THE DORAN-HARDER-THOMPSON CONJECTURE FOR TORIC COMPLETE INTERSECTIONS

CHARLES F. DORAN, JORDAN KOSTIUK, AND FENGLONG YOU

ABSTRACT. Given a Tyurin degeneration of a Calabi-Yau complete intersection in a toric variety, we prove gluing formulas relating the generalized functional invariants, periods, and I-functions of the mirror Calabi-Yau family and those of the two mirror Landau-Ginzburg models. Our proof makes explicit the "gluing/splitting" of fibrations in the Doran-Harder-Thompson mirror conjecture. Our gluing formula implies an identity, obtained by composition with their respective mirror maps, that relates the absolute Gromov-Witten invariants for the Calabi-Yaus and relative Gromov-Witten invariants for the quasi-Fanos.

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1. Introduction

1.1. The Doran-Harder-Thompson conjecture.

Classical mirror symmetry is a conjecture relating properties of a Calabi-Yau variety and properties of its mirror Calabi-Yau variety. The duality has been generalized to Fano varieties. By [EHX97], the mirror to a Fano variety X is a Landau-Ginzburg model $(X^\vee, W)$ such that $X^\vee$ is a Kähler manifold and $W : X^\vee \to \mathbb{C}$ is a proper map called the superpotential.

Mirror symmetry for Landau-Ginzburg models can be generalized to varieties beyond the Fano case. In particular, the Landau-Ginzburg model for a quasi-Fano variety is defined by A. Harder [Har16]. A smooth variety X is quasi-Fano if its anticanonical linear system contains a smooth Calabi-Yau member and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. A degeneration of Calabi-Yau varieties is given by $\mathcal{V} \to \Delta$, where $\Delta \subset \mathbb{C}$ is the unit disk. The degeneration is called a Tyurin degeneration if the total space $\mathcal{V}$ is smooth and the central fiber consists of two quasi-Fano varieties X_1 and X_2 which meet normally along a common smooth anticanonical (Calabi-Yau) divisor X_0.

Motivated in part by a question of A. Tyurin [Tyu04], Doran-Harder-Thompson formulated a conjecture relating mirror symmetry for a Calabi-Yau variety X sarising as a smooth fiber of a Tyurin degeneration $\mathcal{V} \to \Delta$, and mirror symmetry for the quasi-Fano varieties X_1 and X_2 in the central fiber of $\mathcal{V}$. The Doran-Harder-Thompson conjecture states that one should be able to glue the Landau-Ginzburg models $W_i : X_i^\vee \to \mathbb{C}$ of the pair (X_i, X_0) for $i = 1, 2$ to a Calabi-Yau variety $X^\vee$ which is mirror to X with the fibers of the superpotentials gluing to a fibration $W : X^\vee \to \mathbb{P}^1$. Furthermore, the compact fibers of the Landau-Ginzburg models consist of Calabi-Yau manifolds mirror to the common anticanonical divisor X_0.

An enormous amount of evidence has already been collected in favor of this conjecture. The topological version of the conjecture has been proved in [DHT17b] by computing the corresponding Euler numbers. In this case the gluing could be viewed as taking place in the symplectic category, with monodromies aligning with mirror autoequivalences in the bounded derived category on the Tyurin degeneration side. In the case of elliptic curves, the conjecture is proved by Kanazawa [Kan17] via SYZ mirror symmetry. The original paper [DHT17b] included evidence of compatibility of the DHT conjecture with the Dolgachev-Nikulin-Pinkham formulation of mirror symmetry for lattice polarized $K3$ surfaces. This was refined by Doran-Thompson, who obtained mirror notions of lattice polarization for rational elliptic surfaces and for del Pezzo surfaces coming from, respectively, splitting of fibrations and Tyurin degeneration of lattice polarized $K3$ surfaces [DT18].

1.2. The gluing formula.

In this paper, we study the Tyurin degeneration of complete intersections in toric varieties, and derive a gluing formula (in the complex category) for Landau-Ginzburg models.

We first analyze two special examples, namely, a conifold transition $\tilde{Q}_5$ of the quintic threefold and the quintic threefold Q_5 itself. In these two examples, the mirror Calabi-Yau threefold is fibered by M_2-polarized ("mirror quartic") $K3$ surfaces and the mirror LG models are M_2-polarized families of $K3$ surfaces. The key ingredients of the construction are the generalized functional invariants which determine the families of M_2-polarized $K3$ surfaces up to isomorphism. Therefore, gluing LG models must reduce to the gluing of generalized functional invariants.

We show that the gluing formula for generalized functional invariants is simply a product relation between them. Furthermore, we use generalized functional invariants to define the holomorphic periods of LG models as pullbacks of the holomorphic period of mirror quartic $K3$ surfaces. As shown by Doran-Malmendier [DM15] the holomorphic 3-form periods of $K3$ surface fibered Calabi-Yau threefolds can also be computed from the generalized functional invariants and the holomorphic period of the fiber mirror quartic $K3$ surfaces. They are expressed, equivalently, in terms of the Euler integral transform, the Hadamard product of power series, and the middle convolution of ordinary differential equations. As a result, the gluing formula for generalized functional invariants implies the gluing formula for periods as well as the gluing formula for the bases of solutions to the corresponding Picard-Fuchs equations (the full I-functions).

We generalize the gluing formula to toric complete intersections. After suitable identification of variables, we obtain the following relation among periods.

Theorem 1.1 (= Theorem 5.11). *Let X be a Calabi-Yau complete intersection in a toric variety. Consider the Tyurin degeneration of X via a refinement of the nef partition. Let X_1 and $\tilde{X}_2$ be the corresponding quasi-Fano varieties, where X_1 and X_2 are toric complete intersections coming from the refinement of the nef partition and we blow up X_2 to obtain $\tilde{X}_2$. Let X_0 be the smooth anticanonical divisor which lies in the intersection of X_1 and $\tilde{X}_2$. The following Hadamard product relation holds*

$$f_0^X(q) \star_q f_0^{X_0}(q) = \frac{1}{2\pi i} \oint f_0^{X_1}(q, y) \star_q f_0^{\tilde{X}_2}(q, y) \frac{dy}{y}.$$

We also have the Hadamard product relation among the solutions for the Picard-Fuchs equations.

Theorem 1.2 (= Theorem 5.12). *The Hadamard product relation among the bases of solutions to Picard-Fuchs equations is*

$$I^X(q) \star_q I^{X_0}(q) = \frac{1}{2\pi i} \oint I^{X_1}(q, y) \star_q I^{\tilde{X}_2}(q, y) \frac{dy}{y}.$$

We refer to Section 5.3 for explanation of the notation.

In this paper, we focus on toric complete intersections, but the gluing formulas work for more general Calabi-Yau manifolds. In particular, in a forthcoming paper, we will prove the gluing formulas for the Tyurin degenerations of Calabi-Yau threefolds mirror to the M_N-polarized $K3$ surface fibered Calabi-Yau threefolds classified and constructed in full in [DHNT17].

1.3. **Gromov-Witten invariants.** According to the mirror theorem proved by Givental [Giv98] and Lian-Liu-Yau [LLY97], Gromov-Witten invariants of a Calabi-Yau variety are related to periods of the mirror. Following Givental [Giv98], on the A-side, we consider a generating function of Gromov-Witten invariants of the Calabi-Yau variety X, called the J-function. On the B-side, we consider the I-function, which encodes a basis of the corresponding Picard-Fuchs equation. The J-function and I-function are related by the mirror map. A mirror theorem for smooth pairs has recently been proved by Fan-Tseng-You [FTY19] using Givental's formalism for relative Gromov-Witten theory developed by Fan-Wu-You [FWY19b]. Given a quasi-Fano variety X and its anticanonical divisor D, similar to the absolute case, we can consider the relative J-function for the pair (X, D). Under suitable assumptions, the relative I-function is related to the relative J-function via a relative mirror map. In general, the relative I-function lies in Givental's Lagrangian cone for relative Gromov-Witten theory defined in [FWY19b].

The gluing formula for periods implies a relation among absolute Gromov-Witten invariants of the Calabi-Yau variety X and the relative Gromov-Witten invariants of the pairs (X_1, D) and (X_2, D) via mirror maps. On the other hand, absolute Gromov-Witten invariants of X and relative Gromov-Witten invariants of (X_1, D) and (X_2, D) are related by the degeneration formula. While we may consider our gluing formula as the B-model counterpart of the degeneration formula, the precise compatibility between the gluing formula and the degeneration formula is not yet known.

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2. Preparation

2.1. **Generalized functional invariants.** In this section, we recall the definition of generalized functional invariants for threefolds. We also give a definition for generalized functional invariants in all dimensions.

We begin by reviewing some aspects of Kodaira's theory of ellipic surfaces, for which we refer to Kodaira's original papers [Kod60, Kod63] for a complete treatment. Consider the following differential equation:

$$(1) \qquad \frac{d^2 f}{dt^2} + \frac{1}{t}\frac{df}{dt} + \frac{\frac{31}{144}t - \frac{1}{36}}{t^2(t-1)^2}f = 0.$$

Then, (1) is a Fuchsian differential equation with regular singularities at $t = 0, 1, \infty$. It admits a basis of solution ω_1, ω_2 for which the quotient $\tau = \frac{\omega_1}{\omega_2}$ defines a multi-valued function to the upper half-plane:

$$\mathbf{P}^1_t \Rrightarrow \mathfrak{H}.$$

The single-valued inverse function $J\colon \mathfrak{H} \to \mathbf{P}^1_t$ is the classical modular J-function.

The differential equation (1) is the Picard-Fuchs differential equation for the following family of elliptic curves:

$$(2) \qquad\qquad y^2 = 4x^3 - \frac{27t}{t-1}x - \frac{27t}{t-1}.$$

For each t, the elliptic curve E_t is the elliptic curve with J-invariant equal to t.

Given an arbitrary family of elliptic curves $f\colon \mathcal{E} \to T$, the period map $T \Rrightarrow \mathfrak{H}$ is the multi-valued function $\tau = \frac{\omega_1}{\omega_2}$ determined by choosing a suitable basis of period functions. The composition of the period map with the modular J-function is a *rational function* $\mathcal{J}\colon T \to \mathbf{P}^1$ and is called *Kodaira's functional invariant* associated to the family $f\colon \mathcal{E} \to T$. For each $t \in T$, $\mathcal{J}(t)$ is the J-invariant of the elliptic curve E_t.

The *homological invariant* of the family $f\colon \mathcal{E} \to T$ is the local system $\mathcal{G} = R^1 f_* \mathbf{Z}$ of first cohomology groups of the family. Kodaira showed in [Kod60] that the functional and homological invariants determine the isomorphism class of the elliptic surface. The functional invariant $\mathcal{J}$ is sufficient to determine the *projective monodromy* representation of the homological invariant. It follows that the functional invariant, together with a representation $\pi_1(T) \to \{\pm 1\}$ determine the elliptic surface up to isomorphism.

A similar story holds for M_N-polarized families of $K3$ surfaces, as was proved in [DHNT17]. The period domain for such families is equal to $\mathfrak{H}$, and the moduli space is $\mathcal{M}_{M_N} \cong X_0(N)^+$, the quotient of the modular curve $X_0(N)$ by the Fricke involution. To any family $f\colon \mathcal{X} \to B$ of M_N-polarized $K3$ surface, we associate the *generalized functional invariant* $g\colon B \to X_0(N)^+$ which is defined by sending each point $b \in B$ to the corresponding point in moduli of the fiber $f^{-1}(b)$. The generalized functional invariants associated to families of M_N-polarized $K3$ surfaces have even tighter control over the families: a family $f\colon \mathcal{X} \to B$ is determined up to isomorphism by the generalized functional invariant [DHNT17].

Example 2.1. Choose a coordinate λ on the modular curve $X_0(2)^+$ for which $\lambda = 0$ is the cusp, $\lambda = \frac{1}{256}$ is the order 2 orbifold point and $\lambda = \infty$ is the order 4 orbifold point. Let $\mathcal{X}_2$ denote the quartic mirror family of $K3$ surfaces defined by

$$\{\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - 1) + \lambda = 0\}.$$

Then, the fibers of $\mathcal{X}_2$ are M_2-polarized $K3$ surfaces for

$$\lambda \in X_0(2)^+ - \left\{0, \frac{1}{256}, \infty\right\}.$$

The singular fiber types of $\mathcal{X}_2$ are determined in [DHNT16]. The singular fiber at the cusp $\lambda = 0$ is a singular $K3$ surface of type III containing 4 components; the monodromy transformation of the corresponding variation of Hodge structure is maximally unipotent. At $\lambda = \frac{1}{256}$, the singular fiber is a singular $K3$ surface containing an A_1 singularity; the monodromy transformation on the VHS is conjugate to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The singular fiber at $\lambda = \infty$ is a singular fiber with 31 components and the monodromy is conjugate to

$$\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

According to the results described above, any M_2-polarized family of $K3$ surfaces is birational to the pull-back of $\mathcal{X}_2$ via the generalized functional invariant map $g \colon B \to X_0(2)^+$. The types of singular fibers appearing in the pull-back is determined by the ramification profile and is worked out in detail in [DHNT16].

The Picard-Fuchs operator corresponding to the VHS of the quartic mirror family is

(3) $$\delta^3 - 256\lambda \left(\delta + \frac{1}{4}\right)\left(\delta + \frac{1}{2}\right)\left(\delta + \frac{3}{4}\right),$$

and the holomorphic and logarithmic solutions to this ODE are given by

$$f_0^{\mathcal{X}_2} = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4}\lambda^n,$$

$$f_1^{\mathcal{X}_2} = f_0^{\mathcal{X}_2} \cdot \log\lambda + 4\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4} \cdot \left(\sum_{j=n+1}^{4n} \frac{1}{j}\right)\lambda^n.$$

With this as motivation, we make the following definition:

Definition 2.2. Let $f \colon \mathcal{X} \to B$ be a family of Calabi-Yau manifolds. The morphism $g \colon B \to \mathcal{M}$, taking a point in the base to the corresponding point in the moduli space $\mathcal{M}$ of the fiber is called the *generalized functional invariant* of the family.

Remark 2.3. The construction of $\mathcal{M}$ is strongly dependant on the type of Calabi-Yau fibers under consideration. If any confusion is likely to arise, this will be addressed on a case-by-case basis.

Remark 2.4. In this paper, we will be using Hori-Vafa mirror because it can be used to describe both the mirrors of quasi-Fano varieties and the mirrors of Calabi-Yau complete intersections in toric varities. The difference between generalized functional invariants in our paper and generalized functional invariants in previous literature (e.g. [CDL+16]) is the following. First of all, in [CDL+16], authors considered Batyrev mirrors of Calabi-Yau complete intersections in toric varities. Batyrev mirrors and Hori-Vafa mirrors are equivelent under change of variables. Secondly, generalized functional invariants in [CDL+16, Section 4.1] are defined using the M-polarized K3 surface in [CDL+16, Theorem 4.3]. While in this paper, we will consider generalized functional invariant maps to the mirror family of the Calabi-Yau $(n-1)$-fold in the Tyurin degeneration of Calabi-Yau n-fold. This is because we want to find the relation among generalized functional invariants of two LG models and mirror Calabi-Yau n-fold with respect to the mirror of the intersection of two quasi-Fanos. Therefore, after change of variables, generalized functional invariants in our paper will be the same as generalized functional invariants in [CDL+16].

