PICARD-FUCHS UNIFORMIZATION OF MODULAR SUBVARIETIES

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1. Introduction

One highly structured example can inspire diverse mathematical theories when viewed from the right vantage points. Such is undoubtedly the case for the theories of modular curves, uniformization of Riemann surfaces, and the differential equations which govern them.

As a common origin for each of these theories, consider the "Legendre family" of elliptic curves over the thrice punctured sphere, i.e., the family of elliptic curves E_λ presented as double covers of $\mathbb{P}^1(\mathbb{C})$ ramified over four points $x = 0, 1, \infty, \lambda$:

$$y^2 = x(x-1)(x-\lambda),\ \ \lambda \in \mathbb{C} \setminus \{0,1\}\,.$$

For each $\lambda \in \mathbb{C} \setminus \{0,1\}$, integrating the holomorphic 1-form (differential of the first kind)

$$\frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

over an integral 1-cycle of E_λ produces a complex number "period" $\varpi(\lambda)$. To be precise, if we specify the "A"-cycle Σ_1 of the elliptic curve to be the closed cycle in the x-line encircling the branch points at $x = 0$ and $x = \lambda$, then this cycle can be deformed to follow the branch cut from $x = 0$ to $x = \lambda$ on one y-sheet and return on the second. The integral

$$\varpi_1(\lambda) = \oint_{\Sigma_1} \frac{dx}{y} = \int_0^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = (\pi i)\, {}_2F_1\left[\begin{matrix} \frac{1}{2} \; \frac{1}{2} \\ 1 \end{matrix}; \lambda\right]$$

then describes a function that is holomorphic near $\lambda = 0$, with power series expansion for $|\lambda| < 1$ given in terms of the *hypergeometric series*[1]

$${}_2F_1\left[\begin{matrix} \frac{1}{2} \; \frac{1}{2} \\ 1 \end{matrix}; \lambda\right] = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{\lambda^n}{n!}.$$

introduced by Euler (1769) and much studied by Gauss (1812) and Riemann (1857).

This series is annihilated by the second order linear operator

$$L_2 := \theta^2 - \lambda\left(\theta + \frac{1}{2}\right)^2$$

where $\theta := \lambda\, d/d\lambda$, a special case of the hypergeometric differential equation discovered by Gauss in 1800 in his study of elliptic integrals and used by Riemann (1857) to describe for the first time analytic continuation of solutions around singularities in the complex plane and the monodromy group.

The connection to these considerations can be seen by introducing a second differential (differential of the second kind) on our elliptic curves

$$\frac{x(x-1)dx}{2y^3} = \frac{x(x-1)dx}{2\left(x(x-1)(x-\lambda)\right)^{3/2}}$$

and observing that the integral of this over the A-cycle results in another, linearly independent solution to the equation $L_2\varpi(\lambda) = 0$. These two solutions clearly span the solution space in the neighborhood of $\lambda = 0$, but more remarkably they each admit multi-valued analytic continuations everywhere away from the three special points $\lambda = 0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$ (i.e., the complement of $\{0, 1\} \subset \mathbb{C}$).

Elements in the fundamental group of the thrice punctured sphere are represented by paths on the λ-line avoiding the punctures, and along each such path both functions admit analytic continuations; thus when they return to their starting point, as they still represent a basis for the solution space to our second order linear equation, they must be related to the original pair of solutions by an invertible 2×2 matrix. The totality of these matrices as we run over all possible paths forms a group, the *monodromy group*. In this case, the monodromy group can be generated by the matrices corresponding to a loop about $\lambda = 0$ and a loop about $\lambda = 1$, respectively

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

[1]The notation $(q)_n$ here denotes the (rising) Pockhammer symbol which equals 1 for $n = 0$ and the product

$$q(q+1)\cdots(q+n-1)$$

for positive n.

This simple, yet fundamental example may be generalized in three quite distinct ways, depending on which of its features one wishes to amplify.

(I) The class of *hypergeometric series* ${}_2\mathrm{F}_1(a, b; c; \lambda)$ and the *hypergeometric differential operators* which annihilate them admit several natural generalizations, both to higher rank linear ordinary differential operators (single variable, but more than two linearly independent solutions) and also to theories involving differential systems with several independent variables and various ranks.

In this paper, we review in Part 1 one of these theories that provides a striking bridge from the single to multiple valued settings. The ratio of a pair of solutions $\varpi_i(\lambda)$ $(i = 1, 2)$ in the example above — a complex number — may be consistently chosen to lie always in the upper half of the complex plane for any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. This "period mapping" has target space conformally equivalent to the open unit disc in the complex plane. One standard generalization of this is to higher-dimensional complex balls; this is the setting of the story told in Part 1.

(II) The starting point for our example above was a family of algebraic varieties, specifically elliptic curves, to which we attached a differential equation — known more generally as a *Picard-Fuchs differential equation* or *Gauss-Manin connection* — that annihilates the period functions over the base of the family. There are several closely related notions here.

First, one can ask whether the family of varieties is "maximal" in the sense that every isomorphism class of such varieties is represented by at least one member of our family. Such families will admit coverings (possibly ramified) of the *moduli space* for the varieties in question. In our example above, the Legendre family covers the moduli space for complex structures on elliptic curves in a generically $6 : 1$ fashion, corresponding to the fact that the projective monodromy group has index 6 in the modular group $\mathrm{PSL}_2(\mathbb{Z})$. The base of the Legendre family is itself a *modular curve* for elliptic curves equipped with the extra structure of four marked points (i.e., $0, 1, \infty, \lambda$) providing the "level two" structure.

In general it is quite difficult to characterize explicit algebraic "normal forms" for varieties which upon varying parameters form a maximal family. This turns out to be tractable, however, for many classes of K3 surfaces with enhanced algebraic structure (so-called lattice-polarized K3 surfaces). Moreover, given such a normal form realized as, say, a hypersurface in a suitably nice ambient space, then a variant of an algorithm of Dwork, generalized by Griffiths, sometimes permits computation of the Picard-Fuchs operators which annihilate periods.

What if our family is not maximal? The restriction of Picard-Fuchs differential systems for full families to subfamilies which cover proper subvarieties of moduli space yields Picard-Fuchs differential operators which annihilate periods for these subfamilies. One can turn this problem on its head and ask for a differential-algebraic characterization of special subvarieties of moduli space; this is the main theme of Part 2.

(III) Perhaps the most far-ranging and general notion to evolve from the simple example above is due to Henri Poincaré— the notion of a *uniformizing differential equation*. In no small part due to his personal route to uniformization via the work of Fuchs on differential equations, the method and vision of Poincaré, in contrast to his contemporaries, was firmly rooted in the notion of establishing the existence of and characterizing the uniformizing differential equation. A detailed analysis of Poincaré's personal journey to uniformization, with much of the supporting evidence coming from the Mittag-Leffler archives, is presented in a companion paper [Dor17a] and Harvard dissertation [Dor17b] by Connemara Doran. In Part 2, we show in many cases relating to K3 surface moduli how the elusive uniformizing differential equations may be derived from Picard-Fuchs systems.

The two "Parts" of this paper trace a path from a semi-historical introduction to the Deligne-Mostow multi-parameter generalization of hypergeometric functions related to uniformization by complex balls (Part 1) to a wholly new differential-algebraic characterization of Shimura subvarieties in moduli spaces of lattice-polarized K3 surfaces (Part 2). The program for "algebraic uniformization" indicated at the end of Part 1 is in the early stages of development, with lessons from the history of uniformization pointing the way towards an exciting new frontier. Although Part 2 can be read as an independent story involving Picard-Fuchs equations for families of K3 surfaces, with clear roots in the originating example of the Legendre family and a non-trivial overlap with the hypergeometric case, we explicitly link the two Parts in Section 5 with the example of a family of K3-surfaces whose Picard-Fuchs system is a two-parameter (Appell type) hypergeometric system.

Part 1. Hypergeometric Functions and Ball Quotient Uniformization

Classical hypergeometric functions of a single variable provide explicit uniformizations of the thrice-punctured sphere by the upper half plane, equivalently the Poincaré unit disk or "complex 1-ball." Here we discuss somewhat the historical transition from one to multi-variables. We also motivate the question of studying subloci on the quotient variety (moduli space) that are nicely related to subballs on the universal covering locally symmetric spaces. For our purposes, with uniformization in mind, we will focus on the generalization to multidimensional ball quotients in the study of hypergeometric functions.

There are two core ways of generalizing hypergeometric functions to a higher dimensional base: GKZ[2] (toric), and Deligne-Mostow (moduli of points of $\mathbb{P}^1$). The former can be thought of in terms of periods for subvarieties (e.g., the anticanonical Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties much studied in mirror symmetry [CK99]), the latter in terms of periods for curve covers. In each case, intersection cohomology valued in local systems may be seen as central to the theory, which is not too surprising considering the Riemann-Hilbert correspondence. For applications to uniformization, and since it is tied to a more classical history, we will focus on the Deligne-Mostow side. Deligne and Mostow studied uniformization as a route towards understanding the difference between discrete versus arithmetic groups; uniformization forces the monodromy group acting on an Hermitian symmetric domain to be a lattice in a Lie group. However, one can hope for a theory that simultaneously generalizes the two.

[2]Gel'fand-Graev-Kapranov-Zelevinsky

Historically, work of Appell and Picard pushed the parameter space for the independent variable in hypergeometric equations from a (punctured) Riemann sphere to certain algebraic surfaces. They even obtained some uniformization results, but lacked a general theory.

A notable feature in the theory of a single variable is the existence of discrete but non-arithmetic groups. When extending to higher dimensions, however, a result of Margulis tells us for any rank ≥ 2 group the discrete subgroups/lattices are all arithmetic. In the theory of Deligne and Mostow [DM86], the monodromy group for the multi-variate hypergeometric function is a subgroup of $\mathrm{PU}(k, n)$. A simple global condition, the sum of local monodromy data for a rank one local system at marked points on $\mathbb{P}^1$, determines k. One of the great virtues of the Deligne-Mostow theory is that the monodromy group can be controlled and more or less fully understood for moduli of points on $\mathbb{P}^1$.

2. The basic construction

A complex local system of rank n on X is an n-dimensional complex representation of $\pi_1(X)$. In the very special case of a rank one local system L on $X = \mathbb{P}^1 \setminus \{n \text{ points}\}$, L is determined, up to isomorphism (but not up to unique isomorphism) by local monodromy data.

Definition 1. *A Deligne-Mostow local system L is a rank one complex local system on $\mathbb{P}^1 \setminus n$ points, with local monodromy $e^{2\pi i \alpha_k}$ at the kth point, such that $\sum_k \alpha_k = 2$.*

Here we may as well restrict α_k to lie between 0 and 1. By the defining relation among the loops around individual punctures for π_1 on the n-punctured sphere, their sum must be integral. Furthermore, it turns out that the α_k may always be taken to be rational values, for uniformization applications. Although the final condition $\sum_k \alpha_k = 2$ is not strictly necessary, for applications to ball quotients it is required, since it ultimately forces the monodromy action to be a subgroup of $\mathrm{PU}(1, n-2)$. For comparison, if $\sum_k \alpha_k = m$, then the monodromy group will be a subgroup of $\mathrm{PU}(m-1, n-2)$.

As we allow the coordinates of the n points on $\mathbb{P}^1$ to vary, for fixed local monodromies α_k, then the local system L varies as well. One can easily verify that these fit together to form a local system $\mathcal{L}$ over the open parameter space $\mathbb{P}^o_{0,n} \subset (\mathbb{P}^1)^n$, defined by removing all subdiagonals – that is, not allowing collisions of two or more points.

