lemma 2. By assumption, the fundamental set of solutions satisfies a nondegenerate quadratic relation. Since all such quadrics in $\mathbb{P}^2(\mathbb{C})$ are projectively equivalent to

$$y_1 y_3 - y_2^2 = 0$$

the criterion of the lemma applies and $L(y)$ is a symmetric square. $\square$

In this form, using the expression for the projective normal form of a second order Fuchsian differential equation given in Example 2, it is easy to check that:

Proposition 3. *Let L_2 be as above a second order Fuchsian ordinary differential operator, and let $L = \mathrm{Sym}^2(L_2)$ be its symmetric square. Then the projective normal form of L is the symmetric square of the projective normal form of L_2.* $\square$

In fact, it is possible to provide an explicit description of the relationship between the monodromy matrices of the second order "square root" equation and those of the third order symmetric square equation. This is provided by the faithful representation of $\mathrm{SL}(2, \mathbb{C})$ in $\mathrm{SL}(3, \mathbb{C})$ via the symmetric square representation [38].

Finally, we see the relevance of all of this for Picard–Fuchs equations of our rank 19 lattice polarized K3 surface families:

Theorem 5. *The Picard–Fuchs equation of a family of rank 19 lattice polarized K3 surfaces is the symmetric square of a second order homogeneous linear Fuchsian ordinary differential equation.*

Proof. To begin with, the order of the Picard–Fuchs equation is equal to the rank of the transcendental lattice, i.e., $22 - 19 = 3$. By Nikulin's Torelli theorem for lattice polarized K3 surfaces the period domain lies on a nondegenerate quadric in $\mathbb{P}^2$ [6]. Thus, Corollary 3 implies that the third order Picard–Fuchs differential equation is in fact a symmetric square. $\square$

There is another approach to proving this result in the special case of K3 surfaces polarized by a lattice of the form

$$M_n := U \perp U \perp -E_8 \perp -E_8 \perp \langle -2n \rangle\ ,$$

which takes advantage of their presentation as *Shioda-Inose surfaces* coming from a product of two elliptic curves linked by an n-isogeny. See [32] for more details. Such a simple geometric description is lacking in case of a general rank 19 lattice polarization. Nevertheless, our approach via symmetric square(root) Picard–Fuchs equations still applies! This is what allows our transcendental methods to extend beyond the M_n-polarized case to general rank 19 lattice polarized K3 surface families.

We have effectively reduced the question of automorphicity of the mirror map to the case of uniformization of orbifold Riemann surfaces by second order Fuchsian equations already addressed in Sect. 2.1. Our result is

Theorem 6. *The mirror map of a one parameter family of rank 19 lattice polarized K3 surfaces $\pi : \mathcal{X} \to C$ is an automorphic function for a finite index subgroup of Γ_M if and only if the generalized functional invariant $\mathcal{H}_M(z)$ is branched at most over the orbifold divisor $D \in Pic(\mathbb{K}_M)$.*

Proof. By Theorem 5 the Picard–Fuchs equation of such a family of K3 surfaces is a symmetric square. The mirror map of a one parameter family of rank 19 lattice polarized K3 surfaces about a point of maximal unipotent monodromy is identical to that of the projective normalized square root of its Picard–Fuchs equation about the corresponding point: If $\{f, g\}$ is a fundamental set of solutions to the square root equation, say f the locally holomorphic solution, then $\{f^2, fg, g^2\}$ is a fundamental set of solutions to the symmetric square, with f^2 locally holomorphic. The (truncated) projective period mapping for the K3 surface family, is given by $fg/f^2 = g/f$, which is exactly the projective period ratio of the square root equation. Thus the mirror map for the K3 surface family is modular if and only if the projective normalized square root of Picard–Fuchs is a uniformizing differential equation for C. We can now apply the same galois correspondence for branched covers of orbifolds we used in Theorem 2. Now $X = \mathbb{K}_M$, the a_j and b_j are determined by the positions and orders of the fixed points of the action of Γ_M on $D_M \cong \mathbb{H}$, and the total orbifold divisor of $\mathbb{K}_M$ is again $D = \sum_{j=1}^{m} b_j \cdot (a_j) \in \mathrm{Pic}(\mathbb{K}_M)$. □