2.2. **Periods.** In this section, we define periods of LG models using generalized functional invariants. We first recall the definition of Landau-Ginzburg model for quasi-Fano varieties.

Definition 2.5 ([DHT17b], Definition 2.1). A Landau-Ginzburg model of a quasi-Fano variety X is a pair $(X^\vee, W)$ consisting of a Kähler manifold $X^\vee$ satisfying $h^1(X^\vee) = 0$ and a proper map $W : X^\vee \to \mathbb{C}$, where W is called the superpotential.

Definition 2.6. Given a Landau-Ginzburg model and a choice of holomorphic n-form ω, we define the *periods* of the LG model *relative to W and ω* to be the period functions associated to the varying fibers of the LG model obtained by integrating transcendental cycles across the n-form ω.

Remark 2.7. We expect that if $(X^\vee, W)$ is the LG model of X, then the smooth fibers of W should be mirror to generic anticanonical hypersurfaces in X. Since the fibers are Calabi-Yau manifolds, the choice of ω is well-defined up to multiplication by a holomorphic function on the base of the LG model. Scaling the holomorphic form by a function has the effect of scaling the periods by the same function.

Remark 2.8. It is often the case that our LG models will depend on various deformation parameters. We clarify here that the relative period functions we consider are functions of both the base variable of the LG model and the deformation parameters. It will often be the case that we will treat this object as a *deforming family of periods* in which case we will distinguish the base variable of LG model from the deformation parameters.

Remark 2.9. As described in [DHNT17], the geometry of an M_n-polarized family of $K3$ surfaces is determined by the associated generalized functional invariant. Thus, as long as one uses the pull-back of the chosen holomorphic 2-form on $\mathcal{X}_n$ to compute the periods of the fibration, then the periods of the family will be precisely the pull-backs of the periods of $\mathcal{X}_n$ via the functional invariant. Later on in this paper, we will have to scale these period functions appropriately.

More generally, the LG models that we work with in this paper corresponding to higher-dimensional Calabi-Yau manifolds will be constructed via pull-back. Therefore, in these cases too, the periods of the LG model will be determined by pulling back by rational functions and scaling appropriately.

For Fano varieties, the classical period of the Minkowski polynomial associated to the LG model was introduced in [CCG+13]. Note that the periods in [CCG+13] are power series in one variable, which is essentially the base parameter of the LG model. We would like to point out that the periods in Definition 2.6 can be specialized to the classical periods in [CCG+13].

Example 2.10. We consider the period for the LG model of $\mathbb{P}^3$ along with its smooth anticanonical $K3$ surface. The LG model can be written as the fiberwise compactification of the following. The potential is

$$W : X^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y) \mapsto y,$$

where $X^\vee$ is

$$X^\vee = \left\{ (x_1, x_2, x_3, y) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + x_3 + \frac{q}{x_1 x_2 x_3} = y \right\}.$$

Performing a change of variables $x_i = y\tilde{x}_i$, we find that $X^\vee$ is birational to

$$\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - 1) + \frac{q}{y^4} = 0.$$

That is, $X^\vee$ is the pull-back of the family $\mathcal{X}_2$ via the generalized function invariant

$$\lambda = \frac{q}{y^4}.$$

We obtain periods for this LG model via pull-back. For example, the following expression is a period function with respect to the 2-form obtained by pulling back the 2-form on $\mathcal{X}_2$:

$$f_0^{\mathbb{P}^3}(q, y) := f_0^{\mathcal{X}_2}(\lambda) = \sum_{d \geq 0} \frac{(4d)!}{(d!)^4} \left(\frac{q}{y^4} \right)^d.$$

By setting $q = 1$ and $1/y = t$, we recover the classical period

$$f_0^{\mathbb{P}^3}(t) = \sum_{d \geq 0} \frac{(4d)!}{(d!)^4} t^{4d}.$$

Although this is rather a trivial example, in general, the classical period can be obtained from the period defined in Definition 2.6 in a similar way: set all the complex parameters to 1 and set the base parameter y of the LG model to $1/t$.

2.2.1. *The iterative structure of periods.* In this section, we take a detour to explain how periods of a family of Calabi-Yau n-fold, fibered by Calabi-Yau $(n-1)$-fold, can also be computed using generalized functional invariants. Given a family of Calabi-Yau n-fold fibered by Calabi-Yau $(n-1)$-fold, the period of the family of Calabi-Yau n-fold is the residue integral of the pullback of the period of its internal fibration of the Calabi-Yau $(n-1)$-fold by the generalized functional invariant. This iterative structure has already studied in [DM15].

The quintic mirror family $\mathcal{X}$ of Calabi-Yau threefolds is defined by

$$\{x_1 x_2 x_3 y (x_1 + x_2 + x_3 + y - 1) + \psi = 0\}.$$

By setting

$$\tilde{x}_1 = \frac{x_1}{1 - y}, \tilde{x}_2 = \frac{x_2}{1 - y}, \ \tilde{x}_3 = \frac{x_3}{1 - y}, \ \lambda = \frac{\psi}{y(1 - y)^4},$$

the quintic mirror family is written in terms of the quartic mirror family. That is, for each $\psi \in \mathbb{P}^1 - \{0, \frac{1}{5^5}, \infty\}$, $\mathcal{X}_\psi$ is fibered by M_2-polarized $K3$ surfaces and the fibration is governed by the functional invariant

$$\lambda = \frac{\psi}{y(1 - y)^4}.$$

As is shown in [DM15, Proposition 5.1], we can calculate the periods of the quintic mirror family of Calabi-Yau threefold by integrating the "relative" periods corresponding to the fibration structure. We review some of the details below, but refer the readers to [DM15] for more.

First, note that:

$$\frac{1}{y(1-y)}f_0^{\mathcal{X}_2}(\lambda) = \frac{1}{y(1-y)}\sum_{d_1 \geq 0}\frac{(4d_1)!}{(d_1!)^4}\left(\frac{\psi}{y(1-y)^4}\right)^{d_1}$$

$$= \sum_{d_1,d_2 \geq 0}\frac{(4d_1)!}{(d_1!)^4}\left(\frac{\psi^{d_1}}{y^{d_1}}\right)\frac{(4d_1+d_2)!}{(4d_1)!d_2!}y^{d_2-1}$$

$$= \sum_{d_1,d_2 \geq 0}\frac{(4d_1+d_2)!}{(d_1!)^4 d_2!}\psi^{d_1}y^{d_2-d_1-1}.$$

Here, we have used the fact that

$$\frac{1}{(1-y)^{k+1}} = \sum_{d=0}^{\infty}\frac{(d+k)!}{d!k!}y^d.$$

Next, if we integrate around a closed loop around $y = 0$ and use the residue theorem, we find that

$$\frac{1}{2\pi i}\oint\frac{f_0^{\mathcal{X}_2}(\lambda)}{y(1-y)} = \sum_{d_1 \geq 0}\psi^{d_1}\frac{(5d_1)!}{(d_1!)^5}.$$

The expression on the right-hand side is the well-known expression for the holomorphic period on the quintic mirror family of Calabi-Yau threefolds.

We remark that the series expression above is only convergent for $\left|\frac{\psi}{y(1-y)^4}\right| < 1$ and so the series formula present is only valid on the intersection of $\left|\frac{\psi}{y(1-y)^4}\right| < 1$ and the cylinder $|y| < 1$ (which is happily non-empty!).

In summary, we have constructed the standard holomorphic period of the quintic mirror family by integrating the ψ-dependent family of relative periods with respect to y. The scaling factor of $y(1-y)$ in front of the relative periods is present to ensure that the resulting integral corresponds precisely to the specific choice of holomorphic 3-form that one normally makes when studying the quintic mirror.

3. Tyurin degeneration of a conifold transition of quintic threefolds

In this section, we consider the following Calabi-Yau threefold with two Kähler parameters: complete intersection of bidegrees $(4,1)$ and $(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. We denoted it by $\tilde{Q}_5$. It is a conifold transition of a quintic threefold in $\mathbb{P}^4$, see, for example, [CDJ$^+$08]. The Calabi-Yau threefold $\tilde{Q}_5$ admits a Tyurin degeneration

$$\tilde{Q}_5 \rightsquigarrow \tilde{X}_1 \cup_{K3} \tilde{X}_2,$$

where $\tilde{X}_1$ is a hypersurface of bidegree $(4,1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and $\tilde{X}_2$ is a complete intersection of bidegrees $(4,0),(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. Indeed, $\tilde{X}_1$ is the blow-up of $\mathbb{P}^3$ along the complete intersection of two quartic surfaces and $\tilde{X}_2$ is the blow-up of a quartic threefold Q_4 along the complete intersection of two hyperplanes (degree one hypersurfaces in Q_4).

3.1. **Generalized functional invariants.** The mirrors of $\tilde{Q}_5$, $\tilde{X}_1$ and $\tilde{X}_2$ can be written down explicitly following [Giv98]. The mirror $\tilde{Q}_5^\vee$ of $\tilde{Q}_5$ is the compactification of

$$\left\{ (x_1, x_2, x_3, x_4, y) \in (\mathbb{C}^*)^5 \,\middle|\, x_1 + x_2 + x_3 + \frac{q_1}{x_1 x_2 x_3 x_4} + \frac{q_0}{y} = 1; x_4 + y = 1 \right\}.$$

The LG model for $\tilde{X}_1$ is

$$W_1 : \tilde{X}_1^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y_1) \mapsto y_1,$$

where $\tilde{X}_1^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_1) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + x_3 + \frac{q_{1,1}}{x_1 x_2 x_3} + \frac{q_{0,1}}{y_1} = 1 \right\}.$$

The LG model for $\tilde{X}_2$ is

$$W_2 : \tilde{X}_2^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y_2) \mapsto y_2,$$

where $\tilde{X}_2^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_2) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + x_3 + \frac{q_{1,2}}{x_1 x_2 x_3 (1 - q_{0,2}/y_2)} = 1 \right\}.$$

By performing an appropriate change of variables, we see that all three of these families are fibered by quartic mirror $K3$ surfaces. This allows us to conclude that they are families of M_2-polarized $K3$ surfaces and read off their generalized functional invariants. For example, for the family $\tilde{X}_1^\vee$, we make the following change of variables:

$$\tilde{x}_1 = \frac{x_1}{1 - q_{0,1}/y_1}, \tilde{x}_2 = \frac{x_2}{1 - q_{0,1}/y_1}, \tilde{x}_3 = \frac{x_3}{1 - q_{0,1}/y_1}, \lambda_1 = \frac{q_{1,1}}{(1 - q_{0,1}/y_1)^4}.$$

This produces the quartic mirror family of $K3$ surfaces:

$$\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - 1) + \lambda_1 = 0.$$

We read off the generalized functional invariant for $\tilde{X}_1^\vee$ as $\lambda_1 = \frac{q_{1,1}}{(1 - q_{0,1}/y_1)^4}$. Similarly, we compute the generalized functional invariants for $\tilde{Q}_5^\vee$ and $\tilde{X}_2^\vee$ by making appropriate changes of variable to match with the quartic mirror family. We obtain the following generalized functional invariants for $\tilde{Q}_5^\vee$, $\tilde{X}_1^\vee$ and $\tilde{X}_2^\vee$:

$$\lambda = \frac{q_1}{(1 - y)(1 - q_0/y)^4}, \quad \lambda_1 = \frac{q_{1,1}}{(1 - q_{0,1}/y_1)^4}, \quad \lambda_2 = \frac{q_{1,2}}{1 - q_{0,2}/y_2}.$$

Before proceeding, we map out the locations of the branch points and singular fibers. For $\tilde{Q}_5^\vee$ with functional invariant $\lambda = \frac{1}{(1-y)\left(1 - \frac{q_0}{y}\right)^4}$, we have

$$\lambda^{-1}(0) = \{0, \infty\}, \ \lambda^{-1}(\infty) = \{1, q_0\}, \ \lambda^{-1}\left(\frac{1}{256}\right) = \{\text{five points}\}.$$

Similarly, we find

$$\lambda_1^{-1}(0) = \{0\}, \ \lambda_1^{-1}(\infty) = \{q_{0,1}\}, \ \lambda_1^{-1}\left(\frac{1}{256}\right) = \{\text{four points}\}.$$

$$\lambda_2^{-1}(0) = \{0\}, \quad \lambda_2^{-1}(\infty) = \{q_{0,2}\}, \quad \lambda_2^{-1}\left(\frac{1}{256}\right) = \{\text{one point}\}.$$

The point $y_1 = 0$ is a cusp and corresponds to a semistable fiber of type III with 34 components; the fiber at $y_1 = \infty$ is smooth; the four fibers located at the pre-image of $\frac{1}{256}$ are singular $K3$ surfaces containing a single A_1 singularity; the point $y_1 = q_{0,1}$ has a type III fiber. On the second LG model, the point $y_2 = 0$ is a cusp corresponding to a type III fiber with 4 components; the fiber $y_2 = q_{0,2}$ has 31 components; the fiber $y_2 = \infty$ is smooth; the fiber over the pre-image of $\frac{1}{256}$ is a singular $K3$ surface with an A_1 singularity.

The conifold transition itself has type III fibers at $y = 0$ and $y = \infty$ with 34 and 4 components respectively; the fiber $y = 1$ has 31 components; the fiber $y = q_0$ has a type III fiber; the five fibers over the pre-images of $\frac{1}{256}$ are singular $K3$ surfaces with A_1 singularities.

We glue the bases of the two LG models by making the following identification of variables:

$$(4) \qquad\qquad q_1 = q_{1,1} = q_{1,2}, \quad q_0 = q_{0,1}, \quad y = y_1 = q_{0,2}/y_2.$$

Proposition 3.1. *Under the identification (4), the following product relation holds among the functional invariants:*

$$(5) \qquad\qquad \frac{\lambda}{q_1} = \frac{\lambda_1}{q_{1,1}} \cdot \frac{\lambda_2}{q_{1,2}}.$$

Remark 3.2. The parameter q_1 above is a scaling parameter on the modular curve $X_0(2)^+$. Thus, one should think of equation (5) as saying that the generalized functional invariant of $\tilde{Q}_5^\vee$ is the product of the two functional invariants corresponding the LG models $\tilde{X}_1^\vee$ and $\tilde{X}_2^\vee$ after scaling appropriately.