Now the goal is to understand the pairing of holomorphic local-system valued one-forms with one-cycles on X. In the case $\sum_k \alpha_k = 2$ it is easily verified that there is a unique holomorphic form up to scaling. Rather than dealing with compact versus non-compact support and pairings, it is easiest to think in terms of the first intersection cohomology valued in L, which itself is a rank $n-2$ local system $\mathrm{IH}(L)$. The holomorphic form spans a one-dimensional subspace. This also fits together to be local system, $\mathrm{IH}(\mathcal{L})$ now of rank $n-2$ over $\mathbb{P}^o_{0,n}$. Because L is determined only up to isomorphism, there is an ambiguity of rescaling factor, which means the natural object to consider is the projectivization $\mathbb{P}(\mathrm{IH}(\mathcal{L}))$. The holomorphic part becomes a point in each fiber of $\mathbb{P}(\mathrm{IH}(\mathcal{L}))$.

Now, the algebraic group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbb{P}^o_{0,n}$ by means of Möbius transformations on $\mathbb{P}^1$. The projectivized local system then descends to a projective local system with fiber $\mathbb{P}^{n-3}$ on $\mathcal{M}_{0,n}$, the moduli space of n distinct points on $\mathbb{P}^1$. The marked point in each $\mathbb{P}^{n-3}$ fiber, coming from the holomorphic component of the first intersection cohomology group, defines a multi-valued hypergeometric "function" HG from $\mathcal{M}_{0,n}$ to $\mathbb{P}^{n-3}$. This is actually a multi-valued map, which may be thought of as single valued on a branch together with a monodromy group action. Furthermore, the intersection pairing on intersection cohomology must be respected by the monodromy action. This means the monodromy group is a subgroup of $\mathrm{PU}(1, n-2)$, and in particular acts to preserve the open complex ball $\mathbb{B}^{n-3} \subset \mathbb{P}^{n-3}$.

To get uniformization criteria, one needs to understand how this multi-valued function HG extends over boundary components in the moduli of points, and whether there are associated fundamental domains for the action of the monodromy group on the complex ball. This in turn is the same as saying that the monodromy group is acting as a discrete lattice in $\mathrm{PU}(1, n-2)$. One can then ask further whether or not that discrete lattice is actually arithmetic.

Let's be more precise now about the Deligne and Mostow generalization of the uniformization of the thrice punctured sphere to an arbitrary number of points. The automorphism group of $\mathbb{P}^1$ is $\mathrm{PGL}_2(\mathbb{C})$, and it is well-known that an element of the group is uniquely specified by selecting three points and assigning them in order to $\{0, 1, \infty\}$. Thus, one may reinterpret the classical case more "symmetrically" as the moduli space of 4 distinct ordered points on $\mathbb{P}^1$. There is only one compactification as an algebraic variety, namely $\mathbb{P}^1$ itself. In the special case where each $\mu_i = 1/2$ this reproduces the hypergeometric Picard-Fuchs equation for the Legendre family from the Introduction, with Γ the monodromy group of integral cycles on these elliptic curve 4-point branched covers. Their approach also applies to all "triangle group" quotients.

The classical extension due to Picard is then the moduli space of 5 points on $\mathbb{P}^1$, denoted $\mathcal{M}_5$. Geometric Invariant Theory (GIT) provides a finite list of compactifications of $\mathcal{M}_5$. A list of 5 rational numbers $\mu = (\mu_1, \ldots, \mu_5)$ naturally defines a G-linearization and hence a GIT completion of $\mathcal{M}_5$: we denote the stable partial completion by $\mathcal{M}_{5,\mu}$ and the full semi-stable compactification by $\overline{\mathcal{M}}_{5,\mu}$. Deligne and Mostow observed that these were the correct completions and so Picard's arguments could be made rigorous.

Deligne and Mostow then extended the theory of hypergeometric functions to an arbitrary number of points, introducing a condition called (INT) that generalizes a notion of Picard (they name the resulting lattices *Picard lattices*). They consider integrals of the form

$$\mathrm{F}_{gh}(x_2, \ldots, x_{d+1}) = \int_g^h u^{-\mu_0}(u-1)^{-\mu_1} \prod_{i=2}^{d+1} (u - x_i)^{-\mu_i} du$$

where $g, h \in \{\infty, 0, 1, x_2, \ldots, x_{d+1}\}$, so for fixed $\mu_0, \ldots, \mu_{d+1}$ the F_{gh} becomes a multi-valued function on

$$M := \{(x_i) \mid x_i \neq 0, 1, \infty, \text{ and } x_i \neq x_j \text{ for } i \neq j\} \subset (\mathbb{P}^1)^{d+3}.$$

Setting $n = d+3$, this is the HG described above. As the complex linear span of these functions forms a $d+1$ dimensional vectorspace invariant under monodromy, this yields a map from the universal covering space of M to $\mathbb{P}^d$. Let Γ be the image of $\pi_1(M)$ in $\mathrm{PGL}(d+1,\mathbb{C})$. Denote by S the index set $\{\infty, 0, 1, \dots, d+1\}$, and choose μ_∞ so that the sum

$$\sum_{i\in S} \mu_i = 2.$$

Under the further hypothesis that all of these μ_i are real and strictly positive, Deligne and Mostow define the condition

$$(\mathrm{INT}) : (1-\mu_i-\mu_j)^{-1} \text{ is an integer for all } i \neq j \in S \text{ such that } \mu_i+\mu_j < 1.$$

and prove

Theorem 1. *[DM86] If the condition* (INT) *holds for all $i, j \in S$, then Γ is a lattice in the projective unitary group* $\mathrm{PU}(d, 1)$.

So, when μ satisfies condition (INT), they show HG_μ determines an orbifold uniformization of M_n by $\mathbb{B}^{n-3}$.

Mostow then extended the criterion further in order to take advantage of symmetries in μ (where some of the μ_i are equal), showing that the criterion (Σ INT) is almost necessary and sufficient.[3] Specifically, consider a subset $S_1 \subset S$ such that $\mu_s = \mu_t$ for all $s, t \in S_1$. Then define the condition

$$(\Sigma\,\mathrm{INT}) : \begin{array}{l} \text{For all } s \neq t \text{ such that } \mu_s+\mu_t < 1, \\ (1-\mu_s-\mu_t)^{-1} \text{ is } \begin{cases} \text{an integer if } s \text{ or } t \text{ is not in } S_1, \\ \text{a half-integer if } s, t \in S_1. \end{cases} \end{array}$$

Once again, under the hypothesis that the μ_i are real and strictly positive for $i \in S$, Mostow proves

Theorem 2. *[Mos86] If (μ_s) satisfies the condition* $(\Sigma\,\mathrm{INT})$*, then Γ is a lattice in $PU(V) = PU(d, 1)$.*

It turns out there are only finitely many μ that satisfy (Σ INT), and then only for $5 \le n \le 12$.

The Deligne-Mostow conditions (INT) and (Σ INT) characterize discrete uniformization by the complex ball; a further (ARITH) condition is needed to to pick out arithmetic lattices.

Theorem 3. *[DM86] For any $b \in \mathbb{Q}$, let $\langle b\rangle$ denote the fractional part of b, i.e., $0 \le \langle b\rangle < 1$ and $b - \langle b\rangle \in \mathbb{Z}$. Let $\mu = \mu_s, s \in S$ satisfy condition* (INT) *and let δ denote the least common denominator of μ. Then Γ is an arithmetic lattice in* $\mathrm{PU}(V)$ *if and only if*

(ARITH)*: For each integer A relatively prime to δ with $1 < A < \delta - 1$,*
$\sum_{s\in S}\langle A\mu_s\rangle = 1$ *or card(S) = 1.*

[3]The name is designed to be suggestive of the action of a symmetric group Σ.

Applying this theorem in the case when $n = 4$ (i.e., $d = 1$), one recovers the criterion for the arithmetic triangle groups classified by Takeuchi [Tak77]. This criterion also sheds light on the restriction to submoduli spaces, where the structure theory of the arithmetic subgroups yield inherited uniformization for moduli spaces. This clean picture is often obscured when directly considering the families of curves covering the orbifold $\mathbb{P}^1$; in essence it is a feature more of Hodge structure (e.g., related to Jacobians) than the particular geometry of these curve covers per se.

3. Hypergeometric functions for subspaces and other moduli spaces

Given a uniformization of an algebraic variety, one can try to identify sub-locally symmetric spaces of the universal covering space endowed with actions of discrete subgroups and what subvarieties these correspond to. In the other direction, one could look at moduli problems that perhaps relate to the original moduli problem, most especially those moduli spaces that embed as subvarieties of the original moduli space, and try to see how the uniformization naturally restricts.

A crucial point in Deligne and Mostow's approach to uniformization via hypergeometric functions is that the hypergeometric multivalued function for local monodromy data $\alpha = (\alpha_1, \dots, \alpha_n)$, canonically defined on the open set in $(\mathbb{P}^1)^n$ where points do not collide, extends over the GIT semistable set for the $\mathrm{SL}_2(\mathbb{C})$-linearization of $(\mathbb{P}^1)^n$ given by the line bundle $L_1^{\otimes \alpha_1} \otimes \cdots \otimes L_n^{\alpha_n}$. Here L_i is the degree 1 line bundle over the ith $\mathbb{P}^1$ factor. Correspondingly, the function HG_α extends from the interior of the moduli space of points over the boundary loci, for the quotient $(\mathbb{P}^1)^n//_\alpha \mathrm{SL}_2(\mathbb{C}) = \mathcal{M}_n^\alpha$.

By the theory of variation of GIT, there are only finitely many GIT chambers, and hence finitely many semistable loci and corresponding moduli spaces of n points on $\mathbb{P}^1$, for any given n. This number increases very rapidly with n. Necessarily, many different choices $\alpha = (\alpha_1, \dots, \alpha_n)$ define the same GIT chamber. Nonetheless, it is far from the case that any GIT moduli space of points admits a uniformization by HG_α for some α. The aforementioned conditions (INT) and (Σ INT) completely characterize these.

There is a natural partial ordering on the α taken over all n, given by "collision of points" inducing "addition" of local monodromy for the corresponding points. That is, if $\alpha = (\alpha_1, \dots, \alpha_n)$ and we "collide" the ith and jth points, $i < j$, then for $\beta = (\alpha_1, \dots, \alpha_i + \alpha_j, \alpha_{i+1}, \dots \alpha_{j-1}, \alpha_{j+i}, \dots, \alpha_n)$, which has length $n-1$, we have $\alpha > \beta$. This partial ordering is compatible with the partial ordering on semistable sets induced by inclusion for different n. In particular, the restriction of a GIT moduli space of n points to boundary loci corresponding to a GIT moduli space of $n-1$ points compatibly restricts the function HG_α to the function HG_β.

Given a uniformization of moduli of n points on $\mathbb{P}^1$ via HG_α, this induces a uniformization of the boundary moduli space of $n-1$ points via HG_β. Furthermore, the condition (ARITH) is compatible with this restriction to the boundary. In particular, GIT boundary moduli spaces correspond to subball quotients, yielding a chain of nested uniformizations, respecting arithmeticity. Interestingly, there are two major chains of ball quotient structures on GIT moduli of points on $\mathbb{P}^1$, one associated with the monodromy group being defined over the Gaussian integers, and the other with the monodromy group being defined over the Eisenstein integers.

In an amusing triple confluence of historical terminology, what one gets is a lattice (partially ordered set) of lattices (nice discrete subgroup of Lie group) that are automorphisms of lattices (over nice rings of integers).