Using the theorem of Fano reproduced in Sect. 4.3, we can even characterize near modularity properties of one parameter families of rank 18 lattice polarized K3 surfaces. By the nondegenerate quadric structure of the period domain and case 3 of Theorem 9 we know that the fourth order projective normalized Picard–Fuchs equation is a tensor product of two second order Fuchsian equations in projective normal form. If the fundamental solutions, in {hol., log.} pairs, for these factor equations are

$$\{a, b\} \text{ and } \{c, d\}\ ,$$

then the fundamental solutions to the product equation take the form

$$\{ac, bc, ad, bd\}$$

so the truncated projective period mapping consists of the pair $\{b/a, d/c\}$, i.e., the pair of projective solutions to the factor equations. Although it is not natural to describe the mirror map when the dimension of the family is unequal to that of the associated period domain, there is a good notion of “bimodularity”, i.e., when each factor equation is a uniformizing differential equation (necessarily distinct else the lattice polarization rank jumps to 19 and the equation is a symmetric square).

3.2. Multi-parameter families of K3 surfaces. For the multiparameter definition of points of maximal unipotent monodromy and the mirror map we refer the reader to the unified presentation in [5] (§5.2.2 and §6.3.1 respectively). The details of the local description of the mirror map are in fact irrelevant for what follows as we address the *global* question of modularity. In any case, the existence of a point of maximal unipotent monodromy is again guaranteed by the (related) hypotheses:

1. the family is not isotrivial, and
2. the period mapping is surjective.

We define the generalized functional invariant $\mathcal{H}_M : S \to \mathbb{K}_M$ for a family $\pi : \mathcal{X} \to S$ of M-polarized K3 surfaces as in Sect. 3.1 as the composition of the multivalued period morphism $S \Rrightarrow D_M$ and the quotient map to the coarse moduli space $D_M \to \mathbb{K}_M$ coming from the global Torelli theorem.

Under our hypotheses, the Gauss–Manin system for an n parameter family of rank $20-n$ lattice polarized K3 surfaces is a system of linear partial differential equations of rank $n+2$ in n independent variables. Any such system has a (projective) normal form [37]

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = g_{ij} \frac{\partial^2 u}{\partial z_1 \partial z_n} + \sum_{k=1}^{n} A_{ij}^k \frac{\partial u}{\partial z_k} + A_{ij}^0 u \; (1 \le i, j \le n) \,,$$

where

$$g_{ij} = g_{ji} \,, \; A_{ij}^k = A_{ji}^k \,, \; A_{ij}^0 = A_{ji}^0 \,, \; g_{1n} = 1 \,, \; A_{1n}^k = A_{1n}^0 = 0$$

(for $n \ge 3$), or [36]

$$\frac{\partial^2 u}{\partial x^2} = l \frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + pu \,,$$

$$\frac{\partial^2 u}{\partial y^2} = m \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial u}{\partial x} + d \frac{\partial u}{\partial y} + qu$$

(for $n = 2$).

The global Torelli theorem of Nikulin again implies that the periods map to a quadric projective hypersurface. The natural subclass of uniformizing differential equations adapted to the Picard–Fuchs equations of lattice polarized K3 surfaces with surjective period mappings are the *holomorphic conformal connections* (connections modelled after hyperquadrics) introduced by Kobayashi [20]. Once again the question of automorphicity of the inverse to the projective period mapping reduces to the uniformizability of the base S of our family as a branched cover of the modular orbifold $\mathbb{K}_M$.

Fortunately, the galois correspondence for branched covers of orbifold Riemann surfaces has been generalized by Namba [31] to the case of orbifold complex manifolds of higher dimension. We refer to [31, Theorem 1.2.7] for the details, but the only essential difference is that we must add a higher dimensional analogue of the topological condition excluding "$g = 0, m = 1$ or 2" in the Riemann surface case. This topological condition, [31, Condition 1.2.4], says: if $\mu_j^d \in H[\mu^b]$, then $b_j \mid d$ (for all j, $1 \le j \le m$). By applying the galois correspondence as before to our families we find