This gluing formula for generalized functional invariants illustrates how the fibers of the two Landau-Ginzburg models are combined into the internal fibration on the Calabi-Yau threefold. Before exploring the meaning of this gluing in terms of both fiberwise periods (of the M_2-polarized $K3$ surface fibers) and periods of the Calabi-Yau threefold itself, it is natural to ask whether the mirror to the conifold transition can be seen directly in terms of the generalized functional invariant λ. The answer is a resounding "yes!".

3.1.1. *Mirror quintic as a limit.* Re-writing the functional invariant, we have

$$\lambda = \frac{yq_1}{(y-1)(y-q_0)^4}.$$

This corresponds to the family of $h^{2,1} = 2$ Calabi-Yau threefolds that we are studying. In order to obtain the quintic mirror via a limit, we make a change of variables and take an appropriate limit. First, make the transformation:

$$y = (1 - q_0)\tilde{y} + q_0.$$

In this coordinate system, the generalized functional invariant transforms to

$$\lambda = \frac{q_1}{(1 - q_0)^5} \frac{((1 - q_0)\tilde{y} + q_0)}{\tilde{y}^4(\tilde{y} - 1)}.$$

Now pull out a factor of q_0 from the numerator:

$$\lambda = \frac{q_1 q_0}{(1 - q_0)^5} \cdot \frac{\left(\frac{1 - q_0}{q_0}\tilde{y} + 1\right)}{\tilde{y}^4(\tilde{y} - 1)}.$$
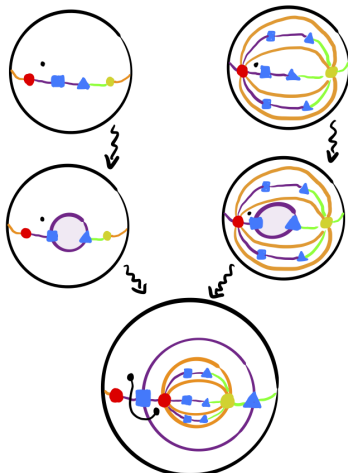
FIGURE 1. The topological gluing of functional invariants.

Set $z_1 = \frac{q_1}{(1-q_0)^4}$, $z_2 = z_1 \frac{q_0}{1-q_0} = \frac{q_1 q_0}{(1-q_0)^5}$. Then, the functional invariant above is re-written as

$$\lambda = \frac{z_1 \tilde{y} + z_2}{\tilde{y}(\tilde{y} - 1)^4}.$$

In the limit $z_1 \to 0$, we obtain the quintic mirror with z_2 as the scaling parameter. In other words, we want $\frac{q_1}{(1-q_0)^4}$ to go to zero in such a way that $\frac{q_1 q_0}{(1-q_0)^5}$ remains finite. This is exactly the limit that was considered in [CDJ+08]. The corresponding limits performed on the level of holomorphic periods are carried out in [CDJ+08].

3.1.2. *Topological Gluing.* Before continuing, let us briefly explain how the multiplication of functional invariants described above relates to the topological gluing described in [DHT17a]. The maps $\lambda_1, \lambda_2, \lambda$ are all covers of the modular curve $X_0(2)^+$ branched over $0, \infty$, with λ branching over two points p_1, p_2 depending on z_1, z_2. On the base curve $X_0(2)^+$, draw arcs joining 0 to p_1, p_1 to p_2, p_2 to ∞ and ∞ to 0. Then, the covers λ_1, λ_2 are determined by graphs drawn on the Riemann sphere. These graphs are depicted in Figure 1, where the red dots are pre-images of 0, the blue squares are pre-images of p_1, the blue triangles are pre-images of p_2, and the yellow dots are pre-images of ∞.

The spheres at the top of Figure 1 correspond to λ_1 on the right and λ_2 on the left. Below, we have cut one of the edges joining a blue square to the adjacent blue triangle and opened it up. Finally, we glue the two spheres along these cuts. The resulting object is a sphere whose graph determines the cover corresponding to λ, which corresponds, in turn, to the $h^{2,1} = 2$ family of Calabi-Yau threefolds.

In the limit as $z_1 \to 0$, the blue squares degenerate into the red dots, which themselves coalesce to produce one red dot corresponding to the cusp of ramification index 5. This cover corresponds to the quintic mirror.

We can recognize the local system of the VHS corresponding to the M_2-polarized K3 surfaces as the gluing of the two individual VHSs corresponding to λ_1, λ_2. This follows from the fact that all three of these local systems are determined by the covers $\lambda, \lambda_1, \lambda_2$ respectively, but we can also be more explicit if we reference Figure 1. Indeed, choose base points on each

of the covering spheres, depicted as the black dots in Figure 1, together with a basis of loops—note that the monodromy around the blue dots is trivial. Then, a basis of loops for the glued sphere is obtained by first taking the basis on the left sphere, and then "dragging" the basis of loops corresponding to the right sphere along the black path. Since there is no monodromy around the blue square, this is well-defined and the resulting global mondromy representation can be thought of as the concatenation of the original two monodromy representations. That is, if the monodromy tuple corresponding to the left sphere is (g_1, g_2, g_3) and the monodromy truple corresponding to the right sphere is $(h_1, h_2, h_3, h_4, h_5, h_6)$, then the monodromy tuple of the gluing is $(g_1, g_2, g_3, h_1, h_2, h_3, h_4, h_5, h_6)$. Moreover, if we make these choices so that g_3 and h_1 correspond the monodromy around the red dots, then the limiting monodromy is obtained by simply multiplying g_3 and h_1 together.

3.2. **Relation among periods.** In Section 3.1, we saw how the generalized functional invariant for $\tilde{Q}_5$ arises by gluing two LG models and their corresponding generalized functional invariants. In this section, we explore what this means on the level of periods.

3.2.1. *Periods for mirror quartic $K3$.* Consider the period associated to quartic mirror

$$(6) \qquad f_0^{K3}(q_1) = \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} q_1^{d_1}.$$

Let $\theta_{q_1} = q_1 \frac{\partial}{\partial q_1}$, then $f_0^{K3}(q_1)$ satisfies the following ODE

$$(7) \qquad (\theta_{q_1}^3 - 4q_1(4\theta_{q_1} + 1)(4\theta_{q_1} + 2)(4\theta_{q_1} + 3))F(q_1) = 0.$$

The basis of solutions for Equation (7) can be taken as the coefficients of powers of H in the following function:

$$(8) \qquad I^{K3}(q_1) = q_1^H \sum_{d_1 \geq 0}^{\infty} q_1^{d_1} \frac{\prod_{k=1}^{4d_1}(4H + k)}{\prod_{k=1}^{d_1}(H + k)^4} \qquad \mod H^3,$$

where $q_1^H = e^{H \log q_1}$.

3.2.2. *Periods for $\tilde{Q}_5^{\vee}$.* The holomorphic period for $\tilde{Q}_5$ can be found in [CDJ+08]. It can also be obtained as a residue integral of the pullback of

$$\frac{1}{(1 - y)(1 - q_0/y)} f_0^{K3}(q_1)$$

by the generalized functional invariant

$$\lambda = \frac{q_1}{(1 - y)(1 - q_0/y)^4}.$$

We have

$$\frac{1}{(1 - y)(1 - q_0/y)} f_0^{K3}(\lambda) = \frac{1}{(1 - y)(1 - q_0/y)} \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_1}{(1 - y)(1 - q_0/y)^4} \right)^{d_1}$$

$$= \frac{1}{(1 - y)} \sum_{d_1, d_{0,1} \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_1}{(1 - q_0/y)^4} \right)^{d_1} \frac{(d_1 + d_{0,1})!}{d_1! d_{0,1}!} (y)^{d_{0,1}}$$

$$= \sum_{d_1, d_{0,1}, d_{0,2} \geq 0} \frac{(4d_1)!}{(d_1!)^4} (q_1)^{d_1} \frac{(d_1 + d_{0,1})!}{d_1! d_{0,1}!} \frac{(4d_1 + d_{0,2})!}{(4d_1)! d_{0,2}!} (y)^{d_{0,1}} (q_0/y)^{d_{0,2}}.$$

After taking a residue, we obtain the well-known formula for the holomorphic period of $\tilde{Q}_5^\vee$:

$$
\begin{aligned}
f_0^{\tilde{Q}_5}(q_1, q_0) &= \frac{1}{2\pi i} \oint \frac{1}{(1-y)(1-q_0/y)} f_0^{K3}(\lambda) \frac{dy}{y} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!}{(d_1!)^4} (q_1)^{d_1} \frac{(d_1 + d_0)!}{d_1! d_0!} \frac{(4d_1 + d_0)!}{(4d_1)! d_0!} (q_0)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!(d_1 + d_0)!}{(d_1!)^5 (d_0!)^2} q_1^{d_1} q_0^{d_0}.
\end{aligned}
$$

Remark 3.3. These series will converge on the intersection $\left| \frac{q_1}{(1-y)(1-\frac{q_0}{y})} \right| < 1$ and $|y| < 1$. Similar convergence considerations will percolate in the work that follows. We will avoid commenting further on domains of convergence unless there is likely to be ambiguity as to which domains we should be using.

The holomorphic period $f_0^{\tilde{Q}_5}(q_1, q_0)$ is the solution for the following system of PDEs.

(9)
$$
\left(\theta_{q_1}\right)^5 - q_1 \left(4\theta_{q_1} + \theta_{q_0} + 1\right)\left(4\theta_{q_1} + \theta_{q_0} + 2\right)\left(4\theta_{q_1} + \theta_{q_0} + 3\right)\left(4\theta_{q_1} + \theta_{q_0} + 4\right)\left(\theta_{q_1} + \theta_{q_0} + 1\right) F(q_1, q_0) = 0
$$
$$
\left(\theta_{q_0}\right)^2 - q_0 \left(4\theta_{q_1} + \theta_{q_0} + 1\right)\left(\theta_{q_1} + \theta_{q_0} + 1\right) F(q_1, q_0) = 0.
$$

The basis of solutions for the system of PDEs is given by

$$
I^{\tilde{Q}_5}(q_1, q_0) = q_1^H q_0^P \sum_{d_1, d_0 \geq 0} q_1^{d_1} q_0^{d_0} \frac{\prod_{k=1}^{4d_1 + d_0}(4H + P + k) \prod_{k=1}^{d_1 + d_0}(H + P + k)}{\prod_{k=1}^{d_1}(H + k)^5 \prod_{k=1}^{d_0}(P + k)^2} \qquad \mod H^3, P^2.
$$

3.2.3. *Periods for LG models.* Following Section 2.2, periods for $\tilde{X}_1^\vee$ can be constructed by pulling back periods of the mirror quartic via the generalized functional invariant map λ_1. Consider the following scaled version of the pull-back of the holomorphic period:

$$
\begin{aligned}
f_0^{\tilde{X}_1}(q_{1,1}, q_{0,1}, y_1) &= \frac{1}{(1 - q_{0,1}/y_1)} f_0^{K3}(\lambda_1) \\
&= \frac{1}{(1 - q_{0,1}/y_1)} \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,1}}{(1 - q_{0,1}/y_1)^4}\right)^{d_1} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!(4d_1 + d_0)!}{(d_1!)^4 (4d_1)! d_0!} q_{1,1}^{d_1} (q_{0,1}/y_1)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 d_0!} q_{1,1}^{d_1} (q_{0,1}/y_1)^{d_0}.
\end{aligned}
$$

Because $q_{0,1}$ and y_1 always appear at the same time with the form $q_{0,1}/y_1$, we will also denote by $f_0^{\tilde{X}_1}(q_{1,1}, x)$ the holomorphic period for $\tilde{X}_1^\vee$ defined above:

$$
f_0^{\tilde{X}_1}(q_{1,1}, x) = \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 d_0!} q_{1,1}^{d_1} x^{d_0}.
$$

The holomorphic period $f_0^{\tilde{X}_1}(q_{1,1}, x)$ is the solution for the following system of PDEs.

$$(10) \quad (\theta_{q_{1,1}})^4 - q_{1,1}(4\theta_{q_{1,1}} + \theta_x + 1)(4\theta_{q_{1,1}} + \theta_x + 2)(4\theta_{q_{1,1}} + \theta_x + 3)(4\theta_{q_{1,1}} + \theta_x + 4)F(q_{1,1}, x) = 0$$
$$(\theta_x) - x(4\theta_{q_{1,1}} + \theta_x + 1)F(q_{1,1}, x) = 0.$$

The basis of solutions for the system of PDEs is given by

$$I^{\tilde{X}_1}(q_{1,1}, x) = q_{1,1}^H x^P \sum_{d_1, d_0 \geq 0}^{\infty} q_{1,1}^{d_1} x^{d_0} \frac{\prod_{k=1}^{4d_1+d_0}(4H + P + k)}{\prod_{k=1}^{d_1}(H + k)^4 \prod_{k=1}^{d_0}(P + k)} \qquad \text{mod } H^3, P^2.$$

We also write

$$I^{\tilde{X}_1}(q_{1,1}, q_{0,1}, y) = q_{1,1}^H (q_{0,1}/y)^P \sum_{d_1, d_0 \geq 0}^{\infty} q_{1,1}^{d_1} (q_{0,1}/y)^{d_0} \frac{\prod_{k=1}^{4d_1+d_0}(4H + P + k)}{\prod_{k=1}^{d_1}(H + k)^4 \prod_{k=1}^{d_0}(P + k)} \qquad \text{mod } H^3, P^2.$$

Similarly, we can pull back the holomorphic period for the mirror quartic via the generalized functional invariant map λ_2 to obtain the holomorphic period for $\tilde{X}_2^\vee$:

$$\frac{1}{(1 - q_{0,2}/y_2)} f_0^{K3}(\lambda_2) = \frac{1}{(1 - q_{0,2}/y_2)} \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,2}}{1 - q_{0,2}/y_2} \right)^{d_1}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!(d_1 + d_0)!}{(d_1!)^4 (d_1)! d_0!} q_{1,2}^{d_1} (q_{0,2}/y_2)^{d_0}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!(d_1 + d_0)!}{(d_1!)^5 d_0!} q_{1,2}^{d_1} (q_{0,2}/y_2)^{d_0}.$$

We define $f_0^{\tilde{X}_2}(q_{1,2}, y)$, the holomorphic period for $\tilde{X}_2^\vee$, to be

$$f_0^{\tilde{X}_2}(q_{1,2}, y) = \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!(d_1 + d_0)!}{(d_1!)^5 d_0!} q_{1,2}^{d_1} y^{d_0}.$$

The holomorphic period $f_0^{\tilde{X}_2}(q_{1,2}, y)$ is the solution for the following system of PDEs.

$$(11) \qquad (\theta_{q_{1,2}})^4 - 4q_{1,2}(4\theta_{q_{1,2}} + 1)(4\theta_{q_{1,2}} + 2)(4\theta_{q_{1,2}} + 3)(\theta_{q_{1,2}} + \theta_y + 1)F(q_{1,2}, y) = 0$$
$$(\theta_y) - y(\theta_{q_{1,2}} + \theta_y + 1)F(q_{1,2}, y) = 0.$$

The basis of solutions for the system of PDEs is given by

$$I^{\tilde{X}_2}(q_{1,2}, y) = q_{1,2}^H y^P \sum_{d_1, d_0 \geq 0}^{\infty} q_{1,2}^{d_1} y^{d_0} \frac{\prod_{k=1}^{4d_1}(4H + k) \prod_{k=1}^{d_1+d_0}(H + P + k)}{\prod_{k=1}^{d_1}(H + k)^5 \prod_{k=1}^{d_0}(P + k)} \qquad \text{mod } H^3, P^2.$$

Remark 3.4. Let us address the scaling factors that appear in our period expressions. Ultimately, we will take the relative periods of $\tilde{X}_1^\vee$ and $\tilde{X}_2^\vee$ to construct the periods for $\tilde{Q}_5$. In order to make sure that the periods for $X_i^\vee$ are "compatible" with each other, we need to scale the periods appropriately. More precisely, the functions $f_0^{X_i}(q_{1,i}, x)$ are $q_{1,i}$-dependent families of periods functions for the base variable x. The scaling factors are chosen to ensure that the characteristic exponents of the corresponding $q_{1,i}$-dependent families of ODEs at the singular points are the same for both $f_0^{\tilde{X}_1}(q_{1,1}, x)$ and $f_0^{\tilde{X}_2}(q_{1,2}, y)$. This should be thought of as a fine-tuning of the holomorphic form used to calculate the period integrals associated to each factor.