Theorem 4. *[Dor03, Dor4b] The GIT moduli space of eight points on $\mathbb{P}^1$ with $\alpha = (\frac{1}{4}, \ldots, \frac{1}{4})$ is the largest dimensional Deligne-Mostow ball quotient with discrete (arithmetic) group realized as automorphisms of a lattice over the Gaussian integers. All others arise by a partially ordered sequence of moduli subspaces that inherit a ball quotient structure. The GIT moduli space of twelve points on $\mathbb{P}^1$ with $\alpha = (\frac{1}{6}, \ldots, \frac{1}{6})$ is the largest dimensional Deligne-Mostow ball quotient with discrete (arithmetic) group realized as automorphisms of a lattice over the Eisenstein integers. All others arise by a partially ordered sequence of moduli subspaces that inherit a ball quotient structure.*

This gives interesting internal structure to the uniformization results of Deligne and Mostow. One illustration of the utility of this approach is that it is easy to find a number of errors of omission in the Deligne-Mostow tables classifying GIT moduli of points on $\mathbb{P}^1$ that are ball quotients via hypergeometric uniformization. (These tables were fixed by different means by Thurston.) Curiously, little structure can be seen in the discrete but non-arithmetic examples, although the patterns suggest there "should" be larger uniformization (requiring more than 12 points) of a GIT moduli space of points on $\mathbb{P}^1$ that realizes these examples as subball quotients.

Other examples can also be approached in this way. After all, moduli of points are especially convenient for encoding other moduli problems. Working with embedded subvarieties, looking for compatibility with the ball quotient structure, is the easiest approach. Hurwitz spaces provide another. The following results have been shown by a few authors independently.

Theorem 5. *[Dor03, Dor4a] The moduli space of cubic surfaces inherits a ball quotient structure from Deligne-Mostow ball quotients by restriction via an embedding into a GIT moduli space of points on $\mathbb{P}^1$. Likewise the moduli of rational elliptic surfaces inherits a ball quotient structure from Deligne-Mostow ball quotients.*

4. Where to go from here

There are a number of "mysteries" associated with the zoo of results in uniformization following Deligne and Mostow. Perhaps the most basic is: why end at twelve points? Although this certainly is the limit, using the (Σ INT) condition, for GIT moduli spaces of points on $\mathbb{P}^1$, there is still lots of internal structure to the complete list of examples that would be better explained by inheriting structure from moduli involving more points. In particular, there is still very little understood about discrete but non-arithmetic lattices, so further internal structure, especially if there are "universal" examples like the Gauss and Eisenstein examples above, would be very valuable. Furthermore, uniformization by ball quotients turns out to work for many other moduli spaces, in particular for moduli of cubic threefolds and moduli of cubic fourfolds. What is going on?

If one can deal with more points, we might see more clearly how the chain of discrete non-arithmetic examples arise. However, it is not clear that directly working with the monodromy action on complex balls (type I symmetric domain) is the way to go — realiziations as subobjects in type IV symmetric domains might be better. Alternatively, one can try to adapt the existing theory to a more uniform moduli space of points — for instance, the Deligne-Mumford compactification $\overline{M}_{0,n}$.

With the idea of finding a simultaneous generalization of toric GKZ and Deligne-Mostow theory in mind, and $\mathbb{M}_{0,n}$ as a core example to guide us, the following result is quite promising.

Theorem 6. *[DG14] Blow-ups along projective arrangements admit an "algebraic uniformization": they arise as non-reductive GIT quotients of affine space by the action of a connected solvable group. In particular, $\overline{M}_{0,n}$, via the Kapranov description as a projective variety, is such a quotient.*

This allows one to treat spaces like $\overline{M}_{0,n}$ as "almost toric" for many purposes. It also opens the door to a clean generalization of Deligne-Mostow to a much wider class of spaces. In this vein, work of Couwenberg, Heckman, and Looijenga [CHL05] should prove a useful guide.

Part 2. Differential-Algebraic Detection of Shimura Subvarieties

The problem that we wish to address in this part, both in the abstract and in concrete situations, is, given a quasiprojective family of K3 surfaces $\mathcal{X}/B$ with generic Picard lattice L which is maximal in the sense that the period map $\mathrm{Per}(\mathcal{X}/B)$ is locally surjective, and given a divisor in B denoted D, along with a parametrization of D, when does the Picard rank of a general fiber along D increase?

The tool that we will make most use of is the Picard-Fuchs equation of the family $\mathcal{X}/B$. In the case where $\mathcal{X}$ is the family of K3 surfaces which are Shioda-Inose partners [SI77, Ino78, Mor84] of products of elliptic curves $E_1 \times E_2$, a solution to this problem is presented in [CDLW09]. Therein, the authors reparametrize this family using a pair of variables called j_1 and j_2 and show that there is a PDE so that if $j_1(t)$ and $j_2(t)$ parametrize a curve C in U, then C supports a family of Picard rank 19 K3 surfaces if and only $j_1(t)$ and $j_2(t)$ satisfy this PDE.

A similar statement was proven in [DHMW16] for a larger family of K3 surfaces which are Shioda-Inose partners of principally polarized abelian surfaces [CD12, Kum08]. We will generalize this result to a general maximal family of K3 surfaces. As we will see in the next subsection, given a Picard-Fuchs differential ideal for $\mathcal{X}/B$, it is not hard to get a prospective system of PDEs whose solutions correspond to interesting subvarieties of B, including divisors where the Picard rank increases. The difficulty comes in showing that all solutions of this system of PDEs correspond to places where the Picard rank of the general fiber increases. In [CDLW09], it is shown that the naïve solution is incorrect, as it also detects curves where one of the factor curves $E_1 \times E_2$ is held constant. In [DHMW16], a numerical computation was performed using explicit realizations of monodromy matrices of the corresponding family of K3 surfaces to show that the only solutions to the corresponding set of PDEs characterizes divisors where generic rank jumps.

The tools used in the previous cases are not available to us in the general situation, therefore, we will make use of abstract results of Moonen [Moo98] and Abdulali [Abd94] characterizing totally-geodesic subvarieties of Shimura varieties. The main theorem of this part is the following.

Theorem 7 (Theorem 14)**.** *There is a strict differential algebraic characterization of Shimura divisors of arithmetic quotients of type IV symmetric domains except in cases which resemble that of [CDLW09].*

On the other hand, it became abundantly clear while writing this paper, that even in relatively simple cases, such as the Humbert surface of discriminant 5, the actual PDEs that one obtains can be very complicated. Writing down the appropriate system of PDEs in this case is a computationally demanding task, and checking whether a specific subvariety gives a solution to this PDE is only feasible in very simple situations. Therefore, as an addendum to this, we have implemented an algorithm in MAGMA [BCP97] which is capable of checking whether parametrized curves and surfaces in the moduli space of N-polarized K3 surfaces support a family of K3 surfaces of rank 19 or 18 respectively. This is simply an instantiation of the well-known Griffiths-Dwork algorithm for computing Picard-Fuchs equations of hypersurfaces now adapted to *weighted* projective spaces.

5. Relation to Part 1

Before we proceed with the technical work needed to prove the results outlined in the previous section, we will take a moment to discuss how our results are related to those of Part 1. The statements in this section are impressionistic at best but can be made precise.

The main theorem in this part of the paper says that there is a differential algebraic condition which can be used to detect "jumping behaviour" in a variation of Hodge structure. The previous part of this paper discusses ball quotients by arithmetic subgroups of $\mathrm{PU}(n, d)$ and their boundary behaviour. This section describes an example which is in the intersection of these two situations. Kondō [Kon07] describes a family of K3 surfaces whose moduli space is in fact a ball quotient by an arithmetic subgroup Γ of $\mathrm{PU}(\Lambda)$ for Λ a lattice of over $\mathbb{Q}(\sqrt{-5})$. These K3 surfaces are written as hypersurfaces in $\mathbb{WP}(1,1,1,3)$ with defining equations

$$x_3^2 = x_1(x_1^5 - x_2x_3(x_2 - x_3)(x_2 - \lambda_1 x_3)(x_2 - \lambda_2 x_3)) \tag{1}$$

So their moduli is directly related to the moduli space of five points in $\mathbb{P}^1$. One can show that these curves are in fact a quotient of a pair of curves $C_1 \times C_2$ where C_1 is the unique curve of genus 2 which admits an order 5 automorphism and C_2 is a genus 4 plane curve written as

$$x_1^5 - x_2x_3(x_2 - x_3)(x_2 - \lambda_1 x_3)(x_2 - \lambda_2 x_3).$$

In this case, the generic K3 surface in Equation 1 has transcendental lattice $A_4 \oplus A_4 \oplus H \oplus H_5$ where H_5 is the unique indefinite lattice of discriminant -5 and rank 2.

There is a differential ideal which annihilates a subset of the holomorphic periods of the family of K3 surfaces in Equation 1, which has generators

$$\frac{\partial^2}{\partial\lambda_1^2} = \frac{-2(-4\lambda_1\lambda_2+5\lambda_2^2+2\lambda_1-3\lambda_2)}{5\lambda_2(\lambda_2-1)(-\lambda_1+\lambda_2)}\frac{\partial}{\partial\lambda_1} + \frac{2\lambda_1(\lambda_1-1)}{5\lambda_2(\lambda_2-1)(-\lambda_1+\lambda_2)}\frac{\partial}{\partial\lambda_2} - \frac{6}{25\lambda_2(\lambda_2-1)}$$
$$\frac{\partial^2}{\partial\lambda_2^2} = \frac{-2\lambda_2(\lambda_2-1)}{5(-\lambda_1+\lambda_2)\lambda_1(\lambda_1-1)}\frac{\partial}{\partial\lambda_1} - \frac{2(-5\lambda_1^2+4\lambda_1\lambda_2+3\lambda_1-2\lambda_2)}{5(-\lambda_1+\lambda_2)\lambda_1(\lambda_1-1)}\frac{\partial}{\partial\lambda_2} - \frac{6}{25\lambda_1(\lambda_1-1)}$$

and

$$\left(-\frac{5\lambda_1}{2}+\frac{5\lambda_2}{2}\right)\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} = -\frac{\partial}{\partial\lambda_1}+\frac{\partial}{\partial\lambda_2}.$$

Note that this is precisely Appell's hypergeometric differential equation with appropriate parameters. Solutions to this set of differential equations are of the form F_{gh} discussed in the previous section for some choice of μ_1, μ_2 and $d = 1$.

A rational curve parametrized by rational functions $(\lambda_1(t), \lambda_2(t))$ then has period map a sub-ball quotient of the two-dimensional ball if and only if the restriction of the solutions of the differential equation above satisfy a differential equation of the form

$$\frac{d^2}{dt^2} = A(t)\frac{d}{dt} + B(t) \tag{2}$$

for some rational functions $A(t)$ and $B(t)$. Following the proof of Theorem 15, one may deduce that there is a nonlinear partial differential equation in terms of $\lambda_1(t)$ and $\lambda_2(t)$ whose rational solutions are exactly the rational curves in the λ_1, λ_2 plane on which the differential equation satisfied by restrictions of the Appel hypergeometric equation above satisfy a differential equation of the form in Equation 2.

These solutions correspond to places where the generic Picard lattice of the family of K3 surfaces in the form of Equation 1 jumps. It is easy to check that such loci include the locus where pairs of points collide – that is, where λ_1 or λ_2 equal $0, 1$ or ∞, and where $\lambda_1 = \lambda_2$.

Therefore the behaviour discussed in the previous section is closely related to what follows, albeit in a slightly different form.

Remark 1. To the reader who is more interested in arithmetic ball quotients than arithmetic quotients of type IV symmetric domains: many of the results in the rest of this paper have analogues in terms of arithmetic ball quotients. That is, if a system of differential equations on an arithmetic ball quotient M uniformizes M, then there is a set of partial differential equations whose algebraic solutions correspond to arithmetic sub-ball quotients. These sub-ball quotients should be thought of as generalizations of points in the moduli space of points in $\mathbb{P}^1$ where points collide.

6. Picard-Fuchs equations and totally-geodesic subvarieties

In this section, we will discuss how we can use a maximal family $\mathcal{X}$ of K3 surfaces over a base B and the corresponding Picard-Fuchs ideal $\mathrm{PF}(\mathcal{X})$ to determine which subvarieties of B have image in the appropriate period space which is a totally-geodesic subvariety.