Theorem 7. *The mirror map of an n parameter family of rank* $20 - n$ *lattice polarized K3 surfaces* $\pi : \mathcal{X} \to S$ *is an automorphic function for a finite index subgroup of* Γ_M *(*$M :=$ *the polarizing lattice) if and only if the generalized functional invariant* $\mathcal{H}_M$ *is branched at most over the orbifold divisor of* $\mathbb{K}_M$. □

4. Calabi–Yau Threefold Families

We have seen that the presence of a global Torelli theorem is a great help in establishing modularity criteria for the mirror map, expressed in terms of natural algebraic invariants of our families of Calabi–Yau manifolds. It is known that in general moduli spaces of polarized Calabi–Yau threefolds lack the structure of a locally symmetric space. Nevertheless, it is possible that a differential algebraic criterion for automorphicity of the mirror map may be obtainable for Calabi–Yau threefold families by making use of the "special geometry" of Calabi–Yau threefold moduli. In fact, one can use the constraints imposed by special geometry on the Picard–Fuchs equation of a one parameter family of Calabi- Yau threefolds with $h^{2,1} = 1$ to derive an auxiliary differential equation (involving the Yukawa couplings, the coefficients of Picard–Fuchs, and the rational function defining the putative uniformizing differential equation) which holds if and only if the mirror map is an automorphic function (Theorem 9 in Sect. 4.2).

4.1. Picard–Fuchs equations of Calabi–Yau threefolds and special geometry. Special geometry arises in global $N = 2$ supersymmetry in four dimensions as a structure on the manifold spanned by the scalars in the vectormultiplets. The moduli space of (2,2) superconformal field theories, and thus the moduli space of Calabi–Yau threefolds, satisfies the same constraint equation for the natural Kähler metric on moduli space.

In the case of one parameter families of Calabi–Yau threefolds with $h^{2,1} = 1$ much is known about the implications of special geometry. In particular, the effect of special geometry on the fourth order Picard–Fuchs ordinary differential equations is well known [19, 3]. We review these results in this section, using notation largely compatible with that in [19, 27]. We will always use primes (e.g., $f'(z)$) to denote derivatives with respect to the base parameter z, and dots (e.g., $\dot{F}(t)$) to denote derivatives with respect to the truncated period mapping parameter t.

Suppose given the Picard–Fuchs equation for a family of $h^{2,1} = 1$ Calabi–Yau threefolds

$$Lf(z) = 0 : \quad f''''(z) + b_3(z)f'''(z) + b_2(z)f''(z) + b_1(z)f'(z) + b_0(z)f(z) = 0$$

with fundamental solutions

$$(\xi_0, \xi_1, \xi_0\dot{F}(\xi_1/\xi_0), \xi_0((\xi_1/\xi_0)\dot{F}(\xi_1/\xi_0) - 2F(\xi_1/\xi_0))).$$

Then $t(z) := \xi_1/\xi_0$ is the truncated period mapping.

By rescaling the solutions $g(z) := f(z)/A(z)$, where

$$A(z) = \exp\left(-\frac{1}{4}\int b_3(z)dz\right),$$

we obtain the projective normalized Picard–Fuchs equation

$$\overline{L}g(z) = 0 : \quad g''''(z) + a_2(z)g''(z) + a_1(z)g'(z) + a_0(z)g(z) = 0$$

with fundamental solutions $1/A(z)$ times the previous ones. In fact, $a_1(z) = a_2'(z)$ (see [19]). Let $u(z) = \xi_0/A$. The quantum Yukawa coupling is related to the holomorphic solution $u(z)$ about the point of maximal unipotent monodromy:

$$K = \dot{F}^{(3)} = \text{const.}A^2/\xi_0^2 = \text{const.}/u^2.$$

By reduction of order applied to the $z \leftrightarrow t$ variables exchanged equation $\tilde{L}g = 0$, we derive a third order variant of the Picard–Fuchs equation in t, satisfied by $u(t)$,

$$PF_t^{(3)}u(t) = 0 : \quad \dot{u}^{(3)}(t) + \tfrac{1}{2}c_2(t)\dot{u}(t) + \tfrac{1}{4}\dot{c}_2(t)u(t) = 0,$$

i.e.,

$$PF_t^{(3)}u = \frac{1}{4}\left(\tilde{L}(u \cdot t) - t \cdot (\tilde{L}u)\right).$$

We recognize $PF_t^{(3)}$ as the symmetric square of the second order "square root" equation

$$PF_t^{(2)}v(t) = 0 : \quad \ddot{v}(t) + \tfrac{1}{8}c_2(t)v(t) = 0$$

satisfied by $v(t) = \sqrt{u(t)}$.