3.2.4. *Relations.*

Theorem 3.5. *The holomorphic periods of LG models can be glued together to form the holomorphic period of the mirror Calabi–Yau $\tilde{Q}_5^\vee$ with correction given by the holomorphic period of the mirror quartic K3 surface. More precisely, the relation is given by the Hadamard product*

$$f_0^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_0^{K3}(q_1) = \frac{1}{2\pi i} \oint f_0^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) \frac{dy}{y},$$

where $\star_{q_1}$ means the Hadamard product with respect to the variable q_1.

Proof. This is a straightforward computation:

$$\frac{1}{2\pi i} \oint f_0^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) \frac{dy}{y}$$

$$= \frac{1}{2\pi i} \oint \sum_{d_1, d_{0,1}, d_{0,2} \geq 0} \frac{(4d_1 + d_{0,1})!}{(d_1!)^4 d_{0,1}!} \frac{(4d_1)!(d_1 + d_{0,2})!}{(d_1!)^5 d_{0,2}!} q_1^{d_1} (q_0/y)^{d_{0,1}} (y)^{d_{0,2}} \frac{dy}{y}$$

$$= \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 d_0!} \frac{(4d_1)!(d_1 + d_0)!}{(d_1!)^5 d_0!} q_1^{d_1} q_0^{d_0}$$

$$= f_0^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_0^{K3}(q_1).$$

$\square$

Remark 3.6. Recall that, we have the following identification of variables:

$$q_1 = q_{1,1} = q_{1,2}, \quad q_0 = q_{0,1}, \quad y = y_1 = q_{0,2}/y_2.$$

Gluing the LG models on the level of periods means taking the residue integral for the periods of LG models over y, where y is the base parameter of the internal fibration of $\tilde{Q}_5^\vee$. The period of $\tilde{Q}_5^\vee$ are also computed by residue integral (over the parameter y) of period of the Calabi-Yau family in one dimensional lower via the generalized functional invariants.

One can also write down a Hadamard product relation among the Picard-Fuchs operators. Moreover, we have the following identity among the bases of solutions to the Picard-Fuchs equations.

Theorem 3.7. *We have the following Hadamard product relation*

$$(12) \qquad I^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} I^{K3}(q_1) = \frac{1}{2\pi i} \oint I^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} I^{\tilde{X}_2}(q_1, y) \frac{dy}{y} \qquad \mod H^3, P^2.$$

In the Hadamard product $\star_{q_1}$, we treat $\log q_1$ as a variable that is independent from q_1. Alternative, one may consider $\bar{I}(q_1, q_0) := I(q_1, q_0)/q_1^H q_0^P$ and write Hadamard product relation for $\bar{I}$.

More explicitly, one can write an identity for each coefficient of $H^a P^b$ for $0 \leq a \leq 2$, $0 \leq b \leq 1$. Let $f_{i,j}^{\tilde{Q}_5}(q_1, q_0)$ be the coefficient of $H^i P^j$ for $I^{\tilde{Q}_5}(q_1, q_0)$. We still write $f_0^{\tilde{Q}_5}(q_1, q_0)$ for the coefficient of $H^0 P^0$, since it is simply the holomorphic period. Similarly, for the coefficients of $I^{K3}(q_1)$, $I^{\tilde{X}_1}(q_1, q_0, y)$ and $I^{\tilde{X}_2}(q_1, y)$. We have the following identity for the coefficient of

H:

$$f_{1,0}^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_0^{K3}(q_1) + f_0^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_1^{K3}(q_1)$$

$$= \frac{1}{2\pi i} \oint \left(f_{1,0}^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) + f_0^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_{1,0}^{\tilde{X}_2}(q_1, y) \right) \frac{dy}{y}.$$

For the coefficient of H^2, we have

$$f_{2,0}^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_0^{K3}(q_1) + f_{1,0}^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_1^{K3}(q_1) + f_0^{\tilde{Q}_5}(q_1, q_0) \star_{q_1} f_2^{K3}(q_1)$$

$$= \frac{1}{2\pi i} \oint \left(f_{2,0}^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) + f_{1,0}^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_{1,0}^{\tilde{X}_2}(q_1, y) + f_0^{\tilde{X}_1}(q_1, q_0, y) \star_{q_1} f_{2,0}^{\tilde{X}_2}(q_1, y) \right) \frac{dy}{y}.$$

Remark 3.8. We want to point out that $I^{\tilde{X}_1}$ and $I^{\tilde{X}_2}$ are not exactly the I-functions for the corresponding relative Gromov-Witten invariants in Section 6.3. We will use $I^{(\tilde{X}_1, K3)}$ and $I^{(\tilde{X}_2, K3)}$ to denote the relative I-functions which are more complicated than $I^{\tilde{X}_1}$ and $I^{\tilde{X}_2}$. Nevertheless, the information of $I^{\tilde{X}_1}$ and $I^{\tilde{X}_2}$ can be extracted from the relative I-functions.

4. Tyurin degeneration of quintic threefolds

4.1. **Blow-up along $\mathbb{P}^3$.** We consider the Tyurin degeneration of a quintic threefold Q_5 into a quartic threefold Q_4 and the blow-up $\mathrm{Bl}_C \mathbb{P}^3$ of $\mathbb{P}^3$ along complete intersection center of degrees 4 and 5 hypersurfaces, that is,

$$Q_5 \rightsquigarrow Q_4 \cup_{K3} \mathrm{Bl}_C \mathbb{P}^3,$$

where C is the complete intersection of degrees 4 and 5 hypersurfaces in $\mathbb{P}^3$, and $K3$ is the common anticanonical $K3$ hypersurface. We would like to write down the LG models for Q_4 and $\mathrm{Bl}_C \mathbb{P}^3$ and glue them to the mirror family of Q_5. Since $\mathrm{Bl}_C \mathbb{P}^3$ can be written as a hypersurface in a toric variety, we can write down the LG models following Givental [Giv98]. For the rest of this section, we write $X_1 := Q_4$ and $\tilde{X}_2 := \mathrm{Bl}_C \mathbb{P}^3$.

The LG model for $X_1 = Q_4$ is

$$W_1 : X_1^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y_1) \mapsto y_1,$$

where $X_1^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_1) \in (\mathbb{C}^*) \,\middle|\, x_1 + x_2 + x_3 + \frac{q_{1,1}}{x_1 x_2 x_3 y_1} = 1 \right\}.$$

Following [CCGK16, Section E], the quasi-Fano variety $\tilde{X}_2 = \mathrm{Bl}_C(\mathbb{P}^3)$ can be constructed as a hypersurface of degree $(4,1)$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3})$.

The LG model of $\tilde{X}_2 = \mathrm{Bl}_C(\mathbb{P}^3)$ is defined as follows

$$W_2 : \tilde{X}_2^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y_2) \to y_2,$$

where $\tilde{X}_2^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_2) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + x_3 + \frac{q_{0,2}}{y_2} + \frac{q_{1,2} y_2}{x_1 x_2 x_3} = 1 \right\}.$$

Finally, the mirror quintic family $Q_5^\vee$ is defined by the following equation

$$x_1 x_2 x_3 y (x_1 + x_2 + x_3 + y - 1) + q_1 = 0.$$

Similar to the computation in Section 3, we obtain the generalized functional invariants for $Q_5^\vee$, $X_1^\vee$ and $\tilde{X}_2^\vee$ respectively:

$$(13) \qquad \lambda = \frac{q_1}{y(1-y)^4}, \quad \lambda_1 = \frac{q_{1,1}}{y_1}, \quad \lambda_2 = \frac{q_{1,2} y_2}{(1 - q_{0,2}/y_2)^4}.$$

We set

$$(14) \qquad q_1 = q_{1,1} = q_{1,2} y_2, \quad y = y_1 = q_{0,2}/y_2.$$

The matching of singular fibers works similarly to the previous section with singular fibers on one LG model being glued to smooth fibers of the other. We have

Proposition 4.1. *Under the identification (14), the following product relation holds among the functional invariants:*

$$(15) \qquad \frac{\lambda}{q_1} = \frac{\lambda_1}{q_{1,1}} \cdot \frac{\lambda_2}{q_{1,2}}.$$

Remark 4.2. Similar to Equation (5), Equation (15) says that the the functional invariant for $Q_5^\vee$ is equal to the product of the functional invariants of the LG models $X_1^\vee$ and $\tilde{X}_2^\vee$ after scaling.

The period for $Q_5^\vee$ is

$$f_0^{Q_5}(q_1) = \sum_{d_1 \geq 0} q_1^{d_1} \frac{(5d_1)!}{(d_1!)^5}.$$

The period for $X_1^\vee$ is

$$f_0^{X_1}(q_{1,1}, y_1) = f_0^{K3}(\lambda_1) = \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,1}}{y_1} \right)^{d_1}.$$

The period for $\tilde{X}_2^\vee$ is

$$\begin{aligned}
f_0^{\tilde{X}_2}(q_{1,2} y_2, q_{0,2}/y_2) &= \frac{1}{1-y} f_0^{K3}(\lambda_2) \\
&= \frac{1}{1 - q_{0,2}/y_2} \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,2} y_2}{(1 - q_{0,2}/y_2)^4} \right)^{d_1} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!}{(d_1!)^4} (q_{1,2} y_2)^{d_1} \frac{(4d_1 + d_0)!}{(4d_1)! d_0!} (q_{0,2}/y_2)^{d_0} \\
&= \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 d_0!} (q_{1,2} y_2)^{d_1} (q_{0,2}/y_2)^{d_0}.
\end{aligned}$$

We may rewrite the period for $\tilde{X}_2^\vee$ as

$$f_0^{\tilde{X}_2}(q_1, y) = \sum_{d_1, d_0 \geq 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 d_0!} (q_1)^{d_1} (y)^{d_0}.$$

Note that $q_1 = q_{1,1} = q_{1,2} y_2$ and $y = y_1 = q_{0,2}/y_2$ when we glue the LG models.

Theorem 4.3. *We have the following Hadamard product relation:*

$$f_0^{Q_5}(q_1) \star_{q_1} f_0^{K3}(q_1) = \frac{1}{2\pi i} \oint f_0^{X_1}(q_1, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) \frac{dy}{y}.$$

Proof. This is a straightforward computation. □

The Picard-Fuchs operators and the bases of solutions to Picard-Fuchs equations are related in a similar way. Let

$$I^{Q_5}(q_1) = e^{H \log q_1} \sum_{d \geq 0} \left(\frac{\prod_{k=1}^{5d}(5H + k)}{\prod_{k=1}^{d}(H + k)^5} \right) q_1^d;$$

$$I^{X_1}(q_1, y) = (q_1/y)^H \sum_{d_1 \geq 0}^{\infty} (q_1/y)^{d_1} \frac{\prod_{k=1}^{4d_1}(4H + k)}{\prod_{k=1}^{d_1}(H + k)^4};$$

$$I^{\tilde{X}_2}(q_1, y) = q_1^H y^P \sum_{d_1, d_0 \geq 0}^{\infty} q_1^{d_1} y^{d_0} \frac{\prod_{k=1}^{4d_1 + d_0}(4H + P + k)}{\prod_{k=1}^{d_1}(H + k)^4 \prod_{k=1}^{d_0}(P + k)}.$$

Theorem 4.4. *We have the following Hadamard product relation*

$$(16) \qquad I^{Q_5}(q_1) \star_{q_1} I^{K3}(q_1) = \frac{1}{2\pi i} \oint I^{X_1}(q_1, y) \star_{q_1} I^{\tilde{X}_2}(q_1, y) \frac{dy}{y} \qquad \mod H^3,$$

where we set $P = H$. *In the Hadamard product* $\star_{q_1}$, *we treat* $\log q_1$ *as a variable that is independent from* q_1. *Alternative, we can consider* $\bar{I}(q_1) := I(q_1)/q_1^H$ *and write Hadamard product relation for* $\bar{I}$.

4.2. **Blow-up along the quartic threefold.** We consider the Tyurin degeneration of Q_5 into $\mathbb{P}^3$ and the blow-up of Q_4 along the complete intersection C of hypersurfaces of degrees 1 and 5:

$$Q_5 \rightsquigarrow \mathbb{P}^3 \cup_{K3} \mathrm{Bl}_C(Q_4).$$

For the rest of this section we write $X_1 := \mathbb{P}^3$ and $\tilde{X}_2 := \mathrm{Bl}_C(Q_4)$.