Remark 2. In what follows, we will always assume that we work with a variation of Hodge structure or family of K3 surfaces over a Zariski open subset of $\mathbb{C}^n$. This is only to simplify arguments and in order to work with Picard-Fuchs ideals in the differential ring $\mathbb{C}[x_1,\ldots,x_n,\partial_1,\ldots,\partial_n]$ instead of having to work with more general D-modules. Furthermore, all examples that we are interested in can be addressed in terms of Picard-Fuchs ideals.

6.1. **Preliminaries.** Here we will let L be an integral lattice of signature $(2, n-2)$ and rank n, which is made up of $\mathbb{Z}^n$ and the bilinear pairing $\langle \bullet, \bullet \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$. We define Γ_L to be the full automorphism group of the lattice L, or in other words, elements γ of $\mathrm{GL}_n(\mathbb{Z})$ satisfying $\langle \gamma u, \gamma v\rangle = \langle u, v\rangle$ for all u, v in $\mathbb{Z}^n$. For a field k of characteristic zero, we will denote $L \otimes k$ by L_k. We will let $\mathcal{P}_L$ be a choice of one of the two connected components of the Hermitian symmetric domain

$$\{z \in \mathbb{P}(L_{\mathbb{C}}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle > 0\} = \mathrm{SO}(2, n-2)/(\mathrm{SO}(2) \times \mathrm{SO}(n-2)).$$

The action of Γ_L on $\mathcal{P}_L$ is arithmetic. Therefore, the right quotient $\mathcal{M}_L := \Gamma_L \backslash \mathcal{P}_L$ can be equipped with the structure of a complex quasi-projective orbifold. Note that $\dim \mathcal{M}_L = n-2$. We will let $\mathcal{X}$ be a smooth projective family of K3 surfaces over a quasiprojective base B. The local system $R^2\pi_*\mathbb{Z}$, decomposes into a transcendental piece $\mathbb{T}_{\mathcal{X}}$ and an algebraic piece $\mathbb{NS}_{\mathcal{X}}$. To describe this decomposition, note that there is a fiber-wise pairing $R^2\pi_*\mathbb{Z} \times R^2\pi_*\mathbb{Z} \to R^4\pi_*\mathbb{Z} \cong \underline{\mathbb{Z}}_B$, and $\mathbb{NS}_{\mathcal{X}}$ is defined to be the orthogonal complement in $R^2\pi_*\mathbb{Z}$ of a local section α of the relative canonical bundle, and $\mathbb{T}_{\mathcal{X}}$ is the orthogonal complement of $\mathbb{NS}_{\mathcal{X}}$. Each fiber of $(\mathbb{T}_{\mathcal{X}})_s$ is equipped with a lattice structure. We will assume this is isomorphic to a lattice L. There is an algebraic period map

$$\mathrm{Per}(\mathcal{X}) : B \to \mathcal{M}_L.$$

In our case, we can describe this map locally as follows. Take a local basis of flat sections $\gamma_1, \ldots, \gamma_n$ in an open subset U around a point $z \in B$ and a local section α of the relative canonical bundle $\omega_{\mathcal{X}/B}$. Then we get a map

$$z \in U \mapsto \left[\int_{\gamma_1} \alpha : \cdots : \int_{\gamma_n} \alpha\right].$$

This map has image in $\mathcal{P}_L$ and is multivalued when extended to B, since monodromy acts nontrivially on the flat sections γ_i. However, the induced map to the quotient $\Gamma_L\backslash\mathcal{P}_L$ is single valued and is precisely the period map $\mathrm{Per}(\mathcal{X})$.

Definition 6.1. The family $(\mathcal{X}, \pi)$ is called *maximal* if the map $\mathrm{Per}(\mathcal{X})$ is locally an isomorphism onto $\mathcal{M}_L$.

There is a natural rank n local system $\mathbb{L}$ on $\mathcal{M}_L$ obtained by taking the quotient of the trivial local system on $\mathcal{P}_L$ with fiber L by the natural action of Γ_L. On $\mathcal{L} = \mathbb{L} \otimes \mathcal{O}_{\mathcal{M}_L}$, there is a natural metric on this bundle called the *Hodge metric*. A subvariety $\mathcal{S}$ of $\mathcal{M}_L$ will be called *totally-geodesic* if it is totally-geodesic with respect to the Hodge metric on $\mathcal{L}$. We refer to Helgason [Hel78, Ch. IV §7] for more information regarding totally-geodesic subvarieties of Hermitian symmetric domains. There are two common types of totally-geodesic complex submanifolds of $\mathcal{P}_L$ obtained by taking components of intersections of $\mathcal{P}_L$ with linear subspaces of $\mathbb{P}(L_\mathbb{C})$.

(1) $\mathcal{S} = H \cap \mathcal{P}_L$ is an open piece of a quadric hypersurface in a linear subspace H of $\mathbb{P}(L_\mathbb{C})$.
(2) $\mathcal{S} = H$ is an open piece of a linear subspace in H of $\mathbb{P}(L_\mathbb{C})$ contained in $\mathcal{P}_L$.

In the first case, $\mathcal{S}$ comes from the embedding of a group $\mathrm{SO}(2, m-2)$ into $\mathrm{SO}(2, n-2)$ as a subgroup fixing some negative definite subspace. The second case comes from the embedding of the special unitary group $\mathrm{SU}(1, m-1)$ into $\mathrm{SO}(2, n-2)$. We will call these geodesics of orthogonal and unitary type respectively. In the next section, we will give a totally-geodesic characterization of totally-geodesic divisors in $\mathcal{M}_L$ once we have a uniformizing differential equation for $\mathcal{M}_L$.

6.2. **Picard-Fuchs equations and their restrictions.** Now let us consider a maximal smooth projective family of K3 surfaces $(\mathcal{X}, \pi)$ fibered over B. For each fiber $\mathcal{X}_s$ of the family $(\mathcal{X}, \pi)$, there is a polarized Hodge structure on $(\mathbb{T}_\mathcal{X})_s$ which can be extended globally to a variation of Hodge structure on $\mathcal{H} = \mathbb{T}_\mathcal{X} \otimes \mathcal{O}_B$

$$\mathcal{F}^2 \subseteq \mathcal{F}^1 \subseteq \mathcal{F}^0 = \mathcal{H}$$

where $\mathcal{F}^2$ is a line bundle on B. The vector bundle $\mathcal{H}$ is equipped with a flat connection called the *Gauss-Manin connection*, and which we denote ∇. As usual, ∇ is a map from $\mathcal{H}$ to $\mathcal{H} \otimes \Omega_B$. If we choose a vector field v on B, then we denote the operator on $\mathcal{H}$ obtained by composing ∇ with contraction by v as ∇_v. If we denote by α a global section of $\mathcal{H}$, then for γ any local section of the dual bundle $\mathcal{H}^\vee = \mathbb{T}_\mathcal{X}^\vee \otimes \mathcal{O}_B$, we denote $\int_\gamma \alpha$ the pairing between α and γ. Locally on B, this produces a holomorphic function. If γ is dual to a *flat* section of $\mathcal{H}$ then if x is a coordinate direction on B then

$$\frac{\partial}{\partial x} \int_\gamma \alpha = \int_\gamma \nabla_x \alpha \tag{3}$$

Following [Pet86], if α is a global section of $\mathcal{H}$ and $\operatorname{rank} \mathcal{H} = n$ then any set of $n+1$ holomorphic sections of $\mathcal{H}$ satisfy an $\mathcal{O}_B$-linear relation. In particular, if we take α and repeatedly apply the operators ∇_{x_i}, then for any $n+1$ such derivatives, there exists a relation. Let us assume that B is some Zariski open subset of $\mathbb{C}^n$. Then we obtain global coordinate vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ which span $T_u B$ at each point $u \in B$. From this we get operators $\nabla_{x_1}, \dots \nabla_{x_n}$. Let us take $\mathbb{C}[x_1, \dots, x_n, \nabla_{x_1}, \dots, \nabla_{x_n}]$ and in it we define a differential ideal called the *Picard-Fuchs ideal* of the family $\mathcal{X}/B$ which we call $\mathrm{PF}(\mathcal{X})$ generated by all relations satisfied by α. For instance, if $n = 1$, then this ideal is principal and generated by a relation of the form

$$0 = \sum_{i=0}^{n+1} g_i(x) \nabla_x^i \alpha.$$

If γ is any flat section of $\mathcal{H}^\vee$, then by Equation 3, we see that the multivalued holomorphic functions $F_\gamma(x) = \int_\gamma \alpha$ are solutions to the differential equation

$$0 = \sum_{i=0}^{n+1} g_i(x) \frac{\mathrm{d}^i F}{\mathrm{d}x^i},$$

and in fact $F_\gamma(x)$ form a local (or, if the reader prefers, multivalued) basis for the set of solutions to this ODE. If we choose a local flat basis $\gamma_1, \ldots, \gamma_n$ of $\mathcal{H}^\vee$, then the map sending a point in B to the points in $\mathcal{P}_L$ determined by the vector of functions $F_{\gamma_1}, \ldots, F_{\gamma_n}$ is just the period map described in the previous section. Following Sasaki-Yoshida [SY89], we have that the Picard-Fuchs ideal of a maximal family of K3s surfaces over $B \subseteq \mathbb{C}^n$ a Zariski open subset takes a very specific form.

Theorem 8 (Sasaki-Yoshida [SY89])**.** *Let $B \subseteq \mathbb{C}^{n-2}$ be a Zariski open subset and let $(\mathcal{X}, \pi)$ be a family of K3 surfaces over B, so that the map*

$$x \mapsto [F_{\gamma_1}(x) : \cdots : F_{\gamma_n}(x)]$$

is locally an isomorphism onto a quadric in $\mathbb{P}^{n+1}$ for a choice of flat local sections $\gamma_1, \ldots, \gamma_{n+2}$, then $\mathrm{PF}(\mathcal{X})$ *is generated by equations of the form:*

$$\frac{\partial^2}{\partial x_i \partial x_j} - g_{ij} \frac{\partial^2}{\partial x_1 \partial x_n} - \sum_{\ell=1}^{n} A_{ij}^{\ell} \frac{\partial}{\partial x_k} - A_{ij}^0$$

for all unordered pairs $i, j \in [0, n]$ with functions $g_{ij} = g_{ji}$ and $A_{ij}^\ell = A_{ji}^\ell$.

We are interested in totally-geodesic subvarieties of $\mathcal{P}_L$ which are either quadric hypersurfaces in a linear subspace of $\mathbb{P}(L_\mathbb{C})$ or a linear subspace of $\{z \in \mathbb{P}(L_\mathbb{C}) : \langle z, z\rangle = 0\}$ intersected with $\mathcal{P}_L$. We now perform a local analytic computation in each situation to identify them in terms of periods.

Theorem 9. *Let U be an analytic open disc in B, and let $F_1, \ldots, F_{n+2}$ be periods of $\mathcal{X}|_U$. Assume that $S \subseteq U$ is a submanifold smoothly parametrized by a map*

$$(y_1, \ldots, y_k) \mapsto (x_1(y_1, \ldots, y_k), \ldots, x_{n-2}(y_1, \ldots, y_k))$$

(1) The periods $F_i|_S$ satisfy a set of equations of the form

$$\frac{\partial^2}{\partial y_i \partial y_j} - h_{ij} \frac{\partial^2}{\partial y_1 \partial y_n} - \sum_{\ell=1}^{k} B_{ij}^{\ell} \frac{\partial}{\partial y_k} - B_{ij}^0$$

if and only if the image of of $[F_1|_S : \cdots : F_n|_S]$ is an open subset of a quadric in a $k+1$-dimensional subspace of $\mathcal{P}_L$.

(2) The periods $F_i|_S$ satisfy a set of equations of the form

$$\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{k=1}^{n-1} A_{ij}^k \frac{\partial}{\partial y_k} + A_{ij}^0$$

for all pairs i and j if and only if the image of $[F_1|_S : \cdots : F_n|_S]$ is an open subset of k-dimensional linear subspace of $\mathcal{P}_L$.