By plugging in the two remaining fundamental solutions, one finds that the resulting system of equations reads

$$c_2(t) = r_2(t)\ ,\ c_0(t) = r_0(t),$$

where

$$c_2(t) = a_2(z)(\dot{z})^2 - \frac{15}{2}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + 5\frac{\dot{z}^{(3)}}{\dot{z}} = a_2(z)(\dot{z})^2 + 5\{z(t), t\},$$

$$r_2(t) = 2\frac{\ddot{K}}{K} - \frac{5}{2}\left(\frac{\dot{K}}{K}\right)^2,$$

and the lengthy expressions for $c_0(t)$ and $r_0(t)$ are found in [19], where they are used to derive nonlinear ordinary differential equations of high order for the mirror map and Yukawa coupling. The $c_0 = r_0$ equation provides no simplification of our approach to modularity in Sect. 4.2, so c_0 and r_0 may be safely ignored.

By reduction of order applied to $\overline{L}g = 0$, we find a third order Picard–Fuchs type equation in z for $T(z) = t'(z)$, $PF_z^{(3)}T(z) = 0$:

$$T'''(z) + 4\frac{u'(z)}{u(z)}T''(z) + \left(6\frac{u''(z)}{u(z)} + a_2(z)\right)T'(z) + \left(4\frac{u'''(z)}{u(z)} + 2a_2(z)\frac{u'(z)}{u(z)} + a_2'(z)\right)T(z) = 0.$$

It is important that the dependence of the coefficients on u is only through the ratios

$$\frac{u'(z)}{u(z)},\ \frac{u''(z)}{u(z)},\ \text{and}\ \frac{u'''(z)}{u(z)}$$

– this is why the constant relating K and u never enters into the equation even if we rewrite it in terms of K. With this in mind, let $r := d\log u$, and $\overline{L}u(z) = 0$ becomes

$$(r''' + 4rr'' + 3(r')^2 + 6r^2r' + r^4) + a_2(r' + r^2) + a_2'r + a_0 = 0. \tag{4}$$

For Calabi–Yau threefold families (assuming special coordinates) Lian and Yau show that the mirror map satisfies a "quantum corrected" version of Schwarz's equation (the $c_2 = r_2$ equation above):

$$2Q(z)(\dot{z})^2 + \{z, t\} = \tfrac{2}{5}\ddot{y} - \tfrac{1}{10}(\dot{y})^2,$$

where $y(t) = \log(K(t))$, $Q(z) = a_2(z)/10$. For reference note as well that

$$c_2(t) = 2\ddot{y} - \frac{1}{2}(\dot{y})^2.$$

4.2. Characterization of modular mirror maps. We will start with the case with no instanton corrections:

The quantum corrections vanish if and only if $c_2(t) = 0$, i.e., when

$$\ddot{x} = (\dot{x})^2,$$

where $x = y/4$. Letting $X = \dot{x}$, this becomes $\dot{X} = X^2$, with solutions $X(t) = -(c-t)^{-1}$ for constant c. So

$$K(t) = \exp(y(t)) = \exp(4x(t)) = \exp(4(\log(c-t) + d)) = \text{const.}(c-t)^4.$$

Whenever $K(t)$ does not take this particular form, we know that the mirror map $z(t)$ is not an automorphic function for the projective monodromy group of the second order ordinary differential equation in projective normal form with coefficient $Q(z)$. Conversely, if $K(t)$ satisfies Eq. (4.2) and $Q(z)$ satisfies the local conditions coming from the Schwarzian derivative for orbifold uniformization (i.e., characteristic exponent differences are proper unit fractions or zero), then the mirror map $z(t)$ will be an automorphic function. A two parameter family of Calabi–Yau threefolds (a subfamily of the 101 parameter family of Calabi–Yau quintic hypersurfaces in $\mathbb{P}^4$) without instanton corrections is described in [3].