The LG model for $X_1 = \mathbb{P}^3$ is given by the fiberwise compactification of

$$W : (\mathbb{C}^*)^3 \to \mathbb{C}$$

$$(x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3 + \frac{q_{1,1}}{x_1 x_2 x_3}.$$

It can be rewritten as follows. The potential is

$$W_1 : X_1^\vee \to \mathbb{C}$$

$$(x_1, x_2, x_3, y_1) \mapsto y_1,$$

where $X_1^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_1) \in (\mathbb{C}^*)^4 \, \middle| \, x_1 + x_2 + x_3 + \frac{q_{1,1}}{x_1 x_2 x_3} = y_1 \right\}.$$

The Blown-up variety $\tilde{X}_2 = \mathrm{Bl}_C(Q_4)$ can be realized as a complete intersection of degrees $(1,1)$ and $(4,0)$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4}) \to \mathbb{P}^4$. The LG model of $\tilde{X}_2 = \mathrm{Bl}_C(Q_4)$ can be written as follows:

The potential is

$$W_2 : \tilde{X}_2^\vee \to \mathbb{C}$$
$$(x_1, x_2, x_3, y_2) \mapsto y_2,$$

where $\tilde{X}_2^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, x_2, x_3, y_2) \in (\mathbb{C}^*)^4 \,\middle|\, x_1 + x_2 + x_3 + \frac{y_2^4 q_{1,2}}{x_1 x_2 x_3 (1 - q_{0,2}/y_2)} = 1 \right\}.$$

The generalized functional invariants for $Q_5^\vee$, $X_1^\vee$ and $\tilde{X}_2^\vee$ are

$$\lambda = \frac{q_1}{y(1-y)^4}, \quad \lambda_1 = \frac{q_{1,1}}{y_1^4}, \quad \lambda_2 = \frac{q_{1,2} y_2^4}{(1 - q_{0,2}/y_2)}.$$

For λ, we consider the change of variable $y \mapsto 1 - y$, then we have

$$\lambda = \frac{q_1}{y^4(1-y)}, \quad \lambda_1 = \frac{q_{1,1}}{y_1^4}, \quad \lambda_2 = \frac{q_{1,2} y_2^4}{(1 - q_{0,2}/y_2)}.$$

We consider the following identification among variables

(17)
$$q_1 = q_{1,1} = q_{1,2} y_2^4, \quad y = y_1 = q_{0,2}/y_2.$$

Proposition 4.5. *Under the identification (17), we have the relation among generalized functional invariants*

$$\frac{\lambda}{q_1} = \frac{\lambda_1}{q_{1,1}} \cdot \frac{\lambda_2}{q_{1,2}}.$$

The period for $X_1^\vee$ is

$$f_0^{X_1}(q_{1,1}, y_1) = f_0^{K3}(\lambda_1) = \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,1}}{y_1^4} \right)^{d_1}.$$

The period for $\tilde{X}_2^\vee$ is

$$f_0^{\tilde{X}_2}(q_{1,2} y_2^4, q_{0,2}/y_2) = \frac{1}{1-y} f_0^{K3}(\lambda_2) = \frac{1}{1 - q_{0,2}/y_2} \sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \left(\frac{q_{1,2} y_2^4}{(1 - q_{0,2}/y_2)} \right)^{d_1}$$
$$= \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \frac{(d_1 + d_0)!}{d_1! d_0!} (q_{1,2} y_2^4)^{d_1} (q_{0,2}/y_2)^{d_0}.$$

We can rewrite the period for $\tilde{X}_2^\vee$ as

$$f_0^{\tilde{X}_2}(q_1, y) = \sum_{d_1, d_0 \geq 0} \frac{(4d_1)!}{(d_1!)^4} \frac{(d_1 + d_0)!}{d_1! d_0!} (q_1)^{d_1} (y)^{d_0}.$$

We have the Hadamard product relation among periods.

Theorem 4.6. *The following relation holds for holomorphic periods:*

$$f_0^{Q_5}(q_1) \star_{q_1} f_0^{K3}(q_1) = \frac{1}{2\pi i} \oint f_0^{X_1}(q_1, y) \star_{q_1} f_0^{\tilde{X}_2}(q_1, y) \frac{dy}{y}.$$

The Picard-Fuchs operators and the bases of solutions to Picard-Fuchs equations are related in a similar way. Let

$$I^{X_1}(q_1, y) = (q_1/y^4)^H \sum_{d_1 \geq 0}^{\infty} (q_1/y^4)^{d_1} \frac{\prod_{k=1}^{4d_1}(4H + k)}{\prod_{k=1}^{d_1}(H + k)^4};$$

$$I^{\tilde{X}_2}(q_1, y) = q_1^H y^P \sum_{d_1, d_0 \geq 0}^{\infty} q_1^{d_1} y^{d_0} \frac{\prod_{k=1}^{4d_1}(4H + k) \prod_{k=1}^{d_1+d_0}(H + P + k)}{\prod_{k=1}^{d_1}(H + k)^5 \prod_{k=1}^{d_0}(P + k)}.$$

Theorem 4.7. *We have the following Hadamard product relation*

$$(18) \qquad I^{Q_5}(q_1) \star_{q_1} I^{K3}(q_1) = \frac{1}{2\pi i} \oint I^{X_1}(q_1, y) \star_{q_1} I^{\tilde{X}_2}(q_1, y) \frac{dy}{y} \qquad \mod H^3,$$

where we set $P = 4H$. In the Hadamard product $\star_{q_1}$, we treat $\log q_1$ as a variable that is independent from q_1. Alternative, we can consider $\bar{I}(q_1) := I(q_1)/q_1^H$ and write Hadamard product relation for $\bar{I}$.

5. Tyurin degeneration of Calabi-Yau complete intersections in toric varieties

5.1. **Set-up.** Let X be a Calabi-Yau complete intersection in a toric variety Y defined by a generic section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s$, where each L_l is a nef line bundle. Let $\rho_l = c_1(L_l)$, then $-K_Y = \sum_{l=0}^{s} c_1(L_l)$. Let $s_l \in H^0(Y, L_l)$ be generic sections determining X. A refinement of the nef partition with respect to L_0 is given by two nef line bundles $L_{0,1}, L_{0,2}$ such that $L_0 = L_{0,1} \otimes L_{0,2}$. Let $\rho_{0,1} = c_1(L_{0,1})$ and $\rho_{0,2} = c_1(L_{0,2})$. We have two quasi-Fano varieties X_1 and X_2 defined by sections of $E_1 = L_{0,1} \oplus L_1 \oplus \cdots \oplus L_s$ and $E_2 = L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s$ respectively. Let $s_{0,1} \in H^0(Y, L_{0,1})$ and $s_{0,2} \in H^0(Y, L_{0,2})$ be the generic sections determining X_1 and X_2. For our construction of a Tyurin degeneration, we blow up X_2 along $X_2 \cap \{s_0 = 0\} \cap \{s_{0,1} = 0\}$. We write the blown-up variety as $\tilde{X}_2$. Two quasi-Fano varieties X_1 and $\tilde{X}_2$ intersect along X_0 which is a Calabi-Yau complete intersection in the toric variety Y defined by a generic section of $L_{0,1} \oplus L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s$.

We can again realize $\tilde{X}_2$ as a complete intersection in a toric variety following [CCGK16, Section E]. Indeed, it is a hypersurface in the total space of $\pi : \mathbb{P}_{X_2}(\mathcal{O} \oplus i^* L_{0,2}^{-1}) \to X_2$ defined by a generic section of the line bundle $\pi^* i^* L_{0,1} \otimes \mathcal{O}(1)$, where $i : X_2 \hookrightarrow Y$ is the inclusion map. In other words, it is a complete intersection in the toric variety $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ given by a generic section of $\pi^* L_{0,2} \oplus \pi^* L_1 \oplus \cdots \oplus \pi^* L_s \oplus (\pi^* L_{0,1} \otimes \mathcal{O}(1))$ where we use the same π for the projection of $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ to the base Y. Then $-K_{\tilde{X}_2} = (P - \pi^* i^* \rho_{0,2})|_{\tilde{X}_2}$ by the adjunction formula, where $P = c_1(\mathcal{O}_{\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})}(1))$.

Remark 5.1. In dimension one, there is no codimension-two subvarieties to blow up. However, we may use the same geometric construction for blow-ups as above to construct a one dimensional subvariety in another toric variety. Then the results in dimension one are parallel to the results in higher dimensions.

Let $p_1, \ldots, p_r \in H^2(Y, \mathbb{Z})$ be a nef integral basis. We write the toric divisors as

$$D_j = \sum_{i=1}^{r} m_{ij} p_i, \quad 1 \leq j \leq m,$$

for some m_{ij}.

Let X be the Calabi-Yau complete intersection in the toric variety Y. The nef partition of the toric divisors gives a partition of the variables $x_1, \ldots, x_m$ into $s + 1$ groups. Let $F_l(x)$ be the sum of x_i in each group $l = 0, \ldots, s$. Following [Giv98], the Hori-Vafa mirror $X^\vee$ of X is Calabi-Yau compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; F_l(x) = 1, l = 0, \ldots, s \right. \right\}.$$

Note that q_i, $1 \le i \le r$, are complex parameters for $X^\vee$ from the ambient toric variety Y.

5.2. **Blowing-up both quasi-Fanos.** We first consider the case when we blow-up both quasi-Fanos. Then, the Calabi-Yau and quasi-Fanos, denoted by $\tilde{X}, \tilde{X}_1$ and $\tilde{X}_2$ respectively, becomes complete intersections in the toric variety $Y \times \mathbb{P}^1$. Let π_1 and π_2 be the projection of $Y \times \mathbb{P}^1$ onto Y and $\mathbb{P}^1$ respectively. Then

- $\tilde{X}$ is the complete intersection in $Y \times \mathbb{P}^1$ defined by generic sections of $\pi_1^* L_1$, $\ldots$, $\pi_1^* L_s$, $\pi_1^* L_{0,1} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\pi_1^* L_{0,2} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$;
- $\tilde{X}_1$ is the complete intersection in $Y \times \mathbb{P}^1$ defined by generic sections of $\pi_1^* L_1$, $\ldots$, $\pi_1^* L_s$, $\pi_1^* L_{0,1}$ and $\pi_1^* L_{0,2} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$;
- $\tilde{X}_2$ is the complete intersection in $Y \times \mathbb{P}^1$ defined by generic sections of $\pi_1^* L_1$, $\ldots$, $\pi_1^* L_s$, $\pi_1^* L_{0,1} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\pi_1^* L_{0,2}$;
- X_0 is the complete intersection in Y defined by generic sections of $L_{0,1}$, $L_{0,2}$, L_1, $\cdots$, L_s.

The mirror for $\tilde{X}$ is the compactification of

$$\left\{ (x_1, \ldots, x_m, y_1, y_2) \in (\mathbb{C}^*)^{m+2} \left| \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; y_1 y_2 = q_0; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) + y_1 = 1, F_{0,2}(x) + y_2 = 1 \right\}$$

$$= \left\{ (x_1, \ldots, x_m, y) \in (\mathbb{C}^*)^{m+1} \left| \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) + y = 1, F_{0,2}(x) + \frac{q_0}{y} = 1 \right\},$$

where $F_{0,1}(x)$ and $F_{0,2}(x)$ correspond to the refinement of the nef partition with respect to L_0. In particular, we have $F_{0,1}(x) + F_{0,2}(x) = F_0(x)$.

The mirror for $\tilde{X}_1$ is a LG model

$$W : \tilde{X}_1^\vee \to \mathbb{C}^*$$

$$(x_1, \ldots, x_m, y_1) \mapsto y_1,$$

where $\tilde{X}_1^{\vee}$ is the fiberwise compactification of

$$\left\{ (x_1, \ldots, x_m, y_1) \in (\mathbb{C}^*)^{m+1} \left| \prod_{j=1}^m x_j^{m_{ij}} = q_{i,1}, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1, F_{0,2}(x) + \frac{q_{0,1}}{y_1} = 1 \right\}.$$

Similarly, the mirror for $\tilde{X}_2$ is a LG model

$$W : \tilde{X}_2^{\vee} \to \mathbb{C}^*$$

$$(x_1, \ldots, x_m, y_2) \mapsto y_2,$$

where $\tilde{X}_2^{\vee}$ is the fiberwise compactification of

$$\left\{ (x_1, \ldots, x_m, y_2) \in (\mathbb{C}^*)^{m+1} \left| \prod_{j=1}^m x_j^{m_{ij}} = q_{i,2}, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) + \frac{q_{0,2}}{y_2} = 1, F_{0,2}(x) = 1 \right\}.$$

The $\tilde{X}_1$ and $\tilde{X}_2$ intersect along X_0 which is a Calabi-Yau variety in one dimensional lower. The mirror for X_0 is $X_0^{\vee}$ which is the compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \left| \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; \right. \right.$$

$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1, F_{0,2}(x) = 1 \right\}.$$

5.2.1. *Generalized functional invariants.* We can compute the generalized functional invariants from the mirrors. Let $F_{0,1}(x) = x_{j_1} + \ldots, x_{j_a}$ and $F_{0,2}(x) = x_{j_{a+1}} + \ldots, x_{j_{a+b}}$. For $\tilde{X}^{\vee}$, we have $F_{0,1}(x) + y = 1$ and $F_{0,2}(x) + \frac{q_0}{y} = 1$. For $y \neq 1$ and $\frac{q_0}{y} \neq 1$, the following change of variables can give us $X_0^{\vee}$.

$$\bar{x}_{j_k} = \frac{x_{j_k}}{1 - y}, k = 1, \ldots, a; \quad \bar{x}_{j_k} = \frac{x_{j_k}}{1 - q_0/y}, k = a + 1, \ldots, a + b;$$

and

(19)
$$\lambda_i = \frac{q_i}{\prod_{k=1}^a (1 - y)^{m_{ij_k}} \prod_{k=a+1}^{a+b} (1 - q_0/y)^{m_{ij_k}}}, i = 1 \ldots, r.$$

Then, $\lambda = (\lambda_1, \ldots, \lambda_r)$ is called the generalized functional invariant for $\tilde{X}^{\vee}$.

The generalized functional invariants for $\tilde{X}_1^{\vee}$ and $\tilde{X}_2^{\vee}$ can be computed in a similar way. Indeed, we have

(20)
$$\lambda_{i,1} = \frac{q_{i,1}}{\prod_{k=a+1}^{a+b} (1 - q_{0,1}/y_1)^{m_{ij_k}}}, i = 1 \ldots, r;$$

(21)
$$\lambda_{i,2} = \frac{q_{i,2}}{\prod_{k=1}^a (1 - q_{0,2}/y_2)^{m_{ij_k}}}, i = 1 \ldots, r.$$

We have the following identification of the variables:

(22) $$y = y_1 = q_{0,2}/y_2, \quad q_0 = q_{0,1}, \quad q_i = q_{i,1} = q_{i,2}, 1 \le i \le r.$$

Proposition 5.2. *Under the identification (22), the generalized functional invariants satisfy the product relation*

$$\frac{\lambda_i}{q_i} = \frac{\lambda_{i,1}}{q_i}\frac{\lambda_{i,2}}{q_i}, \quad i = 1, \dots, r.$$

Remark 5.3. To specialize to Section 3, we set $Y = \mathbb{P}^4$. Then $m = 5$, $r = 1$, $l = 0$, and $m_{1j} = 1$ for $j = 1, \dots, 5$. We also have $F_{0,1} = x_4$ and $F_{0,2} = x_1 + x_2 + x_3 + x_5$, where $x_5 = \frac{Q}{x_1 x_2 x_3 x_4}$. Then the generalized functional invariants specialize to the generalized functional invariants in Section 3.