(3) *The image of* $[F_1|_S : \dots F_n|_S]$ *is not contained in a* $(k+1)$*-dimensional hyperplane of* $\mathbb{P}(L_{\mathbb{C}})$ *if and only if there is some pair of indices* i, j *so that there is no* $\Gamma(S, \mathcal{O}_S)$ *linear relationship between*

$$\frac{\partial^2 F_\ell|_S}{\partial y_1 \partial y_k}, \frac{\partial^2 F_\ell|_S}{\partial y_i \partial y_j}, \frac{\partial F_\ell|_S}{\partial y_1}, \dots, \frac{\partial F_\ell|_S}{\partial y_k}, \text{ and } F_\ell|_S$$

for all ℓ.

Proof. We have a period map $\mathrm{Per}(X)$ which is the projectivization of the locally injective holomorphic map

$$\widetilde{\mathrm{Per}}(\mathcal{X})(\vec{x}) = (F_1(\vec{x}), \dots, F_n(\vec{x}))$$

Let us take S as in the statement of the theorem. The fact that $F_1, \dots F_n$ satisfy the set of differential equations in part (1) of the theorem show that all higher derivatives of $F_i|_S$ can be written as $\Gamma(S, \mathcal{O}_S)$-linear combinations of

$$F_i|_S, \frac{\partial F_i|_S}{\partial y_1}, \dots \frac{\partial F_i|_S}{\partial y_k}, \frac{\partial^2 F_i|_S}{\partial y_1 \partial y_k}.$$

Another way of saying this is that the value of higher order derivatives at a point p in S are $\mathbb{C}$-linear combinations of the values of

$$F_i|_S(0), \frac{\partial F_i|_S}{\partial y_1}(0), \dots \frac{\partial F_i|_S}{\partial y_k}(0), \frac{\partial^2 F_i|_S}{\partial y_1 \partial y_k}(0).$$

We have n functions of k variables, so we may choose a basis of solutions around $p \in S$ so that the power series expansion of $F_i|_S$ so that for $j > k+2$ the constant coefficient, coefficient of y_i and the coefficient of $y_1 y_n$ of $F_j|_S$ are zero. Since all higher coefficients of the power series expansions of $F_j|_S$ are linear combinations of these coefficients, it follows that $F_j|_S = 0$ for $j > k+2$. Thus the image of S must lie in a hyperplane of $\mathbb{C}^{n+2}$ of dimension $k+2$. Since the restriction of the period map to S is an immersion of S into $\mathcal{P}_L$ contained in the intersection with a hyperplane, this map must be an immersion onto an open subset of a quadric in the projective linear subspace. The converse follows by choosing flat sections $\gamma_1, \dots, \gamma_n$ of $\mathcal{H}^\vee$ so that

$$F_j|_S = \int_{\gamma_j} \alpha|_S = 0 \quad \text{ for } j > k+2.$$

Then it is clear that all derivatives of the vector $(F_1|_S, \dots, F_n|_S)$ lie in the same $(k+1)$-dimensional space and thus we obtain the desired linear relations.

In case (2), we have that any coefficient of $F_i|_S$ of degree greater than 1 can be expressed as a linear combination of the lower coefficients, therefore the same argument shows that the image of $\mathrm{Per}(\mathcal{X})$ restricted to S is contained in an open subset of a projective subspace of $\mathbb{P}(L_{\mathbb{C}})$ contained in $\mathcal{P}_L$.

It is easy to deduce the final claim by similar methods. ☐

Now we begin with the Picard-Fuchs equation associated to a maximal family of K3 surfaces $\mathcal{X}$ over a nonempty Zariski open subset of $\mathbb{C}^n$. As usual, our Picard-Fuchs differential equation is written as in Theorem 8. We will show how to compute linear relations between derivatives of the restriction of periods of $\mathcal{X}$ to S where S is a smooth subvariety of B. Proposition 9 then gives us a tool to identify whether S parametrizes a totally-geodesic subvariety of B or not.

Let Z be a divisor in a Zariski open subset $B \subseteq \mathbb{C}^n$, and assume that, around a smooth point of $Z \subseteq B$ we may parametrize Z holomorphically by a map

$$(y_1, \ldots, y_k) \mapsto (x_1(y_1, \ldots, y_k), \ldots, x_n(y_1, \ldots, y_k))$$

where $x_i(y_1, \ldots, y_{n-1})$ are algebraic functions. Locally, this provides a parametrization of S. Let us compute the differential ideal associated to a restriction of a local section α of $\mathcal{F}^2$. We call this restriction α_S. We know that the tangent space of S is spanned by the vector fields

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}.$$

Thus we have that

$$\nabla_{y_i}(\alpha_S) = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} (\nabla_{x_j}\alpha)_S$$

and similarly, we can compute

$$\begin{aligned}\nabla_{y_i,y_j}\alpha_S &= \sum_{p=1}^{n} \left(\frac{\partial^2 x_p}{\partial y_i \partial y_j} (\nabla_{x_p}\alpha)_S + \frac{\partial x_p}{\partial y_i} \sum_{l=1}^{n} \frac{\partial x_l}{\partial y_j} (\nabla_{x_p,x_l}\alpha)_S \right) \\ &= \sum_{p,l,m=1}^{n} \left(\frac{\partial^2 x_p}{\partial y_i \partial y_j} (\nabla_{x_p}\alpha)_S + \frac{\partial x_p}{\partial y_i} \frac{\partial x_l}{\partial y_j} \left(A^0_{pl} + g_{pl} (\nabla_{x_1,x_n}\alpha)_S + A^m_{pl} (\nabla_{x_m}\alpha)_S \right) \right)\end{aligned}$$

The second equation comes by substituting the equations in Theorem 8 into the first equation. The expressions $\nabla_{x_1,x_n}\alpha, \nabla_{x_1}\alpha, \ldots, \nabla_{x_n}\alpha$ and α span the $(n+2)$-dimensional fibers of $\mathcal{H}$ at each point in S, or alternately, they form a set of $\mathbb{C}(B)$-generators for the trivial bundle $\mathcal{H}$. The expression above gives $\nabla_{y_i}\alpha_S$ and $\nabla_{y_i,y_j}\alpha_S$ in terms of this set of generators. It is now easy to determine conditions under which the situations in Proposition 9 occur. The result of this process produces systems of PDEs depending on the expressions for x_i in terms of y_j which can be used to identify whether S is a totally-geodesic subvariety or not by Theorem 9.

Remark 3. The condition of being totally-geodesic is a differential-geometric one to begin with. The coefficients g_{ij} of the equation in Theorem 8 can be identified with the Hodge metric on the space $\mathcal{P}_L$ (up to scaling by a function), so it stands to reason that there should be a direct way of identifying systems of PDEs which identify totally-geodesic subvarieties. We were not able to deduce such equations from first principles.

Returning to the matter at hand, the coefficient of ∇_{x_1,x_n} is

$$\sum_{k,l=1}^{n} \left(g_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j} \right)$$

The coefficient of ∇_{x_m} is

$$\frac{\partial^2 x_m}{\partial y_i \partial y_j} + \sum_{k,l=1}^{n} \left(A^m_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j} \right)$$

and the constant coefficient is

$$\sum_{k,l=1}^{n} A^0_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j}.$$

We now want to understand the $\mathbb{C}(B)$-linear relations between $\nabla_{y_i,y_j}\alpha_S, \nabla_{y_1}\alpha_S, \dots, \nabla_{y_2}\alpha_S$ and α_S. Since S is embedded in B, we can deduce that $\nabla_{y_1}\alpha_S, \dots, \nabla_{y_k}\alpha_S$ and α_S are $\mathbb{C}(S)$ linearly independent in $\mathcal{H}_{|S}$. Thus S parametrizes a unitary complete geodesic if and only if for each $\nabla_{y_i,y_j}\alpha_S$ is in the span of $\nabla_{y_1}\alpha_S, \dots \nabla_{y_k}$. Since $(\nabla_{x_1,x_n}\alpha)_{|S}$ is independent of the span of all $(\nabla_{x_1}\alpha)_{|S}, \dots, (\nabla_{x_n}\alpha)_{|S}$, it follows that this occurs if and only if

$$\sum_{k,\ell=1}^{n} \left(g_{k\ell} \frac{\partial x_k}{\partial y_i} \frac{\partial x_\ell}{\partial y_j} \right) = 0 \tag{4}$$

for all i, j. The submanifold S supports an orthogonal total geodesic if and only if the derivatives of its periods span a a $(k+2)$-dimensional subspace in $\mathcal{H}_{|S}$. According to the facts above, this occurs if and only if the following condition holds.

Condition 1. *There is some r, j such that for each i, j, there is a linear relation*

$$\nabla_{i,j}(\alpha_S) = h_{i,j}\nabla_{r,s}(\alpha_S) + \sum_{k=1}^{t} B^k_{ij}\nabla_{y_k}(\alpha_S) + B^0_{ij}$$

for functions h_{ij} and B^k_{ij} for all $0 \le i, j, k \le t$.

We state this as a Theorem.

Theorem 10. *Let notation be as above.*

1. *The subvariety S parametrizes an orthogonal total geodesic if Equation 4 does not hold for all pairs i, j and Condition 1 holds.*
2. *The subvariety S parametrizes a unitary total geodesic if and only if Equation 4 holds.*
3. *The subvariety S does not parametrize a totally-geodesic subvariety if neither Condition 1 nor Equation 4 hold.*

This should be thought of as a family of non-linear PDEs satisfied by geodesic subvarieties. The algebraic solutions of such PDEs correspond, by the local criterion, to totally-geodesic subvarieties in B of orthogonal or unitary type.

7. Classification of totally-geodesic subvarieties of $\mathcal{M}_L$ of codimension 1

In the previous section, we showed that totally-geodesic subvarieties of $\mathcal{M}_L$ have a characterization in terms of solutions to partial differential equations. Our ultimate goal is to give a way of identifying, not just totally-geodesic subvarieties of $\mathcal{M}_L$, but in fact Hodge subvarieties of $\mathcal{M}_L$. In this section, we will restrict our attention to divisors, and show that if $n \neq 4$, then all solutions to the partial differential equations in Theorem 10 are in fact Hodge subvarieties. In the case $n = 4$, we will show that the same is true if and only if the lattice L has no isotropic elements, i.e. there is no element u of L so that $u \cdot u = 0$. The key is to combine theorem of Moonen [Moo98] with a theorem of Zarhin [Zar83].

In order to state this theorem, we begin in an abstract setting. We take G to be a connected semisimple algebraic group and let K be a maximal compact parabolic subgroup. Then the variety $X = G(\mathbb{R})/K$ is a hermitian symmetric domain. Let Γ to be an arithmetic subgroup of $G(\mathbb{R})$ and let $\Gamma\backslash G(\mathbb{R})/K$ be the right quotient of X by Γ. We quote the Theorem of Moonen [Moo98] in the case where our totally-geodesic subvariety is a divisor and our Shimura variety is associated to an adjoint algebraic group.

Here we will define the Hodge endomorphism algebra of a Hodge structure in the case where the Hodge structure is on $L_{\mathbb{Q}}$ and is of K3 type. General definitions can be found in [GGK12]. A Hodge structure of weight c can be defined as a map $h : \mathrm{U}(1) \to \mathrm{SO}(L_{\mathbb{R}})$ which has eigenvalues on $L_{\mathbb{C}}$ are $z^p \overline{z}^q$ so that $p + q = c$.

Definition 7.1. The Hodge algebra of a polarized Hodge structure on a $\mathbb{Q}$-vector space $L_{\mathbb{Q}}$ is the subalgebra of $\mathfrak{so}(L_{\mathbb{Q}})$ of endomorphisms ψ so that for some $u \in L_{\mathbb{R}}$, we have that $\psi(h(z)u) = h(z)\psi(u)$ and any $z \in \mathrm{U}(1)$.

To a point $p \in \mathcal{P}_L$, let H_p be the corresponding Hodge structure on L.