Assume for the remainder of Sect. 4.2 that there are instanton corrections present. Suppose that there is a rational function $R(z)$ (necessarily unequal to $Q(z)$) with respect to which the mirror map $z(t)$ satisfies the classical Schwarz equation

$$2R(z)(\dot{z})^2 + \{z, t\} = 0 \tag{5}$$

i.e., with respect to which the mirror map is an automorphic function. There is only one such candidate rational function $R(z)$. This is the rational function which defines the uniformizing differential equation with regular singular points with compatible characteristic exponent differences exactly at those of the projective normal form Picard–Fuchs equation. The only subtlety that arises is one of computational effectivity: If there are more than three regular singular points, then the coefficients in the numerator of $R(z)$ are difficult to determine in general from the denominator data – this is the famous "accessory parameter problem" in Riemann–Hilbert theory.

By subtracting the two expressions (4.1) and (5) we have the equation

$$2(Q(z) - R(z))(\dot{z})^2 = \frac{2}{5}\ddot{y} - \frac{1}{10}(\dot{y})^2. \tag{6}$$

Set $P(z) = 5(Q(z) - R(z))$ and $S(z) = (1/4)P(z)$. Then Eq. (6) can be rewritten as

$$S(z)(\dot{z})^2 = \dot{X} - X^2\ , \tag{7}$$

where $X = \dot{x}$ and $x = y/4$ as before. Now apply a Ricatti transformation

$$X(t) = \frac{\dot{w}(t)}{w(t)} = \frac{d}{dt}\log w(t)$$

yielding the linear ordinary differential equation in t

$$\ddot{w}(t) + S(z(t))(\dot{z})^2 w(t) = 0.$$

Now change the independent variable from t to z and we get a second order linear equation in z

$$w''(z) - \frac{T'(z)}{T(z)} w'(z) + S(z)w(z) = 0, \tag{8}$$

where $T(z) = t'(z)$.

This implies an equation for $\frac{T'(z)}{T(z)}$ in terms of $\{w, S\}$, or $\{K, P\}$, or $\{u, a_2, P\}$, or $\{r, a_2, P\}$ (recall $r = d \log u$). The equation in terms of r, a_2, P reads

$$\frac{T'}{T} = \frac{r'}{r} - \frac{1}{2}r - \frac{1}{2}\frac{P}{r}.$$

One can of course substitute $((a_2(z)/2) - 5R(z))$ for $P(z)$ and obtain the expresion in terms of $\{r, a_2, R\}$ as well.

Now apply this to reduce $PF_z^{(3)}$ to an expression of the form $T(z)\varphi(z) = 0$. Since $T(z)$ is not identically zero by assumption (the mirror map is locally invertible), $\varphi(z) = 0$. This is the modularity condition.

If we use the expression for $d \log T$ in terms of u, we can arrange to never have more than a u''' appear (use $PF_z^{(4)}u = 0$). Similarly we can arrange to never have more than a w''' or a K''' or a r'' appear. In the r variant in Theorem 9 below we use Eq. (4). Of course this results in an additional term involving a_0 (there was only a_2 dependence in the higher order equation).

Theorem 8. *Here is the equation characterizing modularity of the mirror map in terms of* r, a_0, a_2, R*:*

$$\begin{aligned} 0 = {} & -a_2^3 - 16a_0r^2 - 6a_2^2r^2 - 12a_2r^4 + 8r^6 + 30a_2^2R \\ & + 80a_2r^2R + 200r^4R - 300a_2R^2 - 200r^2R^2 + 1000R^3 \\ & + 5a_2ra_2' - 6r^3a_2' - 50rRa_2' + 7a_2^2r' \\ & + 8a_2r^2r' - 64r^3r' + 116r^4r' - 140a_2Rr' \\ & - 80r^2Rr' + 700R^2r' - 12ra_2'r' - 16a_2(r')^2 \\ & - 48r^2(r')^2 + 160R(r')^2 + 16(r')^3 - 50a_2rR' + 60r^3R' \\ & + 500rRR' + 120rr'R' - 4r^2a_2'' - 16a_2rr'' \\ & + 80r^3r'' + 160rRr'' + 32rr'r'' + 40r^2R''. \end{aligned}$$

In particular, this modularity equation is a second order nonlinear ordinary differential equation with rational function coefficients which the logarithmic derivative of the holomorphic solution to the Picard–Fuchs equation (4.1) satisfies if and only if the mirror map is an automorphic function.