5.2.2. *Periods.* Let $q = (q_1, \dots, q_r)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$. The holomorphic period for $X_0^\vee$ is

$$f_0^{X_0}(q) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! q^d,$$

where $q^d = q_1^{\langle p_1, d \rangle} \cdots q_r^{\langle p_r, d \rangle}$. We can compute the holomorphic period for $\tilde{X}^\vee$ via generalized functional invariants:

$$\frac{1}{(1-y)(1-q_0/y)} f_0^{X_0}(\lambda)$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \frac{q^d}{(1-y)^{1+\langle \rho_{0,1}, d \rangle}(1-q_0/y)^{1+\langle \rho_{0,2}, d \rangle}}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \sum_{d_{0,1}, d_{0,2} \ge 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \frac{(d_{0,1} + \langle \rho_{0,1}, d \rangle)!(d_{0,2} + \langle \rho_{0,2}, d \rangle)! q^d}{d_{0,1}! d_{0,2}!} y^{d_{0,1}} (q_0/y)^{d_{0,2}}.$$

Then, we obtain the holomorphic period for $\tilde{X}^\vee$:

$$f_0^{\tilde{X}}(q, q_0) = \frac{1}{2\pi i} \oint \frac{1}{(1-y)(1-q_0/y)} f_0^{X_0}(\lambda) \frac{dy}{y}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \sum_{d_0 \ge 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \frac{(d_0 + \langle \rho_{0,1}, d \rangle)!(d_0 + \langle \rho_{0,2}, d \rangle)! q^d}{(d_0!)^2} q_0^{d_0}.$$

The holomorphic periods for $\tilde{X}_1^\vee$ and $\tilde{X}_2^\vee$ are defined as the pullback of the corresponding generalized functional invariants, we have

$$f_0^{\tilde{X}_1}(q, q_{0,1}, y_1) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \sum_{d_0 \ge 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \frac{\langle \rho_{0,1}, d \rangle!(d_0 + \langle \rho_{0,2}, d \rangle)! q^d}{d_0!} (q_{0,1}/y_1)^{d_0};$$

$$f_0^{\tilde{X}_2}(q, y) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \ge 0}} \sum_{d_0 \ge 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \frac{(d_0 + \langle \rho_{0,1}, d \rangle)! \langle \rho_{0,2}, d \rangle! q^d}{d_0!} (y)^{d_0}.$$

Then, by direct computation, we have

Theorem 5.4. *The following Hadamard product relation holds*

$$f_0^{\tilde{X}}(q, q_0) \star_q f_0^{X_0}(q) = \frac{1}{2\pi i} \oint f_0^{\tilde{X}_1}(q, q_0, y) \star_q f_0^{\tilde{X}_2}(q, y) \frac{dy}{y}.$$

Let $\mathrm{NE}(Y) \subset H_2(Y, \mathbb{R})$ be the cone generated by effective curves and $\mathrm{NE}(Y)_{\mathbb{Z}} := \mathrm{NE}(Y) \cap (H_2(Y, \mathbb{Z})/\mathrm{tors})$. We have the Hadamard product relation among Picard-Fuchs equations.

Theorem 5.5. *The Hadamard product relation among the bases of solutions to Picard-Fuchs equations is*

$$I^{\tilde{X}}(q, q_0) \star_q I^{X_0}(q) = \frac{1}{2\pi i} \oint I^{\tilde{X}_1}(q, q_0/y) \star_q I^{\tilde{X}_2}(q, y) \frac{dy}{y},$$

where

$$I^{\tilde{X}}(q, q_0) = e^{\sum_{i=0}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$

$$\cdot \frac{\left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle + d_0} (\rho_{0,1} + p_0 + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle + d_0} (\rho_{0,2} + p_0 + k) \right)}{\prod_{k=1}^{d_0} (p_0 + k)^2} q_0^{d_0} q^d;$$

$$I^{X_0}(q) = e^{\sum_{i=1}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$

$$\cdot \left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right) q^d;$$

$$I^{\tilde{X}_1}(q, q_0) = e^{\sum_{i=0}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$

$$\cdot \frac{\left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle + d_0} (\rho_{0,2} + p_0 + k) \right)}{\prod_{k=1}^{d_0} (p_0 + k)} q_0^{d_0} q^d;$$

$$I^{\tilde{X}_2}(q, q_0) = e^{\sum_{i=0}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$

$$\cdot \frac{\left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle + d_0} (\rho_{0,1} + p_0 + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right)}{\prod_{k=1}^{d_0} (p_0 + k)} q_0^{d_0} q^d.$$

In the Hadamard product $\star_q$, we treat $\log q_i$ as a variable that is independent from q_i. Alternative, we can consider $\bar{I}(q) := I(q)/(\sum p_i \log q_i)$ and write Hadamard product relation for $\bar{I}$.

Remark 5.6. The case when we blow-up both quasi-Fanos can be considered as a special case of the Tyurin degeneration when blowing-up occurs on only one of the quasi-Fanos. Indeed, we can directly consider the Tyurin degeneration for $\tilde{X}$ in $Y \times \mathbb{P}^1$ given by a refinement of the nef partition $(\pi_1^* L_{0,2} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)) \oplus L_1 \oplus \cdots \oplus L_s \oplus (\pi_1^* L_{0,1} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1))$ with respect to either $\pi_1^* L_{0,2} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1))$ or $\pi_1^* L_{0,1} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1))$.

5.3. **General case.** We consider the case when we blow-up one of the quasi-Fanos. We assume that we blow-up X_2. The blown-up variety is denoted by $\tilde{X}_2$. The mirrors for X and X_0 are described in Section 5.1. The mirror for X_1 is the LG model

$$W : X_1^\vee \to \mathbb{C}^*$$
$$(x_1, \ldots, x_m) \mapsto F_{0,2}(x),$$

where $X_1^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \ \middle| \ \prod_{j=1}^m x_j^{m_{ij}} = q_{i,1}, i = 1, \ldots, r; \right.$$
$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) = 1 \right\}.$$

$\tilde{X}_2$ is a complete intersection in the toric variety $\mathbb{P}_Y(\mathcal{O} \oplus L_{0,2}^{-1})$ given by a generic section of $L_{0,2} \oplus L_1 \oplus \cdots \oplus L_s \oplus (\pi^* L_{0,1} \otimes \mathcal{O}(1))$. The LG model for $\tilde{X}_2$ is

$$W : \tilde{X}_2^\vee \to \mathbb{C}^*$$
$$(x_1, \ldots, x_m, y_2) \mapsto y_2,$$

where $\tilde{X}_2^\vee$ is the fiberwise compactification of

$$\left\{ (x_1, \ldots, x_m, y_2) \in (\mathbb{C}^*)^{m+1} \ \middle| \ \left(\prod_{j=1}^m x_j^{m_{ij}} \right) y_2^{-\sum_{k=a+1}^{a+b} m_{ij_k}} = q_{i,2}, i = 1, \ldots, r; \right.$$
$$\left. F_l(x) = 1, l = 1, \ldots, s; F_{0,1}(x) + \frac{q_{0,2}}{y_2} = 1, F_{0,2}(x) = 1 \right\}.$$

5.3.1. *Generalized functional invariants.* Then, we can compute the generalized functional invariants. Recall that $X^\vee$ is the compactification of

$$\left\{ (x_1, \ldots, x_m) \in (\mathbb{C}^*)^m \ \middle| \ \prod_{j=1}^m x_j^{m_{ij}} = q_i, i = 1, \ldots, r; F_l(x) = 1, l = 0, \ldots, s \right\}.$$

Set $F_{0,2}(x) = y$, then $F_{0,1}(x) = 1 - y$. Following the computation in Section 5.2.1, the generalized functional invariant for $X^\vee$ is

$$\lambda_i = \frac{q_i}{\prod_{k=1}^a (1-y)^{m_{ij_k}} \prod_{k=a+1}^{a+b} (y)^{m_{ij_k}}}, i = 1 \ldots, r. \tag{23}$$

For $X_1^\vee$, we also set $F_{0,2}(x) = y_1$, then the generalized functional invariant is

$$\lambda_{i,1} = \frac{q_{i,1}}{\prod_{k=a+1}^{a+b} y_1^{m_{ij_k}}}, i = 1 \ldots, r. \tag{24}$$

The generalized functional invariant for $\tilde{X}_2$ is

$$\lambda_{i,2} = \frac{q_{i,2}}{y_2^{-\sum_{k=a+1}^{a+b} m_{ij_k}} \prod_{k=1}^a (1 - q_{0,2}/y_2)^{m_{ij_k}}}, i = 1 \ldots, r. \tag{25}$$

We set

$$q_i = q_{i,1} = q_{i,2} y_2^{\sum_{k=a+1}^{a+b} m_{ij_k}}, i = 1, \ldots, r, \quad y = y_1 = q_{0,2}/y_2. \tag{26}$$

Proposition 5.7. *Under the identification (26), the generalized functional invariants satisfy the product relation*

$$\frac{\lambda_i}{q_i} = \frac{\lambda_{i,1}}{q_i}\frac{\lambda_{i,2}}{q_i}, \quad i = 1, \ldots, r.$$

Remark 5.8. One can specialize to Section 4 following Remark 5.3.

Remark 5.9. Section 5.2 can be viewed as a special case of the current section as follows. The ambient toric variety in Section 5.2 is $Y \times \mathbb{P}^1$. Let $p_1, \ldots, , p_r \in H^2(Y, \mathbb{Z})$ be a nef integral basis and $p_0 \in H^2(\mathbb{P}^1, \mathbb{Z})$ is the hyperplane class in $\mathbb{P}^1$. We use the same notation p_i to denote the pullbacks of p_i to $H^2(Y \times \mathbb{P}^1, \mathbb{Z})$. We can write the toric divisors as

$$D_j = \sum_{i=1}^{r} m_{ij} p_i + 0 p_0, \quad 1 \le j \le m,$$

for some m_{ij}. And

$$D_{m+1} = D_{m+2} = \sum_{i=1}^{r} 0 p_i + p_0.$$

In other words,

$$D_j = \sum_{i=0}^{r} m_{ij} p_i, \quad 1 \le j \le m+2,$$

where $m_{ij} = 0$ for $1 \le j \le m$ and $i = r+1$; $m_{ij} = 0$ for $j = m+1, m+2$ and $1 \le i \le m$. The Tyurin degeneration is given by the refinement of the nef partition with respect to the part $\pi_1^* L_{0,1} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$. The refinement produces two new parts $F_{0,1}(x) = x_{j_1} + \ldots, x_{j_a}$ and $F_{0,2} = y$. The generalized functional invariants for $X^\vee$ and $X_1^\vee$ are already computed in Section 5.2. The generalized functional invariant for $\tilde{X}_2^\vee$ is Equation (25). In this special case, the factor $y_2^{-\sum_{k=a+1}^{a+b} m_{ij_k}}$ in the denominator of (25) is actually $y_2^0 = 1$, because there is only one element in $F_{0,2}$ for $i \in \{1, \ldots, r\}$ and $j \in \{m+1, m+2\}$. So the generalized functional invariant for $\tilde{X}_2^\vee$ specialize to the one in Section 5.2. Hence, we recover the generalized functional invariants in Section 5.2.

Note that in Section 5.2, we consider X_0 as a complete intersection in Y, while, in the current section, we consider it as a complete intersection in $Y \times \mathbb{P}^1$. This results in a slightly different expressions of its mirror $X_0^\vee$. Therefore, generalized functional invariants for $X^\vee$ and $X_1^\vee$ are changed accordingly. Nevertheless, the product relation among generalized functional invariants always holds.

Remark 5.10. Note that

$$\rho_{0,2} = c_1(L_{0,2}) = \sum_{k=a+1}^{a+b} D_{j_k} = \sum_{k=a+1}^{a+b}\sum_{i=1}^{r} m_{ij_k} p_i = \sum_{i=1}^{r}\left(\sum_{k=a+1}^{a+b} m_{ij_k}\right) p_i.$$

The first Chern class of the normal bundle of the anticanonical divisor X_0 in X_1 is $i^* \rho_{0,2} \in H^2(X_0)$, where $i : X_0 \hookrightarrow Y$ is the inclusion. It gives an explanation for the factor $y_2^{\sum_{k=a+1}^{a+b} m_{ij_k}}$ in Equation (26). This factor does not appear in Section 5.2 because the normal bundle is trivial.

5.3.2. *Periods.* The holomorphic period for $X_0^\vee$ is

$$f_0^{X_0}(q) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! q^d.$$

We can compute the holomorphic period for $X^\vee$ via generalized functional invariants.

$$\frac{1}{(1-y)} f_0^{X_0}(\lambda)$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \frac{q^d}{(1-y)^{1+\langle \rho_{0,1}, d \rangle} y^{\langle \rho_{0,2}, d \rangle}}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \sum_{d_{0,1} \geq 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \frac{(d_{0,1} + \langle \rho_{0,1}, d \rangle)! q^d}{d_{0,1}! \langle \rho_{0,1}, d \rangle! y^{\langle \rho_{0,2}, d \rangle}} y^{d_{0,1}}.$$

We obtain the holomorphic period for $X^\vee$:

$$f_0^X(q) = \frac{1}{2\pi i} \oint \frac{1}{(1-y)} f_0^{X_0}(\lambda) \frac{dy}{y}$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} (\langle \rho_{0,1}, d \rangle + \langle \rho_{0,2}, d \rangle)! q^d.$$

$$= \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \frac{\prod_{l=0}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} q^d.$$

The holomorphic periods for $X_1^\vee$ and $\tilde{X}_2^\vee$ are computed in a similar way, we have

$$f_0^{X_1}(q, y_1) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \langle \rho_{0,1}, d \rangle! \langle \rho_{0,2}, d \rangle! \frac{q^d}{(y_1)^{\langle \rho_{0,2}, d \rangle}};$$

$$f_0^{\tilde{X}_2}(q, y) = \sum_{\substack{d \in H_2(Y;\mathbb{Z}) \\ \forall j, \langle D_j, d \rangle \geq 0}} \sum_{d_0 \geq 0} \frac{\prod_{l=1}^s \langle \rho_l, d \rangle!}{\prod_{j=1}^m \langle D_j, d \rangle!} \frac{(d_0 + \langle \rho_{0,1}, d \rangle)! \langle \rho_{0,2}, d \rangle! q^d}{d_0!} (y)^{d_0}.$$

Recall that $y = y_1 = q_{0,2}/y_2$. Then we have

Theorem 5.11. *The following Hadamard product relation holds*

$$f_0^X(q) \star_q f_0^{X_0}(q) = \frac{1}{2\pi i} \oint f_0^{X_1}(q, y) \star_q f_0^{\tilde{X}_2}(q, y) \frac{dy}{y}.$$

Similarly, we have the Hadamard product relation among Picard-Fuchs operators. Let

$$I^X(q) = e^{\sum_{i=0}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right) q^d;$$

$$I^{X_0}(q) = e^{\sum_{i=1}^r p_i \log q_i} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right) q^d;$$

$$I^{X_1}(q, y) = e^{\sum_{i=1}^r p_i \log \lambda_{i,1}} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle} (\rho_{0,1} + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right) \frac{q^d}{y^{\langle \rho_{0,2}, d \rangle}},$$

where $\lambda_{i,1} = \frac{q_{i,1}}{\prod_{k=a+1}^{a+b} y_1^{m_{ij_k}}}, i = 1 \dots, r;$

$$I^{\tilde{X}_2}(q, y) = e^{\sum_{i=1}^r p_i \log q_i + p_0 \log y} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}, d_0 \geq 0} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + k)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + k)} \right) \left(\prod_{l=1}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + k) \right)$$
$$\cdot \frac{\left(\prod_{k=1}^{\langle \rho_{0,1}, d \rangle + d_0} (\rho_{0,1} + p_0 + k) \prod_{k=1}^{\langle \rho_{0,2}, d \rangle} (\rho_{0,2} + k) \right)}{\prod_{k=1}^{d_0} (p_0 + k)} y^{d_0} q^d.$$

Theorem 5.12. *The Hadamard product relation among the bases of solutions to Picard-Fuchs equations is*

$$I^X(q) \star_q I^{X_0}(q) = \frac{1}{2\pi i} \oint I^{X_1}(q, y) \star_q I^{\tilde{X}_2}(q, y) \frac{dy}{y},$$

where we set $p_0 = \rho_{0,2}$. In the Hadamard product $\star_q$, we treat $\log q_i$ as a variable that is independent from q_i. Alternative, we can consider $\bar{I}(q) := I(q)/(\sum p_i \log q_i)$ and write Hadamard product relation for $\bar{I}$.