Definition 7.2. Fix a subalgebra E of $\mathfrak{so}(L_{\mathbb{Q}})$. Let $\mathcal{D}_E$ be the image in $\mathcal{M}_L$ of the set of points in $\mathcal{P}_L$ so that the action of E on $L_{\mathbb{Q}}$ preserves the Hodge structure H_p.

Theorem 11 (Moonen, [Moo98, Theorem 4.3])**.** *Assume that G is an adjoint algebraic group and K and Γ are as above. Assume that S is a totally-geodesic divisor which is not of Hodge type. Then*

(1) $G \cong G_1 \times G_1$, where $K = K_1 \times K_2$ where $K_1 \subset G_1(\mathbb{R})$ and $K_2 \subset G_2(\mathbb{R})$ are maximal parabolic subgroups,
(2) There is some finite index subgroup Γ' of Γ which splits as the product $\Gamma' = \Gamma_1 \times \Gamma_2$ with $\Gamma_i \in G_i(\mathbb{Z})$.
(3) There is a point $p \in \Gamma_1\backslash G_1(\mathbb{R})/K_1$ (which is not a CM point) so that S is the image of $p \times (\Gamma_2\backslash G_2(\mathbb{R})/K_2)$ under the natural finite map.

Remark 4. A version of this theorem has been proved by Abdulali [Abd94, Theorem 4.1], and it can also be found in work of Möller-Viehweg-Zuo [MVZ12].

We are interested in the situation where the group G is of type B_n or D_n type. The corresponding simple adjoint algebraic groups are SO_{2n+1} for $n \geq 2$ (type B_n) and $\mathrm{PSO}_{2n} := \mathrm{SO}_{2n}/(\pm\mathrm{Id})$ for $n \geq 4$ (type D_n). For low values of n the groups SO_n are isomorphic to other well-known groups (see e.g. [Hel78, Ch. X §6.4]),

(1) $\mathrm{SO}_3 \cong \mathrm{SL}_2$, $(B_1 = A_1)$.
(2) $\mathrm{PSO}_4 \cong \mathrm{PSL}_2 \times \mathrm{PSL}_2$, $(D_2 = A_1 \times A_1)$.
(3) $\mathrm{PSO}_6 \cong \mathrm{PSL}_4$, $(D_3 = A_3)$.

Therefore, the only case in which the conditions of Theorem 11 may be satisfied is when $n = 4$. It follows that this is the only case where we may have totally-geodesic divisors in $\Gamma_L \backslash \mathcal{P}_L$ which are not themselves of Hodge type.

Finally, we must address the splitting of Γ_L into a product when $\operatorname{rank} L = 4$. The arithmetic group Γ_L can be realized as an arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. We must treat three cases separately. These cases are:

0. The lattice L has no isotropic vectors.
II. The lattice L has an isotropic subspace of rank 1, but no isotropic subspaces of rank 2.
III. The lattice L admits an isotropic subspace of rank 2.

We refer to these situations as rank 4 lattices of types (0), (II) and (III).

Example 7.3. The matrices

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

give examples of Gram matrices of rank 4 lattices L of types (0), (II) and (III) respectively.

If we choose an orthogonal basis v_1, v_2, v_3, v_4 of $L_{\mathbb{Q}}$ so that $\langle v_i, v_i \rangle = a_i$, then we may assume that $a_1, a_2 > 0$ and that $a_3, a_4 < 0$. A calculation which can be found in [Bru08, §2.3.1] shows that Γ_L is isomorphic to a subgroup of the group of units in a quaternion algebra $\mathcal{Q} = (-a_1a_2, -a_3a_4)$ over the commutative algebra $k = \mathbb{Q} + \mathbb{Q}v_1v_2v_3v_4$, where k is viewed as a sub-algebra of the even Clifford algebra of $L_{\mathbb{Q}}$. In case (0), the algebra $\mathcal{Q}$ is not a matrix algebra and k is a quadratic extension of $\mathbb{Q}$. Thus Γ_L is not isogenous to a product of groups in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. In case (II), the algebra $\mathcal{Q}$ is isomorphic to $M_2(k)$ and k is a quadratic extension of $\mathbb{Q}$. This means that Γ_L is commensurable with $\mathrm{SL}_2(\mathcal{O}_\Delta)$ for $\mathcal{O}_\Delta$ the real quadratic order of discriminant Δ. In case (III), we have that $\mathcal{Q} = M_2(k)$ and that k splits as well. Therefore, Γ_L is commensurable with $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$.

Summarizing, we have that

Proposition 1. *If $\Gamma_L \backslash \mathcal{P}_L$ admits a totally-geodesic divisor which is not a Hodge subvariety, then L has rank 4 and admits a rank 2 isotropic sublattice.*

The possible Hodge subvarieties of $\Gamma_L \backslash \mathcal{P}_L$ equipped with Γ_L are classified by Zarhin.

Theorem 12 (Zarhin [Zar83, Theorem 1.5.1, Theorem 1.6])**.** *The Hodge endomorphism algebra of an irreducible K3 type Hodge structure is either a totally real extension of $\mathbb{Q}$ (RM case) or totally imaginary quadratic extension of a totally real extension of $\mathbb{Q}$ (CM case).*

Therefore, it follows that proper Hodge subvarieties of $\Gamma \backslash \mathcal{P}_L$ either parametrize irreducible Hodge structures with Hodge endomorphism algebra a CM or RM field, or they are contained in a divisor $\mathcal{D}_\alpha$ which is the image in $\mathcal{M}_L$ of $\mathcal{P}_L \cap \alpha^\perp$ where $\alpha \in L$

is a primitive element of L so that $\langle\alpha,\alpha\rangle < 0$ and $\alpha^\perp$ is defined to be the hyperplane in $\mathbb{P}(L_\mathbb{C})$ of elements orthogonal to α. These points correspond to reducible Hodge structures with a factor $\mathbb{Z}\alpha$ of rank 1 and of type $(1,1)$.

We can show that if $\operatorname{rank} L \neq 4$ and our totally-geodesic subvariety is of codimension 1, then the only option is that the totally-geodesic subvariety is $\mathcal{D}_\alpha$ for some $\alpha \in L$. The following proposition bounds the codimension of CM and RM Hodge subvarieties in $\mathcal{M}_L$.

Proposition 2. *If $\mathcal{S}$ is a Hodge subvariety of $\mathcal{P}_L$ parametrizing Hodge structures with* $\mathrm{End}_{\mathrm{Hdg}}$ *equal to E for E an RM field, then the codimension of $\mathcal{S}$ in $\mathcal{P}_L$ is at least 3. If E is a CM field, then the dimension of $\mathcal{S}$ in $\mathcal{P}_L$ is at most* $(\dim \mathcal{P}_L)/2$*, and this bound is achieved only when E is an imaginary quadratic extension of $\mathbb{Q}$.*

Proof. In the case where E is a RM field, this is proved by van Geemen in [vG08, Lemma 3.2]. In the case where E is a CM field, this is easy to check. Let L be a lattice of signature $(2, n-2)$. Let E be a CM field and let E_0 be the maximal totally real sub-field of E, and denote the action of $\mathrm{Gal}(E/E_0) \cong \mathbb{Z}/2$ on $\alpha \in E$ by $\alpha \mapsto \alpha'$. Furthermore, following Zarhin [Zar83, §2.1], there is an embedding of a CM field E into $\mathrm{End}(L)$ which satisfies

$$\langle\alpha(u), v\rangle = \langle u, \alpha'(v)\rangle$$

We then take $L_\mathbb{C} = L_\mathbb{C}$ and let $L_{\sigma,\mathbb{C}}$ be the eigenspace of $L_\mathbb{C}$ associated to an embedding σ of E into $\mathbb{C}$. Fix such an embedding σ. Fixing this action of E on L, a Hodge structure H_p of K3 type on L has CM type with $\mathrm{End}_{\mathrm{Hdg}}(H_p) = E$ acting on L as above if and only if the corresponding period point $\omega \in \mathbb{P}(L_\mathbb{C})$ is in an eigenvector of E, or in other words, $\alpha(\omega) = \sigma(\alpha)\cdot\omega$ for every $\alpha \in E$ and some embedding σ of E into $\mathbb{C}$. Thus the points in $\mathcal{P}_L$ are simply the intersection of $\mathbb{P}(L_{\sigma,\mathbb{C}})$ with $\mathcal{P}_L$.

The eigenspaces $L_{\sigma,\mathbb{C}}$ of $L_\mathbb{C}$ for varying embeddings σ are conjugate, hence the dimension of $\mathbb{P}(L_{\sigma,\mathbb{C}})$ is at most

$$(\operatorname{rank} L)/2 - 1 = (\operatorname{rank} L - 2)/2 = (\dim \mathcal{P}_L)/2.$$

This bound is achieved only when $E_0 = \mathbb{Q}$. Therefore, the subvariety of $\mathcal{P}_L$ parametrizing marked Hodge structures with CM by E acting as above and corresponding to the embedding σ live in a subvariety of dimension at least $(\dim \mathcal{P}_L)/2$ as required. There is a countable ways in which E may be embedded into $\mathrm{End}(L)$, therefore, there is a countable number of smooth submanifolds of $\mathcal{P}_L$ parametrizing CM type Hodge structures. Therefore, no submanifold of $\mathcal{P}_L$ of dimension greater than $(\dim \mathcal{P}_L)/2$ can be covered by submanifolds parametrizing CM type Hodge structures. □

Remark 5. In fact, one can check that the open subset of $\mathbb{P}(L_{\sigma,\mathbb{C}})$ satisfying $\langle\omega,\overline{\omega}\rangle \geq 0$ is contained in $\mathcal{P}_L$. It is enough to check that $\langle\omega,\omega\rangle$ is satisfied for all $\omega \in \mathbb{P}(L_{\sigma,\mathbb{C}})$. Let α be an element of E so that $\alpha \neq \alpha'$. Then if $\omega \in L_{\sigma,\mathbb{C}}$

$$\sigma(\alpha')\langle\omega,\omega\rangle = \langle\omega,\alpha'(\omega)\rangle = \langle\alpha(\omega),\omega\rangle = \sigma(\alpha)\langle\omega,\omega\rangle$$

thus $\langle\omega,\omega\rangle = 0$.

Corollary 1. *If $\Gamma_L\backslash\mathcal{P}_L$ admits a totally-geodesic divisor $\mathcal{S}$ and* $\operatorname{rank} L > 4$ *then $\mathcal{S}$ is the image of $\alpha^\perp \cap \mathcal{P}_L$ for some $\alpha \in L$ with $\langle\alpha,\alpha\rangle < 0$.*

Now let us address the details when rank $L = 4$. When L does not admit an isotropic subspace of rank 2, it is a priori possible that we have totally-geodesic curves in $\mathcal{M}_L$ which parametrize irreducible Hodge structures with CM Hodge endomorphism algebra. We can show that this situation does not occur if L admits an isotropic subspace of rank 1.