Special geometry is a phenomenon present in multidimensional families of Calabi–Yau manifolds as well [40]. A multiparameter criterion for automorphicity of the mirror map would of course be desirable. Perhaps the recent mathematical reformulation of special geometry by Freed [10] is a natural starting point.

4.3. Algebraic instanton corrections. In case of families of lattice polarized K3 surfaces, by global Torelli there is a homogeneous quadratic relation among the periods, and no instanton corrections. For Calabi–Yau threefold families we can also interpret the absence of instanton corrections as imposing a particular homogeneous algebraic relation among the periods. In Sect. 4.2 we saw a condition for vanishing of instanton corrections was that $c_2(t) = 0$. Equivalently, as described in [3, §2.1] for example, this can be interpreted as the vanishing of the fourth W-algebra generator w_4 in the presence of the vanishing of the third ($w_3 = 0$ being a consequence of special geometry). This implies in particular that the projective normalized Picard–Fuchs equation have a set of fundamental solutions $\{u_1^3, u_1^2 u_2, u_1 u_2^2, u_2^3\}$, where u_1 and u_2 are the fundamental solutions to the cube root equation

$$u''(z) + Q(z)u(z) = 0.$$

In particular the projective periods map to a twisted cubic space curve.

What about other homogeneous algebraic relations among the periods? We call instanton corrections for which the Picard–Fuchs equation still admits homogeneous algebraic relations among the periods *algebraic instanton corrections*. A century ago Fano classified fourth order Fuchsian ordinary differential equations whose fundamental solutions satisfy homogeneous algebraic relations [9, pp. 496–497]. To paraphrase in more modern language

Theorem 9. *The projective solution to a fourth order Fuchsian ordinary diferential equation falls into one of the following classes:*

1. *The projective solution lies on an algebraic (twisted cubic) curve in $\mathbb{P}^3$. These equations are symmetric cubes of second order Fuchsian ordinary differential equations.*
2. *There is a homogeneous quartic relation among the fundamental solutions. Such equations can be transformed by a differential algebraic change of variables $f = \alpha h + \beta h' + \gamma h''$ to a member of the previous class.*
3. *A quadratic relation with nonvanishing discriminant exists among the fundamental solutions. These equations are the tensor product of two distinct second order Fuchsian ordinary differential equations $L_2 \otimes L_2'$.*
4. *A quadratic relation with vanishing discriminant exists. These equations are formed by operator composition of a first order and a third order equation $L_1 \cdot L_3$.*
5. *No homogeneous algebraic relations exist among the fundamental solutions. This is the generic case.*

We can of course reinterpret Fano's result as providing a rough classification of algebraic instanton corrections.

In the first and last cases at least, we know Fano's classification parallels the classification by differential Galois group of the Picard–Fuchs equation. Since the Picard–Fuchs differential equation is a Fuchsian ordinary differential equation, the differential Galois group equals the Zariski closure of the global monodromy group. In the first case this corresponds to the symmetric cube monodromy representation of $\mathrm{SL}(2, \mathbb{C})$ in $\mathrm{Sp}(4)$. In the last case, the monodromy representation is irreducible and the differential Galois group is all of $\mathrm{Sp}(4)$. It should be possible to fill in the other three entries as well.

In fact we can say more about the absence of algebraic relations among the periods in the last case. By special geometry there are no homogeneous algebraic relations among

$$\{u, u \cdot t, u \cdot \dot{F}, u \cdot (t\dot{F} - 2F)\}$$

which implies there are no algebraic relations whatsoever among

$$\{t, \dot{F}, (t\dot{F} - 2F)\}.$$

Hence there are no algebraic relations among $\{t, F, \dot{F}\}$, and thus no algebraic relations between $\{t, F\}$.

Moreover the modularity equation from Theorem 9 takes a particularly simple form in each of the nongeneric cases (e.g., it characterizes "bimodularity" in class 3. above).