6. GROMOV-WITTEN INVARIANTS

In this section, we discuss how the relation among periods is related to the A-model data: Gromov-Witten invariants. The Tyurin degeneration naturally relates absolute Gromov-Witten invariants of a Calabi-Yau variety and the relative Gromov-Witten invariants of the quasi-Fano varieties via the degeneration formula [Li01], [Li02] and [LR01]. The periods for the mirror Calabi-Yau families are related to absolute Gromov-Witten invariants of Calabi-Yau varieties by the mirror theorem of [Giv98] and [LLY97]. The periods for the LG models are related to relative Gromov-Witten invariants of the quasi-Fano varieties by the relative mirror theorem which is recently proved in [FTY19].

6.1. **Definition.** In this section, we give a brief review of the definition of absolute and relative Gromov-Witten invariants.

Let X be a smooth projective variety. We consider the moduli space $\overline{M}_{g,n,d}(X)$ of n-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$ stable maps to X. Let ev_i be the i-th evaluation map, where

$$\mathrm{ev}_i : \overline{M}_{g,n,d}(X) \to X, \quad \text{for } 1 \leq i \leq n.$$

Let $s_i : \overline{M}_{g,n,d}(X) \to \mathcal{C}_{g,n,d}(X)$ be the i-th section of the universal curve, and let $\psi_i = c_1(s_i^*(\omega_{\mathcal{C}_{g,n,d}(X)/\overline{M}_{g,n,d}(X)}))$ be the descendant class at the i-th marked point. Consider

- $\gamma_i \in H^*(X, \mathbb{Q})$, for $1 \le i \le n$;
- $a_i \in \mathbb{Z}_{\ge 0}$, for $1 \le i \le n$.

Absolute Gromov-Witten invariants of X are defined as

$$(27) \qquad \left\langle \prod_{i=1}^{n} \tau_{a_i}(\gamma_i) \right\rangle_{g,n,d}^{X} := \int_{[\overline{M}_{g,n,d}(X)]^{\mathrm{vir}}} \psi_1^{a_1} \, \mathrm{ev}_1^*(\gamma_1) \cdots \psi_n^{a_n} \, \mathrm{ev}_n^*(\gamma_n).$$

In this paper, we will be focusing on genus zero absolute Gromov-Witten invariants of Calabi-Yau varieties.

Let X be a smooth projective variety and D be a smooth divisor. We can study the relative Gromov-Witten invariants of (X, D).

For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \dots, k_{n_1}) \in (\mathbb{Z}_{>0})^{n_1}$ of $\int_d[D]$. That is,

$$\sum_{i=1}^{n_1} k_i = \int_d[D].$$

We consider the moduli space $\overline{M}_{g,\vec{k},n_2,d}(X, D)$ of $(n_1 + n_2)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the contact orders for relative markings are given by the partition $\vec{k}$. We assume the first n_1 marked points are relative marked points and the last n_2 marked points are non-relative marked points. Let ev_i be the i-th evaluation map, where

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n_2,d}(X, D) \to D, \quad \text{for } 1 \le i \le n_1;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n_2,d}(X, D) \to X, \quad \text{for } n_1 + 1 \le i \le n_1 + n_2.$$

Write $\bar{\psi}_i = s^* \psi_i$ which is the class pullback from the corresponding descendant class on the moduli space $\overline{M}_{g,n_1+n_2,d}(X)$ of stable maps to X. Consider

- $\delta_i \in H^*(D, \mathbb{Q})$, for $1 \le i \le n_1$;
- $\gamma_{n_1+i} \in H^*(X, \mathbb{Q})$, for $1 \le i \le n_2$;
- $a_i \in \mathbb{Z}_{\ge 0}$, for $1 \le i \le n_1 + n_2$.

Relative Gromov-Witten invariants of (X, D) are defined as

$$(28)$$
$$\left\langle \prod_{i=1}^{n_1} \tau_{a_i}(\delta_i) \, \middle| \, \prod_{i=1}^{n_2} \tau_{a_{n_1+i}}(\gamma_{n_1+i}) \right\rangle_{g,\vec{k},n_2,d}^{(X,D)} :=$$
$$\int_{[\overline{M}_{g,\vec{k},n_2,d}(X,D)]^{\mathrm{vir}}} \bar{\psi}_1^{a_1} \, \mathrm{ev}_1^*(\delta_1) \cdots \bar{\psi}_{n_1}^{a_{n_1}} \, \mathrm{ev}_{n_1}^*(\delta_{n_1}) \bar{\psi}_{n_1+1}^{a_{n_1+1}} \, \mathrm{ev}_{n_1+1}^*(\gamma_{n_1+1}) \cdots \bar{\psi}_{n_1+n_2}^{a_{n_1+n_2}} \, \mathrm{ev}_{n_1+n_2}^*(\gamma_{n_1+n_2}).$$

We refer to [IP03], [LR01], [Li01] and [Li02] for more details about the construction of relative Gromov-Witten theory. Relative Gromov-Witten theory has recently been generalized to include negative contact orders (allowing $k_i < 0$) in [FWY19b] and [FWY19a]. We refer to [FWY19b] and [FWY19a] for the precise definition of relative Gromov-Witten invariants with negative contact orders. In this paper, we will be focusing on genus zero relative Gromov-Witten invariants of quasi-Fano varieties along with their anticanonical divisors.

6.2. **A mirror theorem for Calabi-Yau varieties.** In this section, we briefly review the mirror theorem for toric complete intersections following [Giv98]. We are focusing on the case of Calabi-Yau varieties. A mirror theorem relates a generating function of Gromov-Witten invariants of a Calabi-Yau variety and periods of the mirror. The generating function considered by Givental [Giv98] is called J-function. The periods are encoded in the so-called I-function in [Giv98].

The J-function is a cohomological valued function defined by

$$J^X(\tau, z) = e^{\tau/z}\left(1 + \sum_{\alpha} \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}} \setminus \{0\}} Q^d \left\langle \frac{\phi_\alpha}{z(z - \psi)} \right\rangle_{0,1,d}^X \phi^\alpha\right),$$

where the notation is explained as follows:

- $\tau = \sum_{i=1}^r p_i \log Q_i \in H^2(X)$.
- $p_1, \ldots, p_r$ is an integral, nef basis of $H^2(X)$.
- $\mathrm{NE}(X) \subset H_2(X, \mathbb{R})$ is the cone generated by effective curves and $\mathrm{NE}(X)_{\mathbb{Z}} := \mathrm{NE}(X) \cap (H_2(X, \mathbb{Z})/\mathrm{tors})$.
- $\{\phi_\alpha\}$ and $\{\phi^\alpha\}$ are dual bases of $H^*(X)$.
- $\langle \frac{\phi_\alpha}{z(z-\psi)} \rangle_{0,1,d}^X$ is Gromov-Witten invariant of X with degree d, 1-marked point and the insertion is read as follows:

$$\frac{\phi_\alpha}{z(z - \psi)} = \sum_{k \geq 0} \phi_\alpha \psi^k z^{-(k+2)}.$$

Let X be a Calabi-Yau complete intersection in a toric variety Y defined by a section of $E = L_0 \oplus L_1 \oplus \cdots \oplus L_s$, where each L_l is a nef line bundle. Recall that $\rho_l = c_1(L_l)$ and $-K_Y = \sum_{l=0}^s c_1(L_l)$. Consider the fiberwise $\mathbb{C}^*$-action on the total space E. Following Coates-Givental [CG07], we consider the universal family

$$\begin{array}{ccc} \mathcal{C}_{0,n,d} & \xrightarrow{\;\mathrm{ev}\;} & Y \;. \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M}_{0,n,d}(Y) & & \end{array}$$

Coates-Givental [CG07] define

$$E_{0,n,d} := R\pi_* \mathrm{ev}^* E \in K^0_{\mathbb{T}}(\overline{M}_{0,n,d}(Y)).$$

Let $e(\cdot)$ denote the $\mathbb{C}^*$-equivariant Euler class. The twisted J-function is

$$J^{e,E}(\tau, z) = e^{\tau/z}\left(1 + \sum_{\alpha} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}} \setminus \{0\}} Q^d \left\langle \frac{\phi_\alpha}{z(z - \psi)} \right\rangle_{0,1,d}^{e,E} \phi^\alpha\right),$$

where

$$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i) \right\rangle_{0,n,d}^{(e,E)} := \int_{[\overline{M}_{0,n,d}(Y)]^{\mathrm{vir}}} \prod_{i=1}^n (\mathrm{ev}_i^* \gamma_i)\bar{\psi}_i^{a_i} e(E_{0,n,d})$$

is called a twisted Gromov-Witten invariant of Y. The twisted J-function admits a non-equivariant limit $J^{Y,X}$, which satisfies

$$i_* J^X(i^*(\tau), z) = J^{Y,X}(\tau, z) \cup \prod_{l=0}^s \rho_l,$$

where $i : X \hookrightarrow Y$ is the inclusion.

Let $D_1, \ldots, D_m$ be the classes in $H^2(Y)$ which are Poincaré dual to the toric divisors in Y. The I-function is cohomological valued function defined by

$$I^X(q, z) = e^{\sum_{i=1}^r p_i \log q_i / z} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + kz)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + kz)} \right) \left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + kz) \right) q^d.$$

The I-function can be expanded as a power series in $1/z$:

$$I^X(q, z) = f_0(q) + \frac{f_1(q)}{z} + O(z^{-1}).$$

The mirror theorem is stated as follows.

Theorem 6.1 ([Giv98], [LLY97]). *The I-function and the J-function are equal after change of variables:*

$$J^{Y,X}(f_1(q)/f_0(q), z) = I^X(q, z)/f_0(q).$$

Remark 6.2. Note that $I^X(q, z)$ takes values in $H^*(Y)$. The restriction $i^* I^X(q, z)$ takes values in $H^*(X)$ and lies in Givental's Lagrangian cone of the Gromov-Witten theory of X. A more common notation for our $I^X(q, z)$ is $I^{Y,X}(q, z)$. We choose not to use this notation to avoid any possible confusion with the relative I-function that we consider in Section 6.3.

Remark 6.3. By setting $z = 1$ for the I-functions, we obtain the bases of solutions to the Picard-Fuchs equations considered in previous sections.

We list the I-functions for some examples of Calabi-Yau varieties that we consider in this paper.

Example 6.4. Let Q_5 be a smooth quintic threefold in $\mathbb{P}^4$. The I-function is

$$I^{Q_5}(q, z) = e^{H \log q / z} \sum_{d \geq 0} \left(\frac{\prod_{k=1}^{5d} (5H + kz)}{\prod_{k=1}^d (H + kz)^5} \right) q^d.$$

Example 6.5. Let $\tilde{Q}_5$ be a complete intersection of bidegrees $(4, 1)$ and $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. The I-function is

$$I^{\tilde{Q}_5}(q_1, q_0, z) = q_1^{H/z} q_0^{P/z} \sum_{d_1, d_0 \geq 0} \left(\frac{\prod_{k=1}^{4d_1 + d_0} (4H + P + kz) \prod_{k=1}^{d_1 + d_0} (H + P + kz)}{\prod_{k=1}^{d_1} (H + kz)^5 \prod_{k=1}^{d_0} (P + kz)^2} \right) q_1^{d_1} q_0^{d_0}.$$

6.3. **A mirror theorem for log Calabi-Yau pairs.** A mirror theorem for a relative pair (X, D) is proved in [FTY19]. In this section, we focus on the case when X is a toric complete intersection and D is a smooth anticanonical divisor of X. The relative mirror theorem in [FTY19] is again formulated in terms of the relation between J-functions and I-functions under Givental's formalism for relative Gromov-Witten theory which is recently developed in [FWY19b].

Following [FWY19b], the ring of insertions for relative Gromov-Witten theory is defined to be

$$\mathfrak{H} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{H}_i,$$

where $\mathfrak{H}_0 = H^*(X)$ and $\mathfrak{H}_i = H^*(D)$ if $i \in \mathbb{Z} \smallsetminus \{0\}$. For an element $\alpha \in \mathfrak{H}_i$, we write $[\alpha]_i$ for its embedding in $\mathfrak{H}$. The pairing on $\mathfrak{H}$ is defined as follows:

$$
(29) \qquad ([\alpha]_i, [\beta]_j) = \begin{cases} 0, & \text{if } i + j \neq 0, \\ \int_X \alpha \cup \beta, & \text{if } i = j = 0, \\ \int_D \alpha \cup \beta, & \text{if } i + j = 0, i, j \neq 0. \end{cases}
$$

Let $t_{0,2} \in H^2(X)$ and $[t_{tw}] \in \mathfrak{H} \smallsetminus \mathfrak{H}_0$. Following [FTY19], we define the J-function.

Definition 6.6. The J-function for relative Gromov-Witten invariants of (X, D) is a $\mathfrak{H}$-valued function

$$
J^{(X,D)}([t], z) :=
$$

$$
e^{t_{0,2}/z} \left(1 + \frac{[t_{tw}]}{z} + \sum_{(n,d) \neq (0,0)} \sum_\alpha \frac{Q^d}{n!} \left\langle \frac{[\phi_\alpha]}{z(z - \bar\psi)}, [t_{tw}], \ldots, [t_{tw}] \right\rangle_{0,n+1,d}^{(X,D)} [\phi^\alpha] \right),
$$

where $[t] = [t_{0,2}] + [t_{tw}] \in \mathfrak{H}$; $t_{0,2} = \sum_{i=1}^r p_i \log Q_i \in H^2(X)$; $\{[\phi_\alpha]\}$ and $\{[\phi^\alpha]\}$ are dual bases of $\mathfrak{H}$ under the pairing (29).