Proposition 3. *If L is a rank 4 lattice with signature $(2,2)$, which does not admit a rank 2 isotropic subspace but admits a rank 1 isotropic subspace, then there are no curves in $\mathcal{M}_L$ parametrizing irreducible Hodge structures with Hodge endomorphism algebra a quadratic CM field.*

Proof. Assume that L admits a period with complex multiplication by some CM field E. Therefore $L_{\mathbb{Q}}$ becomes a vector space over E. If there is a 1-dimensional space of irreducible Hodge structures on L with CM action by E, then it follows that $\dim_{\mathbb{Q}} E = 2$ and thus E is a quadratic extension of $\mathbb{Q}$, and we may treat $L_{\mathbb{Q}}$ as a 2-dimensional vector space over E. Furthermore, the pairing $\langle \cdot, \cdot \rangle$ behaves reasonably with respect to this structure. According to Zarhin [Zar83, §2.1], we can define a pair in $\Phi : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \mapsto E$ so that

$$\Phi(x, y) \mapsto \alpha, \quad \text{where} \quad \langle ex, y \rangle = \mathrm{tr}_{E/\mathbb{Q}}(e\alpha) \quad \text{for all } e \in E$$

This pairing is non-degenerate linear, and $\Phi(x, y) = \Phi(y, x)'$. Since L has rank 4 and signature $(2, 2)$, it has an isotropic element, therefore, there is some $x \in L$, we have $\Phi(x, x) = 0$. We then let V be the E-span of x in $L_{\mathbb{Q}}$, which is a rank 2 $\mathbb{Q}$-subspace. It is clear that $\Phi(ex, ex) = \mathrm{nrm}_{E/\mathbb{Q}}(e)\Phi(x, x) = 0$ for any element of V, thus this provides the required rank 2 isotropic subspace. $\square$

Finally, we have proved:

Theorem 13. *If $\operatorname{rank} L \geq 5$ or $\operatorname{rank} L = 4$ and L contains no rank 2 isotropic sublattice but contains a rank 1 isotropic sublattice, then totally-geodesic codimension 1 subvarieties of $\mathcal{M}_L$ are just the divisors $\mathcal{D}_{L'}$ for a lattice L' of rank equal to $\operatorname{rank} L - 1$.*

The main theorem of this Part is then:

Theorem 14 (Reinterpretation in terms of PFDEs)**.** *Let $\mathcal{X}$ be a maximal projective family of K3 surfaces over a Zariski open subset B of $\mathbb{C}^n$ so that the lattice $L_{\mathcal{X}}$ satisfies the conditions of Theorem 13. Then there is a bijection between rational solutions to the set of PDEs in Theorem 10 and rational divisors in B on which the generic rank of $\mathrm{Pic}(\mathcal{X}_t)$ goes up by 1.*

Remark 6. It is shown by Müller-Stach-Viehweg-Zuo [MSVZ09] that one can detect Shimura subvarieties of $\mathcal{M}_L$ in terms of the behaviour of specific Higgs bundles. It might be interesting to reinterpret our results in terms of Higgs bunds and determine the relation between our results and the results of [MSVZ09] directly.

7.1. **Explicit form when $\dim B = 2$.** Here we give our results in explicit details when the dimension of the base of the maximal family $\mathcal{X}$ is 2. We give a clean differential geometric classification of all geodesic subvarieties in terms of Picard-Fuchs equations and their restrictions.

Using the notation of Sasaki-Yoshida [SY88] in the following computations. They write the uniformizing differential equations in the case where $n = 2$ as

$$\frac{\partial^2}{\partial x^2} = \ell(x,y)\frac{\partial^2}{\partial x \partial y} + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + p(x,y)$$
$$\frac{\partial^2}{\partial y^2} = m(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y} + q(x,y)$$

Let us define

$$\begin{aligned} B_1 &= (A + \ell q)/\mathrm{dis} & C_1 &= (C + q)/\mathrm{dis} \\ B_2 &= (B + p)/\mathrm{dis} & C_2 &= (D + mp)/\mathrm{dis} \\ B_3 &= (\ell_y + a + bm + \ell(m_x + d + c\ell))/\mathrm{dis} & C_3 &= (m_x + d + c\ell + m(\ell_y + a + bm))/\mathrm{dis} \end{aligned}$$

where

$$\mathrm{dis} = 1 - \ell m$$

and

$$\begin{aligned} A &= a_y + bc + \ell(c_x + ac) & B &= b_y + \ell(d_y + bc) \\ C &= c_x + ac + m(a_y + bc) & D &= d_x + bc + m(b_y + bd) \end{aligned}$$

with

$$\widetilde{B}_i = B_i x_t^2(\ell x_t + 3y_t), \widetilde{C}_i = C_i y_t^2(m y_t + 3x_t)$$

and let

$$\begin{aligned} A_{1,1} &= \widetilde{B}_3 + \widetilde{C}_3 + x_t^3(a\ell + b + \ell_x) + y_t^3(dm + c + m_y) \\ &\quad + 3(x_t x_{tt}\ell + y_t y_{tt} m + y_t x_{tt} + x_t y_{tt}) \\ A_{1,2} &= \widetilde{B}_1 + \widetilde{C}_1 + x_t^3(a^2 + a_x + p) + y_t^3(cd + c_y) + 3(a x_t x_{tt} + c y_t y_{tt}) + x_{ttt} \\ A_{1,3} &= \widetilde{B}_2 + \widetilde{C}_2 + x_t^3(ab + b_x) + y_t^3(d^2 + d_y + q) + 3(b x_t x_{tt} + d y_t y_{tt}) + y_{ttt} \end{aligned}$$

We may then deduce the following theorem:

Theorem 15. *Let us take a maximal projective family of K3 surfaces $\mathcal{X}$ over a quasi-affine base B and let S be a parametrized curve inside of B. The curve S is of unitary type if and only if $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$. If the determinant of the matrix*

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ \ell x_t^2 + m y_t^2 + 2x_t y_t & a x_t^2 + c y_t^2 + x_{tt} & b x_t^2 + d y_t^2 + y_{tt} \\ 0 & x_t & y_t \end{pmatrix} \tag{5}$$

is zero and $\ell x_t^2 + m y_t^2 + 2x_t y_t \neq 0$, then the image of S is a divisor $\mathcal{D}_\alpha$ inside of B.

Corollary 2. *If L has no isotropic vectors, then solutions to $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$ are isolated. If the lattice L has no rank 2 isotropic subspace but has a rank 1 isotropic subspace, then the equation $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$ has no algebraic solutions. If L has a rank 2 isotropic subspace then there are two one parameter spaces of solutions to $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$.*

Remark 7. In the case where $L = \mathrm{H} \oplus \mathrm{H}$, this recovers very easily the box equation described in [CDLW09]. In this case, there is a fiber-wise Hodge-theoretic isometry between a family $\mathcal{E}_{j_1} \times \mathcal{E}_{j_2}$ of of products of elliptic curves over the product $\mathbb{P}^1_{j_1} \times \mathbb{P}^1_{j_2}$ and the fibers of a family of K3 surfaces $\mathcal{X}_{j_1,j_2}$. The explicit Picard-Fuchs equation for this family is presented in [CDLW09, §3.3].

The solutions to the equation $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$ in this case are precisely places where one of j_1 or j_2 is constant. This is generally the case. In the case where such solutions exist, it is because there is a finite map $(\Gamma_1\backslash\mathbb{H}) \times (\Gamma_2\backslash\mathbb{H})$ for subgroups Γ_1 and Γ_2 arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Then algebraic solutions to $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$ are the image of curves $p \times (\Gamma_2\backslash\mathbb{H})$ for some point $p \in (\Gamma_1\backslash\mathbb{H})$.

Such families of K3 surfaces are related to the so-called "basic" families of Peters-Saito [SZ91], which exhibit non-rigid variations of Hodge structure of K3 type.

Remark 8. In Section 8.3, we will write down examples of Picard-Fuchs ideals in the normal form of Sasaki-Yoshida written above. The reader is invited to experiment with the resulting PDE obtained by substituting the resulting equations into Theorem 15.

Remark 9. It can be shown that $\ell x_t^2 + m y_t^2 + 2x_t y_t$ is the product of the coefficient of the Hodge metric on the parametrized curve and the so-called *Griffiths-Yukawa coupling* of the family $\mathcal{X}_\Phi$. This establishes a link between our work and the work of Müller-Stach-Viehweg-Zuo [MSVZ09].

Remark 10. The situation when L admits no isotropic sublattice and has rank 4 is a mysterious one. It is the only situation in which rigid solutions to the equation $\ell x_t^2 + m y_t^2 + 2x_t y_t = 0$ can possibly exist. To our knowledge, no maximal family of K3 surfaces with generic transcendental lattice L of this type exists in the literature.

8. Computational techniques for detecting jumps in Picard rank

As mentioned earlier, the theorems described in the previous section does not seem useful in practice. If we are given $\mathrm{PF}(\mathcal{X})$ for a maximal family of K3 surfaces, then in most cases the system of PDEs satisfied by parametrizations of Shimura subvarieties of codimension 1 can be very complicated to write down. The only exception to this that we know of is the case described in [CDLW09] and generalizations thereof, where the family $\mathcal{X}$ is a family of K3 surfaces over a product of modular curves.

In these cases, checking whether a given curve satisfies these equations can be a challenging computational task if the parametrized curve is the slightest bit complicated. Therefore, our goal in this section is to describe a situation of interest and describe how different computational tools can be used to perform this task in a specific example.

8.1. **Modular subvarieties of $\mathcal{A}_2$.** Here we give an effective method of computing whether a subvariety of $\mathcal{A}_2$, the moduli space of principally polarized abelian surfaces supports a Humbert surface or a Shimura curve. Recall that, according to Morrison [Mor84], if we begin with an abelian surface A, then the associated Kummer surface $\mathrm{Kum}(A)$ admits a double cover $\mathrm{SI}(A)$ satisfying the property that $\mathrm{T}(A)$, the transcendental lattice of A is Hodge isometric to the transcendental lattice of $\mathrm{SI}(A)$. We call $\mathrm{SI}(A)$ the Shioda-Inose partner of A. Gritsenko-Hulek [GH98] and Peters [Pet86] show that there is an isomorphism between the moduli space of K3 surfaces lattice polarized by the lattice $H \oplus E_8 \oplus E_8 \oplus \langle -2 \rangle$. This correspondence can be realized explicitly. Starting with a genus 2 curve C Kumar gives a procedure for writing the K3 surfaces $\mathrm{SI}(J(C))$ in terms of its Igusa invariants, I_2, I_4, I_6 and I_{10}. These invariants form weighted projective coordinates on a birational model of $\mathcal{A}_2$, and have weights corresponding to their subscripts. In particular, following Kumar [Kum08], we write $\mathrm{SI}(J(C))$ as

$$y^2 = x^3 - t^3\left(\frac{I_4}{12}t + 1\right)x + t^5\left(\frac{I_{10}}{4}t^2 + \frac{(I_2I_4 - 3I_6)}{108}t + \frac{I_2}{24}\right).$$

as affine hypersurfaces, or alternately these surfaces may be regardes as elliptically fibered surfaces over $\mathbb{P}^1$. In [DHMW16], we showed how to write these surfaces as generic hypersurfaces in $\mathrm{WP}(3,4,10,13)$ as

$$x_0^{10} + bx_0^6x_1^3 + \left(\frac{d}{4}\right)x_0^2x_1^6 + 3ax_0^4x_1^2x_2 - \left(\frac{c}{2}\right)x_1^5x_2 + x_0^2x_1x_2^2 + 2x_0x_1x_2x_3 - 4x_2^3 + x_1x_3^2 = 0$$

with

$$a = \frac{I_4}{36}, \quad b = -\frac{I_2I_4}{216} + \frac{I_6}{72}, \quad c = \frac{I_{10}}{4}, \quad d = \frac{I_2I_{10}}{96}.$$

Here a, b, c, d are coordinates coming from the work of Clingher-Doran [CD12]. We call this family $\mathcal{X}_{\mathrm{Igusa}}$. Assume we have a parametrized subvariety

$$\Phi : U \subseteq \mathbb{C}^k \to \mathbb{WP}(1,2,3,5)$$

for $k = 1,2,3$, and we define $\mathcal{X}_\Phi$ to be the family of K3 surfaces obtained by pulling back $\mathcal{X}_{\mathrm{Igusa}}$ along the map Φ. One may use computational tools to compute the Picard-Fuchs equation of $\mathcal{X}_\Phi$ using the Griffiths-Dwork technique. This technique is described in detail in a number of places (see Cox-Katz [CK99], Clingher-Doran-Lewis-Whitcher [CDLW09] and Doran-Harder-Movasati-Whitcher [DHMW16] for instance), so we do not review it here. This algorithm is guaranteed to produce the Picard-Fuchs ideal of $\mathcal{X}_\Phi$ if the general member of $\mathcal{X}_\Phi$ is a quasi-smooth hypersurface in $\mathbb{WP}(3,4,10,13)$.