References

1. Atkin, A.O.L. and Swinnerton-Dyer, H.P.F.: Modular forms on noncongruence subgroups. In: *Combinatorics, Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968*, Providence: Am. Math. Soc. 1971, pp. 1–25
2. Candelas, P.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. Reprinted in [43, pp. 31–95]
3. Ceresole, A., D'Auria, R., Ferrara, S., Lerche, W., Louis, J., Regge, T.: Picard–Fuchs equations, special geometry, and target space duality. In: [12, pp. 281–354]
4. Conway, J.H. and Norton, S.P.: Monstrous moonshine. Bull. London Math. Soc. **11**, 308–339 (1979)
5. Cox, D.A. and Katz, S.: *Mirror Symmetry and Algebraic Geometry*. Providence: Am. Math. Soc., 1999
6. Dolgachev, I.V.: Mirror symmetry for lattice polarized $K3$ surfaces. Algebraic geometry, 4. J. Math. Sci. **81**, 2599–2630 (1996)
7. Doran, C.F.: *Picard–Fuchs Uniformization and Geometric Isomonodromic Deformations: Modularity and Variation of the Mirror Map*. Ph.D. thesis, Harvard University, April 1999
8. Doran, C.F.: Picard–Fuchs uniformization: Modularity of the mirror map and Mirror-Moonshine. In B. Gordon, et al., Eds., *The Arithmetic and Geometry of Algebraic Cycles*: Proceedings of the CRM Summer School, June 7–19, 1998, Banff, Alberta, Canada. Centre de Recherches Mathématiques, CRM Proceedings and Lecture Notes. **24**, 2000, pp. 257–281
9. Fano, G.: Über lineare homogene Differentialgleichungen mit algebraische Relationen zwischen den Fundamentallösungen. Math. Ann. **53**, 493–590 (1900)
10. Freed, D.: Special Kähler manifolds. Commun. Math. Phys. **203**, 31–52 (1999)
11. Fricke, R. and Klein, F.: *Vorlesungen über die Theorie der elliptischen Modulfunktionen*. I, II. Leipzig: Teubner, 1890, 1892
12. Greene, B. and Yau, S.-T. (eds.): *Mirror Symmetry II*. In: AMS/IP Studies in Advanced Mathematics. **1**. Providence and Cambridge: Amer. Math. Soc. and International Press, 1997
13. Gunning, R.C.: *Lectures on Riemann surfaces*. Princeton Mathematical Notes. Princeton: Princeton University Press, 1966
14. Hosono, S., Klemm, A., Theisen, S., Yau, S.-T.: Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces. Commun. Math. Phys. **167**, 301–350 (1995)
15. Hosono, S., Klemm, A., Theisen, S., Yau, S.-T.: Mirror symmetry, mirror map and applications to complete intersection Calabi–Yau spaces. Nucl. Phys. B. **433**, 501–552 (1995). Reprinted in [12, pp. 545–606]
16. Hosono, S., Lian, B.H., Yau, S.-T.: GKZ-generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces. Commun. Math. Phys. **182**, 535–578 (1996)
17. Hosono, S., Lian, B.H., Yau, S.-T.: Maximal degeneracy points of GKZ systems. J. Am. Math. Soc. **10**, 427–443 (1997)
18. Hosono, S., Lian, B.H., Yau, S.-T.: Calabi–Yau varieties and pencils of K3 surfaces. LANL archive preprint, alg-geom/9603020
19. Klemm, A., Lian, B.H., Roan, S.-S., Yau, S.-T.: A note on ODEs from mirror symmetry. In: Functional Analysis on the Eve of the 21st Century, Vol. II (New Brunswick, NJ, 1993). Progr. Math. **132**, Boston: Birkhäuser, 1996, pp. 301–323.
20. Kobayashi, S. and Ochiai, T.: Holomorphic structures modeled after hyperquadrics. Tôhoku Math. J. **34**, 587–629 (1982)
21. Lee, M.-H.: Picard–Fuchs equations for elliptic modular varieties. Appl. Math. Lett. **4** 91–95 (1991)
22. Lian, B.H., Liu, K., Yau, S.-T.: The Candelas-de la Ossa-Green- Parkes formula. String Theory, Gauge Theory and Quantum Gravity (Trieste, 1997). Nuclear Phys. B Proc. Suppl. **67**, 106–114 (1998)
23. Lian, B.H. and Yau, S.-T.: Mirror symmetry, rational curves on algebraic manifolds and hypergeometric series. In: XIth International Congress of Mathematical Physics (Paris, 1994). Cambridge: International Press, 1995, pp. 163–184
24. Lian, B.H. and Yau, S.-T.: Arithmetic properties of mirror map and quantum coupling. Commun. Math. Phys. **176**, 163–191 (1996)