For simplicity, we only write down the I-function for the log Calabi-Yau pair (X, D), where X is a complete intersection in the toric variety Y. We consider the extended I-function for relative Gromov-Witten theory. It may be considered as a "limit" of the extended I-function for root stacks $X_{D,r}$ for $r \to \infty$. This is because of the relation between genus zero relative and orbifold Gromov-Witten invariants proved in [ACW17], [TY18] and [FWY19b]. By [FTY19], root stacks are hypersurfaces in toric stack bundles with fibers being weighted projective lines $\mathbb{P}^1_{1,r}$. Let $\sum_{i>0}^b x_i[\mathbf{1}]_i$ represent the extended data. In the language of orbifold Gromov-Witten theory, x_i corresponds to the box element of age i/r in the extended stacky fan of $\mathbb{P}^1_{1,r}$. Note that the age i/r in orbifold Gromov-Witten theory corresponds to contact order i in relative Gromov-Witten theory. The limit is taken by first choosing k_i, d and b, then taking a sufficiently large r. Since it works for any $b > 0$, we may formally take $b \to \infty$ after taking $r \to \infty$.

Definition 6.7. The extended I-function for relative invariants is defined as follows

$$
I^{(X,D)}(q, x, t, z) = I_+ + I_-,
$$

where

$$
I_+ := e^{\sum_{i=1}^r p_i \log q_i / z} \sum_{\substack{d \in \overline{\mathrm{NE}}(Y), k_i \geq 0 \\ \sum i k_i < \langle D, d \rangle}} \left(\prod_{j=1}^m \frac{\prod_{k=-\infty}^0 (D_j + kz)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + kz)} \right) \left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + kz) \right)
$$

$$
\frac{\prod_{0 < k \leq \langle D, d \rangle} (D + kz)}{D + (\langle D, d \rangle - \sum i k_i) z} \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod (k_i!)} [\mathbf{1}]_{-\langle D, d \rangle + \sum i k_i} q^d,
$$

and

$$I_- := e^{\sum_{i=1}^r p_i \log q_i/z} \sum_{\substack{d \in \overline{\mathrm{NE}}(Y), k_i \geq 0 \\ \sum i k_i \geq \langle D, d \rangle}} \left(\prod_{j=1}^m \frac{\prod_{k=-\infty}^0 (D_j + kz)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + kz)} \right) \left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + kz) \right)$$

$$\left(\prod_{0 < k \leq \langle D, d \rangle} (D + kz) \right) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod (k_i!)} [\mathbf{1}]_{-\langle D, d \rangle + \sum i k_i} q^d.$$

Theorem 6.8 ([FTY19], Theorem 1.5). *The extended relative I-function $I^{(X,D)}$ lies in Givental's Lagrangian cone for relative invariants which is defined in* [FWY19b, Section 7.5].

For the purpose of this paper, we take the part of the J-function that takes values in $H^*(X)$. Therefore, we consider the function of the form

$$J_0^{(X,D)}([t], z) :=$$

$$e^{t_{0,2}/z} z \left(1 + \sum_{(n,d) \neq (0,0)} \sum_\alpha \frac{Q^d}{n!} \left\langle \frac{[\phi_\alpha]_0}{z - \psi}, [t_{tw}], \dots, [t_{tw}] \right\rangle_{0, n+1, d}^{(X,D)} [\phi^\alpha]_0 \right),$$

where $[\phi_\alpha]_0$ and $[\phi^\alpha]_0$ are dual bases of $H^*(X)$. The corresponding I-function, denoted by I_0, is the part of I_- that takes values in $\mathfrak{H}_0 = H^*(X)$. Note that the I-function that we consider in Definition 6.7 is actually a non-equivariant limit of the twisted I-function from the ambient toric variety Y similar to the absolute case in Section 6.2. Therefore, I_0 takes values in $H^*(Y)$ and $i^* I_0$ takes values in $H^*(X)$ where $i : X \hookrightarrow Y$ is the inclusion.

Definition 6.9. Let X be a complete intersection in a toric variety Y defined by a section of $E = L_0 \oplus L_1 \oplus \dots \oplus L_s$, where each L_l is a nef line bundle. Let $\rho_l = c_1(L_l)$. Assuming D is nef, the I-function is the $H^*(Y)$-valued function defined as

$$(30) \qquad I_0^{(X,D)}(q, x, z) = e^{\sum_{i=1}^r p_i \log q_i/z} \sum_{d \in \mathrm{NE}(Y)_{\mathbb{Z}}} \sum_{k_i \geq 0, \sum i k_i = \langle D, d \rangle} \prod_{j=1}^m \left(\frac{\prod_{k=-\infty}^0 (D_j + kz)}{\prod_{k=-\infty}^{\langle D_j, d \rangle} (D_j + kz)} \right)$$

$$\left(\prod_{l=0}^s \prod_{k=1}^{\langle \rho_l, d \rangle} (\rho_l + kz) \right) \prod_{k=1}^{\langle D, d \rangle} (D + kz) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod k_i!} q^d.$$

Remark 6.10. The nefness assumption on D can be dropped when D is coming from a toric divisor on Y. In this case, the relative Gromov-Witten theory of (X, D) can be considered as a limit of the Gromov-Witten theory of the complete intersection on the root stack $Y_{D,r}$. Since $Y_{D,r}$ is a toric orbifold, the I-function for a complete intersection in toric orbifolds is known in [CCIT14].

Now, we consider some examples that we study in this paper.

Example 6.11. Let X be a complete intersection of bidegrees $(4, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and D is its smooth anticanonical divisor. The I-function I_0 is

$$I_0^{(X,D)}(q_1, q_0, x, z) =$$

$$q_1^{H/z} q_0^{P/z} \sum_{d_1, d_0 \geq 0} \sum_{k_i \geq 0, \sum i k_i = d_0} \left(\frac{\prod_{k=1}^{4d_1+d_0} (4H + P + kz)}{\prod_{k=1}^{d_1} (H + kz)^4 \prod_{k=1}^{d_0} (P + kz)} \right) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod k_i!} q_1^{d_1} q_0^{d_0}.$$

The extended I-function is

$$I^{(X,D)}(q_1, q_0, x, z) = I_+ + I_-,$$

where

$$I_+ := q_1^{H/z} q_0^{P/z} \sum_{\substack{d_1, d_0 \geq 0, k_i \geq 0, \\ \sum i k_i < d_0,}} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right)$$

$$\frac{1}{P + (d_0 - \sum i k_i)z} \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod(k_i!)} [\mathbf{1}]_{-d_0 + \sum i k_i} q_1^{d_1} q_0^{d_0},$$

and

$$I_- := q_1^{H/z} q_0^{P/z} \sum_{\substack{d_1, d_0 \geq 0, k_i \geq 0 \\ \sum i k_i \geq d_0}} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right)$$

$$\frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod(k_i!)} [\mathbf{1}]_{-d_0 + \sum i k_i} q_1^{d_1} q_0^{d_0}.$$

The mirror map is given by the quotient of the coefficients of z^0 and z^{-1}. The z^0-coefficient is $\sum_{d_1 \geq 0} \frac{(4d_1)!}{(d_1!)^4} q_1^{d_1}$. The coefficient of z^{-1} is

$$\sum_{d_1 \geq 1} \frac{(4d_1)!}{(d_1!)^4} q_1^{d_1} \left(\sum_{k=d_1+1}^{4d_1} \frac{1}{k} \right)(4H + P) + H \log q_1 + P \log q_0$$

$$+ \sum_{d_1 \geq 0, d_0 > 0} \frac{(4d_1 + d_0)!}{(d_1!)^4 (d_0)! d_0} [\mathbf{1}]_{-d_0} q_1^{d_1} q_0^{d_0} + \sum_{i=1}^{\rho} \left(\sum_{d_1 \geq 0, 0 \leq d_0 \leq i} \frac{(4d_1 + d_0)!}{(d_1!)^4 (d_0)!} q_1^{d_1} q_0^{d_0} [\mathbf{1}]_{-d_0 + i} \right) x_i.$$

Example 6.12. Let X be a complete intersection of bidegrees $(4, 0)$ and $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. Let D be its smooth anticanonical divisor. The I-function I_0 is

$$I_0^{(X,D)}(q_1, q_0, x, z) =$$

$$q_1^{H/z} q_0^{P/z} \sum_{d_1, d_0 \geq 0} \sum_{k_i \geq 0, \sum i k_i = d_0} \left(\frac{\prod_{k=1}^{4d_1}(4H + kz) \prod_{k=1}^{d_1+d_0}(H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^5 \prod_{k=1}^{d_0}(P + kz)} \right) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod k_i!} q_1^{d_1} q_0^{d_0}.$$

Example 6.13. Let Q_4 be the quartic threefold in $\mathbb{P}^4$ and $K3$ be the smooth anticanonical divisor. The I-function I_0 is

$$I_0^{(Q_4, K3)}(q_1, x, z) = e^{H \log q_1 / z} \sum_{d \geq 0} \sum_{k_i \geq 0, \sum i k_i = d} \left(\frac{\prod_{k=1}^{4d}(4H + kz)}{\prod_{k=1}^{d}(H + kz)^4} \right) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod k_i!} q_1^d.$$

Example 6.14. Let X be the blow-up of $\mathbb{P}^3$ along the complete intersection of degrees 4 and 5 hypersurfaces. The extended I-function for relative invariants is defined as follows

$$I^{(X,D)}(q_1, q_0, x, t, z) = I_+ + I_-,$$

where

$$I_+ := q_1^{H/z} q_0^{P/z} \sum_{\substack{d_1, d_0 \geq 0, k_i \geq 0 \\ \sum i k_i < d_0 - d_1,}} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right)$$

$$\frac{1}{P - H + (d_0 - d_1 - \sum i k_i)z} \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod(k_i!)} [\mathbf{1}]_{-d_0 + d_1 + \sum i k_i} q_1^{d_1} q_0^{d_0},$$

and

$$I_- := q_1^{H/z} q_0^{P/z} \sum_{\substack{d_1, d_0 \geq 0, k_i \geq 0 \\ \sum i k_i \geq d_0 - d_1}} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right)$$

$$\frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod(k_i!)} [\mathbf{1}]_{-d_0+d_1+\sum i k_i} q_1^{d_1} q_0^{d_0}.$$

In particular, $I_0^{(X,D)}$ is the part of I_- that takes values in $H^\star(X)$ (that is, when $d_0 - d_1 = \sum i k_i$). We have

$$I_0^{(X,D)}(q_1, q_0, x, z) := q_1^{H/z} q_0^{P/z} \sum_{\substack{d_1, d_0 \geq 0, k_i \geq 0 \\ \sum i k_i = d_0 - d_1}} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right) \frac{\prod x_i^{k_i}}{z^{\sum k_i} \prod(k_i!)} q_1^{d_1} q_0^{d_0}.$$

6.4. **Relation with periods.** Since Gromov-Witten invariants are related to periods via mirror maps, the relation among periods given by the gluing formula implies a relation among different Gromov-Witten invariants via their respective mirror maps.

Relative periods can be extracted from the I-functions for relative Gromov-Witten theory. We will illustrate this with an explicit example. Recall that, $\tilde{Q}_5$ is a complete intersection of bidegrees $(4, 1)$ and $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. The Calabi-Yau threefold $\tilde{Q}_5$ admits the following Tyurin degeneration

$$\tilde{Q}_5 \rightsquigarrow X_1 \cup_{K3} X_2,$$

where X_1 is a hypersurface of bidegree $(4, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and X_2 is a complete intersection of bidegrees $(4, 0), (1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^1$. Note that the I-function for relative invariants includes some extra variables x_i corresponding to the contact order of relative markings.

There are many ways to extract periods from the relative I-function. To simplify the discussion, we can extract the coefficient of $x_{d_0} q_1^{d_1} q_0^{d_0}$ for each $d_1, d_0 \geq 0$ from $I_0^{(X_1, K3)}$ and $I_0^{(X_2, K3)}$. Setting $x_i = 1$, then the resulting I-functions are

$$I_0^{(X_1, K3)}(q_1, q_0, z) =$$
$$q_1^{H/z} q_0^{P/z} \sum_{d_1, d_0 \geq 0} \left(\frac{\prod_{k=1}^{4d_1+d_0}(4H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^4 \prod_{k=1}^{d_0}(P + kz)} \right) \frac{q_1^{d_1} q_0^{d_0}}{z^{\min(1,d_0)}},$$

and

$$I_0^{(X_2, K3)}(q_1, q_0, z) =$$
$$q_1^{H/z} q_0^{P/z} \sum_{d_1, d_0 \geq 0} \left(\frac{\prod_{k=1}^{4d_1}(4H + kz) \prod_{k=1}^{d_1+d_0}(H + P + kz)}{\prod_{k=1}^{d_1}(H + kz)^5 \prod_{k=1}^{d_0}(P + kz)} \right) \frac{q_1^{d_1} q_0^{d_0}}{z^{\min(1,d_0)}},$$

which, after setting $z = 1$, are exactly the bases of solutions for the Picard-Fuchs equations in Section 3.2.3. One can then extract the corresponding invariants from the J-functions with the corresponding mirror maps. The gluing formula for periods yields a formula relating the corresponding invariants by pluging in the corresponding J-functions along with mirror maps to Equation (12).

Remark 6.15. The resulting gluing formula for Gromov-Witten invariants relates absolute Gromov-Witten invariants of the Calabi-Yau manifolds X and X_0 to the relative Gromov-Witten invariants of the pairs (X_1, X_0) and (X_2, X_0). This is clearly a sort of "B-model

motivated" degeneration formula for genus zero Gromov-Witten invariants of Tyurin degenerations. Nevertheless, we do not know at present how to relate our gluing formula for periods to the degeneration formula for Gromov-Witten invariants as described in [Li01, Li02, LR01].

More so than the complexity of the mirror maps and Birkhoff factorizations, there are several other obstructions to directly checking compatibility. For example, the relative mirror theorem only involves relative Gromov-Witten invariants whose insertions at the relative markings are cohomology classes pulled back from the ambient space. On the other hand, the degeneration formula involves relative Gromov-Witten invariants whose insertions at relative markings having cohomology classes that are not pulled back from the ambient space. Moreover, there may be relative invariants with negative contact orders involved in the relative mirror theorem, while the degeneration formula only has relative invariants with markings of positive contact orders.

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DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA AND CENTER OF MATHEMATICAL SCIENCES AND APPLICATIONS, HARVARD UNIVERSITY

E-mail address: charles.doran@ualberta.ca

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY

E-mail address: jordan_kostiuk@brown.edu

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA AND FIELDS INSTITUTE FOR RESEARCH IN THE MATHEMATICAL SCIENCES

E-mail address: fenglong@ualberta.ca