8.2. **Detecting Humbert surfaces.** Let Φ be a locally injective map from a Zariski open subset U of $\mathbb{C}^2$ to $\mathbb{WP}(1,2,3,5)$ given by

$$\Phi : (u,v) \mapsto [I_2(u,v) : I_4(u,v) : I_6(u,v) : I_{10}(u,v)].$$

Let the family $\mathcal{X}_\Phi$ be the family of K3 surfaces over U pulled back from $\mathcal{X}_{\mathrm{Igusa}}$ along Φ. A Humbert surface in $\mathcal{A}_2$ is a subvariety $\mathcal{H}_\Delta$ which parametrizes abelian surfaces with real multiplication by the real quadratic order $\mathcal{O}_\Delta$ of discriminant Δ in a real quadratic extension of $\mathbb{Q}$. For out purposes, these can be defined as subloci $\mathcal{D}_\alpha$ of $\mathcal{M}_{H\oplus H\oplus\langle -2\rangle}$. The divisors $\mathcal{D}_\alpha$ are in bijection with Humbert surfaces and this correspondence is made precise in several places, e.g. [EK14]. The Humbert surface of discriminant Δ is a finite cover of $\mathcal{M}_{L_\Delta}$ where

$$L_\Delta = H \oplus \begin{pmatrix} -2 & i \\ i & 2a \end{pmatrix}$$

where $4a - i^2 = -\Delta > 0$ and $i = 0$ if $\Delta \equiv 0 \bmod 4$ and $i = 1$ if $\Delta \equiv 3 \bmod 4$. We will call a curve C in $\mathcal{A}_2 = \mathcal{M}_{H\oplus H\oplus\langle -2\rangle}$ a Shimura curve if there is some Humbert surface

$\mathcal{H}_\Delta$ so that the image of C in the corresponding space $\mathcal{M}_{L_\Delta}$ is a divisor $\mathcal{D}_\alpha$ for some α. This name is appropriate since such curves are quotients of $\mathbb{H}$ by arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})$.

As a Corollary to Theorem 14 and Theorem 10 we have:

Proposition 4. *The image of Φ is a Humbert surface if and only if for all periods F of $\mathcal{X}_\Phi$, there is a $\mathbb{C}(u,v)$ linear relations between F_{uu}, F_{uv}, F_u, F_v and F and a $\mathbb{C}(u,v)$-linear relation between F_{vv}, F_{uv}, F_u, F_v and F.*

Therefore it suffices to compute relations between periods using the Griffiths-Dwork method to detect whether a subvariety is a Humbert surface. This can be implemented on a computer. For instance the authors have had success performing such computations with MAGMA and Macaulay2, though implementation in MAGMA tends to be faster due to the fact that it uses more efficient Gröbner basis algorithms.

8.3. **Examples of Humbert surfaces.** Elkies and Kumar [EK14] have given equations for Humbert surfaces of square-free discriminant less than 100, and subsequently Kumar [Kum08] has given parametrizations for a large number of Humbert surfaces with square discriminant. Therefore, this proposition does not have much practical value, since any Humbert surface that current machinery can efficiently check has already been parametrized. We may, however use this to corroborate their computations. We have written MAGMA code (available upon request) that is capable of, given a prospective 2-parameter map, checking whether the image lies in a Humbert surface. We can use this code to produce uniformizing differential equations of Humbert surfaces of low discriminant, using the parametrizations of Elkies and Kumar. These are listed for discriminants up to 17.

8.4. $\mathcal{H}_8$**.** let

$$D_8 = (4st + 4t^2 + s - 2t - 2)$$

Then

$$l = -\frac{t(4st + 4t^2 - s + 4t + 1)}{sD_8} \qquad m = \frac{s(16st + 12t^2 + 3s - 2t - 4)}{tD_8}$$
$$a = -\frac{t(4t - 1)}{2sD_8} \qquad c = \frac{(8st + 10t^2 + s - 8t - 2)}{tD_8}$$
$$b = -\frac{t(16st + 12t^2 + 3s - 2t - 4)}{2sD_8} \qquad d = \frac{2s(s + 2t - 3)}{tD_8}$$
$$p = -\frac{t}{sD_8} \qquad q = \frac{(s + 2t - 2)}{tD_8}$$

8.5. $\mathcal{H}_{12}$**.** Let

$$D_{12} = (s-1)(s+1)(3s^2 - 4).$$

Then

$$l = -\frac{(9s^4 - 27s^2 + 27t + 16)}{sD_{12}} \qquad m = -\frac{(s^4 - s^2 - 3t)}{stD_{12}}$$
$$a = -\frac{9(2s^2 - 3)}{D_{12}} \qquad c = -\frac{(4s^2 - 5)(s-1)(s+1)}{tD_{12}}$$
$$b = -\frac{(24s^4 - 44s^2 + 27t + 16)}{2sD_{12}} \qquad d = \frac{3(s-1)(s+1)}{2stD_{12}}$$
$$p = -\frac{2(3s^2 - 4)}{D_{12}} \qquad q = 0$$

8.6. $\mathcal{H}_{13}$. Let

$$D_{13} = (-16t^2 + 49ts + 6s^2 - 32t - 22s + 20)$$

Then

$$l = \frac{-2t(16t^2 - 58ts + 3s^2 - 16t - 10s + 4)}{sD_{13}} \qquad m = \frac{-2s(4t^2 - 10ts + 3s^2 + 46t - 12s + 12)}{tD_{13}}$$
$$a = \frac{-2t(-29t + 3s - 5)}{sD_{13}} \qquad c = \frac{2(-20t^2 + 46ts + 3s^2 - 46t - 11s + 10)}{tD_{13}}$$
$$b = \frac{4(-8t^2 + 31ts + 3s^2 - 6t - 8s + 5)}{sD_{13}} \qquad d = \frac{2s(-4t + 5s - 23)}{tD_{13}}$$
$$p = \frac{26t}{sD_{13}} \qquad q = \frac{2(-4t + 5s - 10)}{tD_{13}}$$

8.7. $\mathcal{H}_{17}$. Let

$$D_{17} = (26t^3 + 49t^2 + 184ts + 28t + 93s + 5)$$

Then

$$l = \frac{2(4t^4 + 12t^3 + 54t^2s + 13t^2 + 47ts + 8s^2 + 6t + 10s + 1)}{sD_{17}} \qquad m = \frac{4s(19t^2 + 12t + 68s + 1)}{D_{17}}$$
$$a = \frac{(54t^2 + 47t + 16s + 10)}{sD_{17}} \qquad c = \frac{2(39t^2 + 41t + 136s + 10)}{D_{17}}$$
$$b = \frac{(34t^3 + 65t^2 + 340ts + 38t + 171s + 7)}{sD_{17}} \qquad d = \frac{4s(19t + 6)}{D_{17}}$$
$$p = \frac{17(2t + 1)}{sD_{17}} \qquad q = \frac{2(13t + 5)}{D_{17}}$$

8.8. **Detecting Shimura curves.** Again, let $\mathcal{X}_\Phi$ be the family of K3 surface obtained by pulling back $\mathcal{X}_{\mathrm{Igusa}}$ along the map $\Phi(t)$, which for this section will be a locally injective map from a Zariski open subset of $\mathbb{C}$ to $\mathbb{WP}(1, 2, 3, 5)$. We will assume for a generic point t in the image of Φ, the K3 surface $\mathcal{X}_{\Phi(t)}$ is quasismooth in $\mathbb{WP}(3, 4, 10, 13)$.

Proposition 5. *Assume that the map Φ is generically injective into $\mathcal{A}_2$. Then the image of Φ is a Shimura curve if and only if there is a $\mathbb{C}(t)$-linear relation between F, F_t, F_{tt} and F_{ttt} for any period function F of $\mathcal{X}_\Phi$ and there is no $\mathbb{C}(t)$-linear relation between F, F_t and F_{tt}.*

Such curves can be obtained by taking intersections of Humbert surfaces as we describe in the following section.

8.9. **Parametrizing Shimura curves and computing their Picard-Fuchs equations.** A good way of obtaining parametrizations of Shimura curves is by taking intersections of Humbert surfaces. This approach was pioneered by Hashimoto-Murabayashi in [HM95], and has more recently been applied by Gruenwald [Gru08] and Nagano [Nag15]. The current state of affairs is, to our knowledge, as follows. In his thesis, Gruenwald worked out hypersurface equations for many Humbert surfaces in terms of a specific parametrization of $\mathcal{A}_2$. These hypersurfaces can be rewritten in terms of Igusa invariants. Using the parametrization of Kumar and Elkies of a Humbert surface $\mathcal{H}_{d_2}$, one can pull back the hypersurface equation for $\mathcal{H}_{d_1}$ to parametrized Humbert surfaces. The result is then implicit equations in terms of (u, v) coordinates for curves of intersection between $\mathcal{H}_{d_1}$ and $\mathcal{H}_{d_2}$. After obtaining a parametrization of any of these curves of intersection, the result is a parametrization of a Shimura curve.

Gruenwald then explains how to identify specific Shimura curves in terms of their associated Eichler orders by noting that, in general, a given Shimura curve can be identified by arithmetic methods as the unique curve of intersection between some number of Humbert surfaces. If one chooses a Shimura curve $\mathcal{C}$ so that $\mathcal{C} = \cap_{d_i \in S} \mathcal{H}_{d_i}$ for some set of discriminants S, then all one needs to identify $\mathcal{C}$ is a parametrization of $\mathcal{H}_{d_1}$ and hypersurface equations for $\mathcal{H}_{d_i}$ with $i \neq 1$. Then one can proceed to identify $\mathcal{C}$. If $\mathcal{C}$ has genus 0, then one may easily obtain a parametrization.

We can use the Griffiths-Dwork technique then to compute the Picard-Fuchs uniformizing equation for the relevant Shimura curve. For instance, we can look at a curve in $\mathcal{H}_8 \cap \mathcal{H}_{12}$. The hypersurface $\mathcal{H}_{12}$ has Igusa parametrization

$$\begin{aligned}
I_2(e,f) &= \frac{-8(2f^3 - 2f^2 - 3f + 3e + 3)}{f-1} \\
I_4(e,f) &= 4(f^4 + 15fe + 9e) \\
I_6(e,f) &= \frac{4(6f^7 - 6f^6 - 8f^5 + 67f^4e + 8f^4 - 38f^3e - 141f^2e + 102fe^2 + 48fe + 90e^2 + 72e)}{f-1} \\
I_{10}(e,f) &= -4(f-1)e^3.
\end{aligned}$$

The hypersurface equation for $\mathcal{H}_8$ has a complicated expression in terms of Igusa invariants, but has homogeneous degree 60. Substituting the parametrization above into the implicit equation for $\mathcal{H}_8$, one sees that the intersection of the parametrized part of $\mathcal{H}_{12}$ and $\mathcal{H}_8$ is comprised of seven curves. Here we will describe the simplest such curve, which arises when $f = 0$. This curve is parametrized by the variable t, and in particular, we get a parametrization

$$I_2(t) = 24t + 24 \qquad I_4(t) = 36t$$
$$I_6(t) = 360t^2 + 288t \qquad I_{10}(t) = 4t^3$$

The Picard-Fuchs operator for this curve is

$$t^2 \frac{\mathrm{d}^3}{\mathrm{d}t^3} + \frac{t(39t-2)}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3} + \frac{27t^2}{2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t(27t+16)}{3}.$$

In the literature, Picard-Fuchs equations for modular or Shimura curves are presented as rank two equations, whereas this equation is given by a rank 3 equation. The usual presentation is the *symmetric square root* of the equation above. The algebraic relation between this equation and its symmetric square root is explained in detail by C.F.D. in [Dor00]. In Hashimoto-Murabayashi [HM95] and Nagano [Nag15], it is shown that this curve is the image of the Shimura curve associated to the maximal order of discriminant 6.

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