25. Lian, B.H. and Yau, S.-T.: Mirror maps, modular relations and hypergeometric series. I. LANL archive preprint, hep-th/9507151
26. Lian, B.H. and Yau, S.-T.: Mirror maps, modular relations and hypergeometric series. II. S-duality and Mirror Symmetry (Trieste, 1995). Nuclear Phys. B Proc. Suppl. **46**, 248–262 (1996)
27. Lian, B.H. and Yau, S.-T.: A note on ODEs from mirror symetry II. In preparation.
28. Matsumoto, K., Sasaki, T., Yoshida, M.: Recent progress of Gauss- Schwarz theory and related geometric structures. Memoirs of the Faculty of Science, Kyushu University. Ser. A. **47**, 283–381 (1993)
29. Miranda, R.: The moduli of Weierstrass fibrations over $\mathbb{P}^1$. Math. Ann. **255**, 379–394 (1981)
30. Miranda, R.: *The Basic Theory of Elliptic Surfaces*. Dottorato di Ricerca in Matematica. Pisa: ETS Editrice (1989)
31. Namba, M.: *Branched Coverings and Algebraic Functions*. Pitman Res. Notes Math. Ser. **161**. Harlow: Longman Scientific & Technical, 1987
32. Peters, C.: Monodromy and Picard–Fuchs equations for families of $K3$-surfaces and elliptic curves. Ann. Sci. École Norm. Sup. (4) **19**, 583–607 (1986)
33. Peters, C. and Stienstra, J.: A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and fermi surfaces for potential zero. In: *Arithmetic of Complex Manifolds (Erlangen, 1988)*, Lecture Notes in Math. **1399**. Berlin: Springer, 1989, pp. 110–127
34. Phong, D.H., Vinet, L., Yau, S.-T. (eds.): *Mirror Symmetry III*. AMS/IP Studies in Advanced Mathematics, **10**. Providence, Cambridge, and Montreal: Am. Math. Soc., International Press, and Centre de Recherches Mathématiques, 1999
35. Sasai, T.: Monodromy representations of homology of certain elliptic surfaces. J. Math. Soc. Japan **26**, 296–305 (1974)
36. Sasaki, T. and Yoshida, M.: Linear differential equations in two variables of rank four. I. Math. Ann. **282**, 69–93 (1988)
37. Sasaki, T. and Yoshida, M.: Linear differential equations modeled after hyperquadrics. Tôhoku Math. J. **41**, 321–348 (1989)
38. Singer, M.: Algebraic relations among solutions of linear differential equations: Fano's theorem. Am. J. Math. **110**, 115–143 (1988)
39. Stiller, P.: On the uniformization of certain curves. Pacific J. Math. **107**, 229–244 (1983)
40. Strominger, A.: Special geometry. Commun. Math. Phys. **133**, 163–180 (1990)
41. Venkov, A.B.: Examples of the effective solution of the Riemann–Hilbert problem on the reconstruction of a differential equation from a monodromy group in the framework of the theory of automorphic functions. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **162**, Avtomorfn. Funkts. i Teor. Chisel. III, 5–42, **189** (1987)
42. Verrill, H. and Yui, N.: Thompson series, and the mirror maps of pencils of $K3$ surfaces. In B. Gordon, et al., Eds., *The Arithmetic and Geometry of Algebraic Cycles*: Proceedings of the CRM Summer School, June 7–19, 1998, Banff, Alberta, Canada. Centre de Recherches Mathématiques, CRM Proceedings and Lecture Notes. **24**, 2000, pp. 399–432
43. Yau, S.-T. (ed.): *Mirror Symmetry I*. AMS/IP Studies in Advanced Mathematics, **9**. Cambridge: International Press, 1998
44. Yoshida, M.: *Fuchsian differential equations. With special emphasis on the Gauss-Schwarz theory*. Aspects of Mathematics, E11. Braunschweig: Friedr. Vieweg & Sohn, 1987

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